Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 9 Issue 3(2017), Pages 16-30.

HADAMARD AND FEJÉR-HADAMARD INEQUALITIES AND RELATED RESULTS VIA CAPUTO FRACTIONAL DERIVATIVES

GHULAM FARID, ANUM JAVED, SAIRA NAQVI

Abstract. In this paper we prove the Hadamard and the Fejér-Hadamard inequalities for convex functions via Caputo fractional derivatives. We also derive some related inequalities for n-time differentiable functions $f^{(n)}$ such that $|f^{(n)}|^q, q \geq 1$ is convex, by using Caputo fractional derivatives.

1. Introduction

A function $f : [a, b] \to \mathbb{R}$ is said to be convex if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

holds, for all $x, y \in [a, b]$ and $\lambda \in [0, 1]$. If $-f$ is convex, then f is called concave function and vice versa.

In literature double integral inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

where $f : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, is known as the Hadamard inequality. If f is concave, then the above inequalities hold in the reverse direction.

In [14] Fejér gave the following generalization of the Hadamard inequality.

Theorem 1.1. *Let $f : [a, b] \to \mathbb{R}$ be a convex function over $[a, b]$ with $a < b$ and $g : [a, b] \to \mathbb{R}$ is a nonnegative, integrable and symmetric to $\frac{a+b}{2}$. Then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \leq \frac{1}{b-a}\int_a^b f(x)g(x)dx \leq \frac{f(a)+f(b)}{2}\int_a^b g(x)dx. \tag{1.2}$$

In literature above inequality is known as the Fejér-Hadamard inequality.
The Hadamard inequality and the Fejér-Hadamard inequality got the attention of many mathematicians and many generalizations and refinements have been found so far, for details see, [1, 2, 3, 5, 6, 7, 8, 9, 12] and the references therein.

2010 *Mathematics Subject Classification.* 26A51, 26A33, 26D15, 26D10.
Key words and phrases. Convex functions; Hadamard inequality; Fractional derivatives.

Submitted March 3, 2017. Published August 8, 2017.
Communicated by Jozsef Sandor.

In [5] Dragomir et al. proved the following results which give the bounds of a difference of the Hadamard inequality.

Theorem 1.2. *Let $f : I^o \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I^o, $a, b \in I^o$ and $a < b$. If $f' \in L(a,b)$, then the following inequality holds:*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \right| \leq \frac{(b-a)}{8}\left(|f'(a)| + |f'(b)|\right). \tag{1.3}$$

Theorem 1.3. *Let $f : I^o \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I^o, $a, b \in I^o$ with $a < b$, $f' \in L(a,b)$. If the mapping $|f'|^{\frac{p}{p-1}}$, where $p > 1$, is convex on $[a,b]$, then the following inequality holds:*

$$\begin{aligned} &\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \right| \\ &\leq \frac{b-a}{2(p+1)^{\frac{1}{p}}}\left[\frac{|f'(a)|^{\frac{p}{p-1}} + |f'(b)|^{\frac{p}{p-1}}}{2}\right]^{\frac{p-1}{p}}. \end{aligned} \tag{1.4}$$

Fractional calculus is as much important as calculus. Actually, fractional calculus is a natural extension of classical calculus. Fractional integration and fractional differentiation appear as tools in the subject of partial differential equations [10, 11]. In 1967, M. Caputo made the most significant contribution to fractional calculus. One of the main drawback of the Riemann-Liouville definition of fractional derivative is the strange set of initial conditions. Caputo reformulated the more classic definition of the Riemann-Liouville fractional derivative in order to use classical initial conditions [4].
In the following we give the definition of Caputo fractional derivatives [10].

Definition 1.4. *Let $\alpha > 0$ and $\alpha \notin \{1, 2, 3, ...\}$, $n = [\alpha] + 1$, $f \in AC^n[a,b]$, the space of functions having nth derivatives absolutely continuous. The right-sided and left-sided Caputo fractional derivatives of order α are defined as follows:*

$$({}^C D^{\alpha}_{a+} f)(x) = \frac{1}{\Gamma(n-\alpha)}\int_a^x \frac{f^{(n)}(t)}{(x-t)^{\alpha-n+1}}dt, x > a \tag{1.5}$$

and

$$({}^C D^{\alpha}_{b-} f)(x) = \frac{(-1)^n}{\Gamma(n-\alpha)}\int_x^b \frac{f^{(n)}(t)}{(t-x)^{\alpha-n+1}}dt, x < b. \tag{1.6}$$

If $\alpha = n \in \{1, 2, 3, ...\}$ and usual derivative $f^{(n)}(x)$ of order n exists, then Caputo fractional derivative $({}^C D^{n}_{a+} f)(x)$ coincides with $f^{(n)}(x)$ whereas $({}^C D^{n}_{b-} f)(x)$ coincides with $f^{(n)}(x)$ with exactness to a constant multiplier $(-1)^n$. In particular we have

$$({}^C D^{0}_{a+} f)(x) = ({}^C D^{0}_{b-} f)(x) = f(x) \tag{1.7}$$

where $n = 1$ and $\alpha = 0$.

Recently, Fractional integral inequalities have been studied extensively via fractional integral operators. These inequalities provide upper as well as lower bounds for solutions of the fractional boundary value problems. In this paper in Section 2 we give the Hadamard and the Hadamard type inequalities for convex functions via Caputo fractional derivatives. In Section 3 we derive the Fejér-Hadamard and the Fejér-Hadamard type inequalities for Caputo fractional derivatives.

In the whole paper we consider $C^n[a,b]$ the space of functions $f:[a,b]\to\mathbb{R}$ which are n-time differentiable and $f^{(n)}$ are continuous on $[a,b]$.

2. Hadamard and Hadamard type inequalities via Caputo fractional derivatives

First we prove the following lemmas which will be useful to prove the required results.

Lemma 2.1. *Let $g:[a,b]\to\mathbb{R}$, $a<b$, be a function such that $g\in C^n[a,b]$. If $g^{(n)}$ is symmetric to $\frac{a+b}{2}$, then we have*

$$^C D^{\alpha}_{a+}g(b)=(-1)^{n\,C}D^{\alpha}_{b-}g(a)=\frac{1}{2}[^C D^{\alpha}_{a+}g(b)+(-1)^{n\,C}D^{\alpha}_{b-}g(a)].$$

Proof. By symmetricity of $g^{(n)}$ we have $g^{(n)}(a+b-x)=g^{(n)}(x)$, where $x\in[a,b]$. In the following integral we have

$$\begin{aligned}
^C D^{\alpha}_{a+}g(b) &= \frac{1}{\Gamma(n-\alpha)}\int_a^b \frac{g^{(n)}(x)}{(b-x)^{\alpha-n+1}}dx\\
&= \frac{1}{\Gamma(n-\alpha)}\int_a^b \frac{g^{(n)}(a+b-x)}{(x-a)^{\alpha-n+1}}dx\\
&= \frac{1}{\Gamma(n-\alpha)}\int_a^b \frac{g^{(n)}(x)}{(x-a)^{\alpha-n+1}}dx\\
&= (-1)^{n\,C}D^{\alpha}_{b-}g(a).
\end{aligned}$$

From which we get the required equality. □

Lemma 2.2. *Let $f:[a,b]\to\mathbb{R}$, $0\le a<b$, be the function such that $f\in C^n[a,b]$. Also let $f^{(n+1)}$ be positive and convex function on $[a,b]$. Then the following equality for Caputo fractional derivatives holds:*

$$\begin{aligned}
&\frac{f^{(n)}(a)+f^{(n)}(b)}{2}-\frac{\Gamma(n-\alpha+1)}{2(b-a)^{n-\alpha}}\left[^C D^{\alpha}_{a+}f(b)+(-1)^{n\,C}D^{\alpha}_{b-}f(a)\right] \qquad (2.1)\\
&=\frac{b-a}{2}\int_0^1\left((1-t)^{n-\alpha}-t^{n-\alpha}\right)f^{(n+1)}(ta+(1-t)b)dt.
\end{aligned}$$

Proof. One can note that

$$\begin{aligned}
&\frac{b-a}{2}\int_0^1\left((1-t)^{n-\alpha}-t^{n-\alpha}\right)f^{(n+1)}(ta+(1-t)b)dt\\
&=\frac{b-a}{2}\int_0^1\left((1-t)^{n-\alpha}\right)f^{(n+1)}(ta+(1-t)b)dt\\
&-\frac{b-a}{2}\int_0^1\left(t^{n-\alpha}\right)f^{(n+1)}(ta+(1-t)b)dt,
\end{aligned}$$

where by simple calculation one can get

$$\begin{aligned}
&\frac{b-a}{2}\int_0^1 (1-t)^{n-\alpha} f^{(n+1)}(ta+(1-t)b)dt \\
&= \frac{b-a}{2}\left[\frac{f^{(n)}(b)}{b-a} - (n-\alpha)\int_a^b \left(\frac{x-a}{b-a}\right)^{n-\alpha-1} \frac{f^{(n)}(x)}{b-a}dx\right] \\
&= \frac{f^{(n)}(b)}{2} - \frac{\Gamma\left(n-\alpha+1\right)}{2(b-a)^{n-\alpha}}(-1)^{n} {}^{C}D^{\alpha}_{b-}f(a)
\end{aligned}$$

and

$$\begin{aligned}
&-\frac{b-a}{2}\int_0^1 t^{n-\alpha} f^{(n+1)}(ta+(1-t)b)dt \\
&= \frac{b-a}{2}\left[\frac{f^{(n)}(a)}{b-a} - (n-\alpha)\int_a^b \left(\frac{b-x}{b-a}\right)^{n-\alpha-1} \frac{f^{(n)}(x)}{b-a}dx\right] \\
&= \frac{f^{(n)}(a)}{2} - \frac{\Gamma\left(n-\alpha+1\right)}{2(b-a)^{n-\alpha}} {}^{C}D^{\alpha}_{a+}f(b)
\end{aligned}$$

Hence (2.1) can be established. □

In the following we give the Hadamard inequality for Caputo fractional derivatives.

Theorem 2.3. *Let $f : [a, b] \to \mathbb{R}, 0 \leq a < b$ be the function such that $f \in C^n[a, b]$. Also let $f^{(n)}$ be positive and convex function on $[a, b]$. Then the following inequality for Caputo fractional derivatives holds:*

$$\begin{aligned}
&f^{(n)}\left(\frac{a+b}{2}\right) \\
&\leq \frac{\Gamma(n-\alpha+1)}{2(b-a)^{n-\alpha}}\left[{}^{C}D^{\alpha}_{a+}f^{(n)}(b) + (-1)^{n} {}^{C}D^{\alpha}_{b-}f^{(n)}(a)\right] \\
&\leq \frac{f^{(n)}(a)+f^{(n)}(b)}{2}.
\end{aligned} \tag{2.2}$$

Proof. It is given that $f^{(n)}$ is convex, therefore for $x, y \in [a, b]$ we have

$$f^{(n)}\left(\frac{x+y}{2}\right) \leq \frac{f^{(n)}(x)+f^{(n)}(y)}{2}. \tag{2.3}$$

Let $x = ta + (1-t)b, y = (1-t)a + tb$ for $t \in [0, 1]$. Then $x, y \in [a, b]$ and (2.3) gives

$$2f^{(n)}\left(\frac{a+b}{2}\right) \leq f^{(n)}(ta+(1-t)b) + f^{(n)}((1-t)a+tb), \tag{2.4}$$

multiplying both sides of above inequality with $t^{n-\alpha-1}$, and integrating over $[0, 1]$ we get

$$\begin{aligned}
&2f^{(n)}\left(\frac{a+b}{2}\right)\int_0^1 t^{n-\alpha-1}dt \\
&\leq \int_0^1 t^{n-\alpha-1} f^{(n)}(ta+(1-t)b)dt + \int_0^1 t^{n-\alpha-1} f^{(n)}((1-t)a+tb)dt.
\end{aligned}$$

It follows by change of variables

$$f^{(n)}\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(n-\alpha+1)}{2(b-a)^{n-\alpha}}\left[{}^{C}D^{\alpha}_{a+}f(b)+(-1)^{n C}D^{\alpha}_{b-}f(a)\right]. \tag{2.5}$$

On the other hand convexity of $f^{(n)}$ gives

$$\begin{aligned} &f^{(n)}(ta+(1-t)b)+f^{(n)}((1-t)a+tb) \\ &\leq tf^{(n)}(a)+(1-t)f^{(n)}(b)+(1-t)f^{(n)}(a)+tf^{(n)}(b), \end{aligned} \tag{2.6}$$

multiplying both sides of above inequality with $t^{n-\alpha-1}$, and integrating over $[0,1]$ we have

$$\begin{aligned} &\int_0^1 t^{n-\alpha-1}f^{(n)}(ta+(1-t)b)dt+\int_0^1 t^{n-\alpha-1}f^{(n)}((1-t)a+tb)dt \\ &\leq \left[f^{(n)}(a)+f^{(n)}(b)\right]\int_0^1 t^{n-\alpha-1}dt, \end{aligned}$$

from which one can have

$$\frac{\Gamma(n-\alpha+1)}{2(b-a)^{n-\alpha}}\left({}^{C}D^{\alpha}_{a+}f(b)+(-1)^{n C}D^{\alpha}_{b-}f(a)\right) \leq \frac{f^{(n)}(a)+f^{(n)}(b)}{2}. \tag{2.7}$$

Inequalities (2.5) and (2.7) give the inequality (2.2). □

Theorem 2.4. *Let $f:[a,b]\to\mathbb{R}$, $0\leq a<b$ be the function such that $f\in C^{n+1}[a,b]$. Also let $|f^{(n+1)}|$ is convex on $[a,b]$. Then the following inequality for Caputo fractional derivatives holds:*

$$\begin{aligned} &\left|\frac{f^{(n)}(a)+f^{(n)}(b)}{2}-\frac{\Gamma(n-\alpha+1)}{2(b-a)^{n-\alpha}}\left[{}^{C}D^{\alpha}_{a+}f(b)+(-1)^{n C}D^{\alpha}_{b-}f(a)\right]\right| \\ &\leq \frac{b-a}{2(n-\alpha+1)}\left(1-\frac{1}{2^{n-\alpha}}\right)\left[|f^{(n+1)}(a)|+|f^{(n+1)}(b)|\right]. \end{aligned} \tag{2.8}$$

Proof. From Lemma 2.2 and the convexity of $|f^{(n+1)}|$, we have,

$$\begin{aligned} &\left|\frac{f^{(n)}(a)+f^{(n)}(b)}{2}-\frac{\Gamma(n-\alpha+1)}{2(b-a)^{\alpha-n}}\left[{}^{C}D^{\alpha}_{a+}f(b)+(-1)^{n C}D^{\alpha}_{b-}f(a)\right]\right| \\ &\leq \frac{b-a}{2}\int_0^1 \left|\left((1-t)^{n-\alpha}-t^{n-\alpha}\right)\right|\left|f^{n+1}(ta+(1-t)b)\right|dt. \\ &\leq \frac{b-a}{2}\int_0^1 \left|\left((1-t)^{n-\alpha}-t^{n-\alpha}\right)\right|\left(t\left|f^{(n+1)}(a)\right|+(1-t)\left|f^{(n+1)}(b)\right|\right)dt. \\ &= \frac{b-a}{2}\int_0^{\frac{1}{2}} \left((1-t)^{n-\alpha}-t^{n-\alpha}\right)\left(t\left|f^{(n+1)}(a)\right|+(1-t)\left|f^{(n+1)}(b)\right|dt\right) \\ &+\frac{b-a}{2}\int_{\frac{1}{2}}^1 \left((1-t)^{n-\alpha}-t^{n-\alpha}\right)\left(t\left|f^{(n+1)}(a)\right|+(1-t)\left|f^{(n+1)}(b)\right|dt\right). \end{aligned} \tag{2.9}$$

Now

$$\int_0^{\frac{1}{2}} \left((1-t)^{n-\alpha} - t^{n-\alpha}\right) \left(t \left|f^{(n+1)}(a)\right| + (1-t) \left|f^{(n+1)}(b)\right| dt\right)$$
$$= |f^{(n+1)}(a)| \left[\int_0^{\frac{1}{2}} t(1-t)^{n-\alpha} dt - \int_0^{\frac{1}{2}} t^{n-\alpha+1} dt\right]$$
$$+ |f^{(n+1)}(b)| \left[\int_0^{\frac{1}{2}} (1-t)^{n-\alpha+1} - \int_0^{\frac{1}{2}} (1-t)t^{n-\alpha} dt\right]$$
$$= |f^{(n+1)}(a)| \left[\frac{1}{(n-\alpha+1)(n-\alpha+2)} - \frac{(\frac{1}{2})^{n-\alpha+1}}{n-\alpha+1}\right]$$
$$+ |f^{(n+1)}(b)| \left[\frac{1}{n-\alpha+2} - \frac{(\frac{1}{2})^{n-\alpha+1}}{n-\alpha+1}\right].$$

Similarly we have

$$\int_{\frac{1}{2}}^1 \left((1-t)^{n-\alpha} - t^{n-\alpha}\right) \left(t \left|f^{(n+1)}(a)\right| + (1-t) \left|f^{(n+1)}(b)\right| dt\right)$$
$$= |f^{(n+1)}(a)| \left[\frac{1}{n-\alpha+2} - \frac{(\frac{1}{2})^{n-\alpha+1}}{n-\alpha+1}\right]$$
$$+ |f^{(n+1)}(b)| \left[\frac{1}{(n-\alpha+1)(n-\alpha+2)} - \frac{(\frac{1}{2})^{n-\alpha+1}}{n-\alpha+1}\right].$$

.

Therefore from (2.9) we have

$$\left|\frac{f^{(n)}(a) + f^{(n)}(b)}{2} - \frac{\Gamma(n-\alpha+1)}{2(b-a)^{n-\alpha}} \left[{}^C D^{\alpha}_{a+} f(b) + (-1)^n {}^C D^{\alpha}_{b-} f(a)\right]\right|$$
$$\leq \frac{b-a}{2} \Big[|f^{(n+1)}(a)| \left(\frac{1}{(n-\alpha+1)(n-\alpha+2)} - \frac{(\frac{1}{2})^{n-\alpha+1}}{n-\alpha+1}\right)$$
$$+ |f^{(n+1)}(b)| \left(\frac{1}{n-\alpha+2} - \frac{(\frac{1}{2})^{n-\alpha+1}}{n-\alpha+1}\right)$$
$$+ |f^{(n+1)}(a)| \left(\frac{1}{n-\alpha+2} - \frac{(\frac{1}{2})^{n-\alpha+1}}{n-\alpha+1}\right)$$
$$+ |f^{(n+1)}(b)| \left(\frac{1}{(n-\alpha+1)(n-\alpha+2)} - \frac{(\frac{1}{2})^{n-\alpha+1}}{n-\alpha+1}\right) \Big].$$

From which after a little computation one can have (2.8).

□

3. Fejér-Hadamard inequalities for convex functions via Caputo fractional derivatives

In this section we assume that $\|g^{(n)}\|_\infty = Sup_{x\in[a,b]}|g^{(n)}(x)|$, where $g : [a, b] \to \mathbb{R}$ be such that $g \in C^n[a, b]$. Also we define the following convolution $f * g$ of functions

f and g for Caputo fractional derivatives

$$^{C}D^{\alpha}_{a+}(f*g)(x)=\frac{1}{\Gamma(n-\alpha)}\int_a^x\frac{f^{(n)}(t)g^{(n)}(t)}{(x-t)^{\alpha-n+1}}dt, x>a \tag{3.1}$$

and

$$^{C}D^{\alpha}_{b-}(f*g)(x)=\frac{(-1)^n}{\Gamma(n-\alpha)}\int_x^b\frac{f^{(n)}(t)g^{(n)}(t)}{(t-x)^{\alpha-n+1}}dt, x<b. \tag{3.2}$$

Lemma 3.1. [15] *For $0<\lambda\leq 1$ and $0\leq a<b$, we have*

$$|a^{\lambda}-b^{\lambda}|\leq (b-a)^{\lambda}.$$

Theorem 3.2. *Let $f:[a,b]\to\mathbb{R}, a<b$ be the function such that $f\in C^n[a,b]$. Also let $f^{(n)}$ be positive and convex function on $[a,b]$. If $g:[a,b]\to\mathbb{R}$ is a function such that $g\in C^n[a,b]$ and $g^{(n)}$ is nonnegative, integrable and symmetric to $\frac{a+b}{2}$, then following inequalities for Caputo fractional derivatives hold:*

$$\begin{aligned}
&f^{(n)}\left(\frac{a+b}{2}\right)\left[{}^{C}D^{\alpha}_{a+}g(b)+(-1)^{n}{}^{C}D^{\alpha}_{b-}g(a)\right]\\
&\leq \left[{}^{C}D^{\alpha}_{a+}(f*g)(b)+(-1)^{n}{}^{C}D^{\alpha}_{b-}(f*g)(a)\right]\\
&\leq \frac{f^{(n)}(a)+f^{(n)}(b)}{2}\left[{}^{C}D^{\alpha}_{a+}g(b)+(-1)^{n}{}^{C}D^{\alpha}_{b-}g(a)\right]
\end{aligned} \tag{3.3}$$

Proof. By convexity of $f^{(n)}$ we have

$$\begin{aligned}
&f^{(n)}\left(\frac{a+b}{2}\right)=f^{(n)}\left(\frac{ta+(1-t)b+tb+(1-t)a}{2}\right)\\
&\leq\frac{f^{(n)}(ta+(1-t)b)+f^{(n)}(tb+(1-t)a)}{2},
\end{aligned}$$

where $t\in[0,1]$. Multiplying both sides of above inequality with $2t^{n-\alpha-1}g^{(n)}(tb+(1-t)a)$ and integrating the resulting inequality with respect to t over $[0,1]$ we get

$$\begin{aligned}
&2f^{(n)}\left(\frac{a+b}{2}\right)\int_0^1 t^{n-\alpha-1}g^{(n)}(tb+(1-t)a)dt\\
&\leq\int_0^1 t^{n-\alpha-1}f^{(n)}(ta+(1-t)b)g^{(n)}(tb+(1-t)a)dt\\
&+\int_0^1 t^{n-\alpha-1}f^{(n)}(tb+(1-t)a)g^{(n)}(tb+(1-t)a)dt.
\end{aligned}$$

By setting $tb + (1-t)a = x$ we have

$$\begin{aligned}
&\frac{2}{(b-a)^{n-\alpha}} f^{(n)}\left(\frac{a+b}{2}\right) \int_a^b (x-a)^{n-\alpha-1} g^{(n)}(x)dx \\
&\leq \frac{1}{(b-a)^{n-\alpha}} \left[\int_a^b (x-a)^{n-\alpha-1} f^{(n)}(a+b-x) g^{(n)}(x)dx \right. \\
&\left. + \int_a^b (x-a)^{n-\alpha-1} f^{(n)}(x) g^{(n)}(x)dx\right] \\
&= \frac{1}{(b-a)^{n-\alpha}} \left[\int_a^b \frac{f^{(n)}(x) g^{(n)}(a+b-x)}{(x-a)^{\alpha-n+1}} dx + \int_a^b \frac{f^{(n)}(x) g^{(n)}(x)}{(x-a)^{\alpha-n+1}} dx\right] \\
&= \frac{1}{(b-a)^{n-\alpha}} \left[\int_a^b \frac{f^{(n)}(x) g^{(n)}(x)}{(x-a)^{\alpha-n+1}} dx + \int_a^b \frac{f^{(n)}(x) g^{(n)}(x)}{(x-a)^{\alpha-n+1}} dx\right].
\end{aligned}$$

By using Lemma 2.1, we get first inequality of (3.3).

For second inequality of (3.3) we proceed as follows:

Convexity of $f^{(n)}$ gives

$$f^{(n)}(ta + (1-t)b) + f^{(n)}(tb + (1-t)a) \leq f^{(n)}(a) + f^{(n)}(b),$$

where $t \in [0,1]$. Multiplying both sides of above equation with $t^{n-\alpha-1} g^{(n)}(tb + (1-t)a)$ and integrating the resulting inequality with respect to t over $[0,1]$ we get,

$$\begin{aligned}
&\int_0^1 t^{n-\alpha-1} f^{(n)}(ta + (1-t)b) g^{(n)}(tb + (1-t)a)dt \\
&+ \int_0^1 t^{n-\alpha-1} f^{(n)}(tb + (1-t)a) g^{(n)}(tb + (1-t)a)dt \\
&\leq (f^{(n)}(a) + f^{(n)}(b)) \int_0^1 t^{n-\alpha-1} g^{(n)}(tb + (1-t)a)dt,
\end{aligned}$$

which after little computation gives the required result. □

Next we need the following lemma.

Lemma 3.3. *Let $f : [a,b] \to \mathbb{R}, a < b$ be a function such that $f \in C^{n+1}[a,b]$. Also let that $f^{(n+1)}$ be positive and convex function on $[a,b]$. If $g : [a,b] \to \mathbb{R}$ is a function such that $g \in C^n[a,b]$ and $g^{(n)}$ is nonnegative, integrable and symmetric to $\frac{a+b}{2}$, then following equality for Caputo fractional derivatives hold:*

$$\begin{aligned}
&\frac{f^{(n)}(a) + f^{(n)}(b)}{2} \left({}^C D_{a+}^{\alpha} g(b) + (-1)^{n\,C} D_{b-}^{\alpha} g(a)\right) \\
&- \left({}^C D_{a+}^{\alpha} (f * g)(b) + (-1)^{n\,C} D_{b-}^{\alpha} (f * g)(a)\right) \\
&= \frac{1}{\Gamma(n-\alpha)} \int_a^b \left[\int_a^t (b-s)^{n-\alpha-1} g^{(n)}(s)ds - \right. \\
&\left. \int_t^b (s-a)^{n-\alpha-1} g^{(n)}(s)ds\right] f^{(n+1)}(t)dt
\end{aligned}$$

with $\alpha \geq 0$.

Proof. One can note that

$$\frac{1}{\Gamma(n-\alpha)}\int_a^b\left[\int_a^t (b-s)^{n-\alpha-1}g^{(n)}(s)ds\right. \tag{3.4}$$
$$\left.-\int_t^b (s-a)^{n-\alpha-1}g^{(n)}(s)ds\right] f^{(n+1)}(t)dt$$
$$=\frac{1}{\Gamma(n-\alpha)}\left[\int_a^b\left(\int_a^t (b-s)^{n-\alpha-1}g^{(n)}(s)ds\right) f^{(n+1)}(t)dt\right. \tag{3.5}$$
$$\left.+\int_a^b\left(-\int_t^b (s-a)^{n-\alpha-1}g^{(n)}(s)ds\right) f^{(n+1)}(t)dt\right].$$

By simple calculation one can get

$$\int_a^b\left(\int_a^t (b-s)^{n-\alpha-1}g^{(n)}(s)ds\right) f^{(n+1)}(t)dt$$
$$=\left[\left(\int_a^b \frac{g^{(n)}(s)}{(b-s)^{\alpha-n+1}}ds\right) f^{(n)}(b)-\int_a^b \frac{f^{(n)}(t)g^{(n)}(t)}{(b-t)^{\alpha-n+1}}dt\right]$$
$$=\Gamma(n-\alpha)\left[f^{(n)}(b)^C D_{a+}^{\alpha}g(b)-{}^C D_{a+}^{\alpha}(fg)(b)\right]$$
$$=\Gamma(n-\alpha)\left[\frac{f^{(n)}(b)}{2}[{}^C D_{a+}^{\alpha}g(b)+(-1)^{n C} D_{b-}^{\alpha}g(a)]-{}^C D_{a+}^{\alpha}fg(b)\right]$$

and

$$\int_a^b\left(-\int_t^b (s-a)^{n-\alpha-1}g^{(n)}(s)ds\right) f^{(n+1)}(t)dt$$
$$=\left(\int_a^b \frac{g^{(n)}(s)ds}{(s-a)^{\alpha-n+1}}\right) f^{(n)}(a)-\int_a^b \frac{f^{(n)}(t)g^{(n)}(t)}{(t-a)^{\alpha-n+1}}dt$$
$$=\Gamma(n-\alpha)\left[\frac{f^{(n)}(a)}{2}[{}^C D_{a+}^{\alpha}g(b)+(-1)^{n C} D_{b-}^{\alpha}g(a)]-{}^C D_{a+}^{\alpha}fg(a))\right].$$

Hence one can establish the required equality. □

Theorem 3.4. *Let $f:[a,b]\to\mathbb{R}, a<b$ be a function such that $f\in C^{n+1}[a,b]$. If $|f^{(n+1)}|$ is convex on $[a,b]$ and $g:[a,b]\to\mathbb{R}$ is a function such that $g\in C^n[a,b]$ and $g^{(n)}$ is nonnegative, integrable and symmetric to $\frac{a+b}{2}$, then following inequality for Caputo fractional derivatives hold:*

$$\left|\frac{f^{(n)}(a)+f^{(n)}(b)}{2}[{}^C D_{a+}^{\alpha}g(b)+(-1)^{n C} D_{b-}^{\alpha}g(a)]\right.$$
$$\left.-[{}^C D_{a+}^{\alpha}(f*g)(b)+(-1)^{n C} D_{b-}^{\alpha}(f*g)(a)]\right|$$
$$\leq\frac{(b-a)^{\alpha+1}\|g^{(n)}\|_\infty}{(\alpha+1)\Gamma(n-\alpha)}\left(1-\frac{1}{2^\alpha}\right)\left[|f^{(n+1)}(a)|+|f^{(n+1)}(b)|\right]$$

with $\alpha\geq 0$.

Proof. Using Lemma 3.3 we have

$$\begin{aligned}
&\frac{f^{(n)}(a)+f^{(n)}(b)}{2}\left[{}^{C}D^{\alpha}_{a+}g(b)+(-1)^{n\,C}D^{\alpha}_{b-}g(a)\right]\\
&-\left[{}^{C}D^{\alpha}_{a+}(f*g)(b)+(-1)^{n\,C}D^{\alpha}_{b-}(f*g)(a)\right]\\
&=\frac{1}{\Gamma(n-\alpha)}\int_a^b\left[\int_a^t(b-s)^{n-\alpha-1}g^{(n)}(s)ds\right.\\
&\left.-\int_t^b(s-a)^{n-\alpha-1}g^{(n)}(s)ds\right]f^{(n+1)}(t)dt. \qquad (3.6)
\end{aligned}$$

Using convexity of $|f^{(n+1)}|$ we have

$$|f^{(n+1)}(t)|\leq\frac{b-t}{b-a}|f^{(n+1)}(a)|+\frac{t-a}{b-a}|f^{(n+1)}(b)|, \qquad (3.7)$$

where $t\in[a,b]$.

From symmetricity of $g^{(n)}$ we have

$$\begin{aligned}
\int_t^b(s-a)^{n-\alpha-1}g^{(n)}(s)ds&=\int_a^{a+b-t}\frac{g^{(n)}(a+b-s)}{(b-s)^{\alpha-n+1}}ds\\
&=\int_a^{a+b-t}(b-s)^{n-\alpha-1}g^{(n)}(s)ds.
\end{aligned}$$

This gives

$$\begin{aligned}
&\left|\int_a^t(b-s)^{n-\alpha-1}g^{(n)}(s)ds-\int_t^b(s-a)^{n-\alpha-1}g^{(n)}(s)ds\right|\\
&=\left|\int_t^{a+b-t}(b-s)^{n-\alpha-1}g^{(n)}(s)ds\right|\\
&\leq\begin{cases}\int_t^{a+b-t}|(b-s)^{n-\alpha-1}g^{(n)}(s)|ds, & t\in[a,\frac{a+b}{2}]\\ \int_{a+b-t}^{t}|(b-s)^{n-\alpha-1}g^{(n)}(s)|ds, & t\in[\frac{a+b}{2},b].\end{cases} \qquad (3.8)
\end{aligned}$$

By virtue of the Lemma 3.3 and inequalities (3.7) and (3.8) we have

$$\begin{aligned}
&\left|\left(\frac{f^{(n)}(a)+f^{(n)}(b)}{2}\right)\left[{}^{C}D^{\alpha}_{a+}g(b)+(-1)^{n\,C}D^{\alpha}_{b-}g(a)\right]\right. \\
&\left.-\left[{}^{C}D^{\alpha}_{a+}(f*g)(b)+(-1)^{n\,C}D^{\alpha}_{b-}(f*g)(a)\right]\right| \\
&\leq \frac{1}{\Gamma(n-\alpha)}\left[\int_a^{\frac{a+b}{2}}\left(\int_t^{a+b-t}\left|(b-s)^{n-\alpha-1}g^{(n)}(s)\right|ds\right)\right. \\
&\left(\frac{b-t}{b-a}|f^{(n+1)}(a)|+\frac{t-a}{b-a}|f^{(n+1)}(b)|\right)dt \\
&+\int_{\frac{a+b}{2}}^{b}\left(\int_{a+b-t}^{t}\left|(b-s)^{n-\alpha-1}g^{(n)}(s)g^{(n)}(s)ds\right|\right) \\
&\left.\left(\frac{b-t}{b-a}|f^{(n+1)}(a)|+\frac{t-a}{b-a}|f^{(n+1)}(b)|\right)dt\right] \\
&\leq \frac{\|g^{(n)}\|_{\infty}}{\Gamma(n-\alpha)(b-a)}\left[\int_a^{\frac{a+b}{2}}\left(\frac{1}{(b-t)^{\alpha}}-\frac{1}{(t-a)^{\alpha}}\right)\right. \\
&\left((b-t)|f^{(n+1)}(a)|+(t-a)|f^{(n+1)}(b)|\right)dt \\
&\left.+\int_{\frac{a+b}{2}}^{b}\left(\frac{1}{(t-a)^{\alpha}}-\frac{1}{(b-t)^{\alpha}}\right)\left((b-t)|f^{(n+1)}(a)|+(t-a)|f^{(n+1)}(b)|\right)dt\right].
\end{aligned} \tag{3.9}$$

Now one can have

$$\begin{aligned}
&\int_a^{\frac{a+b}{2}}\left(\frac{1}{(b-t)^{\alpha}}-\frac{1}{(t-a)^{\alpha}}\right)(b-t)dt \\
&=\int_{\frac{a+b}{2}}^{b}\left(\frac{1}{(t-a)^{\alpha}}-\frac{1}{(b-t)^{\alpha}}\right)(t-a)dt \\
&=\frac{(b-a)^{2-\alpha}}{1-\alpha}\left(\frac{1-\alpha}{2-\alpha}-\frac{1}{2^{1-\alpha}}\right)
\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}
&\int_a^{\frac{a+b}{2}}\left(\frac{1}{(b-t)^{\alpha}}-\frac{1}{(t-a)^{\alpha}}\right)(t-a)dt \\
&=\int_{\frac{a+b}{2}}^{b}\left(\frac{1}{(t-a)^{\alpha}}-\frac{1}{(b-t)^{\alpha}}\right)(b-t)dt \\
&=\frac{(b-a)^{2-\alpha}}{(1-\alpha)}\left(\frac{1}{2-\alpha}-\frac{1}{2^{1-\alpha}}\right).
\end{aligned} \tag{3.11}$$

Using (3.10), (3.11) in (3.9) we get the required result. □

Theorem 3.5. *Let $f:[a,b]\to\mathbb{R}, a<b$ be a function such that $f\in C^{n+1}[a,b]$. If $|f^{(n+1)}|^q$, $q>1$ is convex on $[a,b]$ and $g:[a,b]\to\mathbb{R}$ is a function such that $g\in C^n[a,b]$ and $g^{(n)}$ is nonnegative, integrable and symmetric to $\frac{a+b}{2}$, then following* *inequality for Caputo fractional derivatives hold:*

$$\begin{aligned}
&\Bigg|\frac{f^{(n)}(a)+f^{(n)}(b)}{2}\left[{}^{C}D^{\alpha}_{a+}g(b)+(-1)^{n\,C}D^{\alpha}_{b-}g(a)\right] \\
&-\left[{}^{C}D^{\alpha}_{a+}(f*g)(b)+(-1)^{n\,C}D^{\alpha}_{b-}(f*g)(a)\right]\Bigg| \\
&\le \frac{2(b-a)^{1-\alpha}\|g^{(n)}\|_{\infty}\,(1-2^{\alpha})}{(1-\alpha)\Gamma(n-\alpha)}\left(\frac{|f^{(n+1)}(a)|^{q}+|f^{(n+1)}(b)|^{q}}{2}\right)^{\frac{1}{q}} \qquad (3.12)
\end{aligned}$$

with $\alpha \geq 0$ *and* $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By using Lemma 3.3, Hölder inequality, inequality (3.8) and convexity of $|f^{(n+1)}|^{q}$ respectively we have

$$\begin{aligned}
&\Bigg|\frac{f^{(n)}(a)+f^{(n)}(b)}{2}\left[{}^{C}D^{\alpha}_{a+}g(b)+(-1)^{n\,C}D^{\alpha}_{b-}g(a)\right] \\
&-\left[{}^{C}D^{\alpha}_{a+}(f*g)(b)+(-1)^{n\,C}D^{\alpha}_{b-}(f*g)(a)\right]\Bigg| \\
&\le \frac{1}{\Gamma(n-\alpha)}\left[\int_a^b\left|\int_t^{a+b-t}(b-s)^{n-\alpha-1}g^{(n)}(s)ds\right|dt\right]^{1-\frac{1}{q}} \\
&\left[\int_a^b\left|\int_t^{a+b-t}(b-s)^{n-\alpha-1}g^{(n)}(s)ds\right||f^{(n+1)}(t)|^{q}dt\right]^{\frac{1}{q}} \\
&\le \frac{1}{\Gamma(n-\alpha)}\left[\int_a^{\frac{a+b}{2}}\left(\int_t^{a+b-t}|(b-s)^{n-\alpha-1}g^{(n)}(s)|ds\right)dt\right. \\
&\left.+\int_{\frac{a+b}{2}}^{b}\left(\int_{a+b-t}^{t}|(b-s)^{n-\alpha-1}g^{(n)}(s)g^{(n)}(s)|ds\right)dt\right]^{1-\frac{1}{q}} \\
&\left[\int_a^{\frac{a+b}{2}}\left(\int_t^{a+b-t}\left|(b-s)^{n-\alpha-1}g^{(n)}(s)\right|ds\right)|f^{(n+1)}(t)|^{q}dt\right. \\
&\left.+\int_{\frac{a+b}{2}}^{b}\left(\int_{a+b-t}^{t}|(b-s)^{n-\alpha-1}g^{(n)}(s)|ds\right)|f^{(n+1)}(t)|^{q}dt\right]^{\frac{1}{q}} \\
&\le \frac{\|g^{(n)}\|_{\infty}}{\Gamma(n-\alpha)}\left[\left(\frac{2(b-a)^{1-\alpha}}{(1-\alpha)}(1-2^{\alpha})\right)^{1-\frac{1}{q}}\right. \\
&\left.\left(\frac{(|f^{(n+1)}(a)|^{q}+|f^{(n+1)}(b)|^{q})(b-a)^{1-\alpha}}{(1-\alpha)}(1-2^{\alpha})\right)^{\frac{1}{q}}\right].
\end{aligned}$$

From which after a little computation one can have the required result. □

Theorem 3.6. *Let* $f:[a,b]\to\mathbb{R}, a<b$ *be a function such that* $f\in C^{n+1}[a,b]$. *Also let* $|f^{(n+1)}|^{q}$, $q>1$ *be convex on* $[a,b]$ *and* $g:[a,b]\to\mathbb{R}$ *be a function such that* $g\in C^{n}[a,b]$ *and* $g^{(n)}$ *is nonnegative, integrable and symmetric to* $\frac{a+b}{2}$, *then* *following inequalities for Caputo fractional derivatives hold:*

$$(i)\left|\frac{f^{(n)}(a)+f^{(n)}(b)}{2}\left[{}^CD^{\alpha}_{a+}g(b)+(-1)^{n\,C}D^{\alpha}_{b-}g(a)\right]\right.$$
$$\left.-\left[{}^CD^{\alpha}_{a+}(f*g)(b)+(-1)^{n\,C}D^{\alpha}_{b-}(f*g)(a)\right]\right|$$
$$\leq\frac{2^{\frac{1}{p}}(b-a)^{1-\alpha}\|g^{(n)}\|_{\infty}}{(1-\alpha p)^{\frac{1}{p}}\Gamma(n-\alpha)}(1-2^{\alpha p})^{\frac{1}{p}}\left(\frac{|f^{(n+1)}(a)|^q+|f^{(n+1)}(b)|^q}{2}\right)^{\frac{1}{q}} \tag{3.13}$$

with $\alpha\geq 0$.

$$(ii)\left|\frac{f^{(n)}(a)+f^{(n)}(b)}{2}\left[{}^CD^{\alpha}_{a+}g(b)+(-1)^{n\,C}D^{\alpha}_{b-}g(a)\right]\right.$$
$$\left.-[{}^CD^{\alpha}_{a+}(f*g)(b)+(-1)^{n\,C}D^{\alpha}_{b-}(f*g)(a)]\right|$$
$$\leq\frac{(b-a)^{1-\alpha}\|g^{(n)}\|_{\infty}}{(1-\alpha p)^{\frac{1}{p}}\Gamma(n-\alpha)}\left(\frac{|f^{(n+1)}(a)|^q+|f^{(n+1)}(b)|^q}{2}\right)^{\frac{1}{q}} \tag{3.14}$$

with $0\leq\alpha\leq 1$. *where* $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Making use of Lemma 4, Hölder inequality, inequality (3.6) and convexity of $|f^{(n+1)}|^q$ we have

$$\left|\left(\frac{f^{(n)}(a)+f^{(n)}(b)}{2}\right)\left[{}^CD^{\alpha}_{a+}g(b)+(-1)^{n\,C}D^{\alpha}_{b-}g(a)\right]\right.$$
$$\left.-[{}^CD^{\alpha}_{a+}(f*g)(b)+(-1)^{n\,C}D^{\alpha}_{b-}(f*g)(a)]\right|$$
$$\leq\frac{1}{\Gamma(n-\alpha)}\left(\int_a^b\left|\int_t^{a+b-t}\frac{g(s)}{(b-s)^{\alpha-n+1}}ds\right|^p dt\right)^{\frac{1}{p}}\left(\int_a^b|f^{(n+1)}(t)|^q dt\right)^{\frac{1}{q}}$$
$$\leq\frac{1}{\Gamma(n-\alpha)}\left[\int_a^{\frac{a+b}{2}}\left(\int_t^{a+b-t}|\frac{g(s)}{(b-s)^{\alpha-n+1}}|ds\right)dt\right.$$
$$\left.+\int_{\frac{a+b}{2}}^b\left(\int_{a+b-t}^t|\frac{g(s)}{(b-s)^{\alpha-n+1}}|ds\right)dt\right]^{\frac{1}{p}}$$
$$\times\int_a^b\left(\left(\frac{b-t}{b-a}|f^{(n+1)}(a)|^q+\frac{t-a}{b-a}|f^{(n+1)}(b)|^q\right)dt\right)^{\frac{1}{q}}$$
$$\leq\frac{\|g^{(n)}\|_{\infty}}{\Gamma(n-\alpha)}\left[\int_a^{\frac{a+b}{2}}\left(\frac{1}{(b-t)^{\alpha}}-\frac{1}{(t-a)^{\alpha}}\right)^p dt\right.$$
$$\left.+\int_{\frac{a+b}{2}}^b\left(\frac{1}{(t-a)^{\alpha}}-\frac{1}{(b-t)^{\alpha}}\right)^p dt\right]^{\frac{1}{p}}$$
$$\times\left[\int_a^b\left(\frac{b-t}{b-a}|f^{(n+1)}(a)|^q+\frac{t-a}{b-a}|f^{(n+1)}(b)|^q\right)dt\right]^{\frac{1}{q}}. \tag{3.15}$$

Now

$$(A-B)^q\leq A^q-B^q,\quad A\geq B\geq 0$$

gives

$$\left[\frac{1}{(b-t)^{\alpha}}-\frac{1}{(t-a)^{\alpha}}\right]^{p}\leq\frac{1}{(b-t)^{\alpha p}}-\frac{1}{(t-a)^{\alpha p}} \tag{3.16}$$

for $t\in[a,\frac{a+b}{2}]$, and

$$\left[\frac{1}{(t-a)^{\alpha}}-\frac{1}{(b-t)^{\alpha}}\right]^{p}\leq\frac{1}{(t-a)^{\alpha p}}-\frac{1}{(b-t)^{\alpha p}} \tag{3.17}$$

for $t\in[\frac{a+b}{2},b]$. Using (3.16) and (3.17) in inequality (3.15) and solving we get required result. □

Acknowledgement

The research work of first author is supported by Higher Education Commission of Pakistan under NRPU 2016, Project No. 5421.

References

[1] A. G. Azpeitia, *Convex functions and the Hadamard inequality*, Rev. Colomb. Mat., **28**(1) (1994), 7–12.

[2] M. K. Bakula, M. E. Ozdemir, J. Pečarić, *H*adamard type inequalities for m-convex and (α,m)-convex functions, J. Ineq. Pure Appl. Math., **9**(4) (2008), Art. Id. 96.

[3] M. K. Bakula, J. Pečarić, *Note on some Hadamard type inequalities*, J. Ineq. Pure Appl. Math., **5**(3) (2004), Art. Id. 74.

[4] M. Caputo, *Linear models of dissipation whose Q is almost frequency independent Part II*, Geophys. J. Int., **13**(5) (1967), 529–539.

[5] S. S. Dragmoir, R. P. Agarwal, *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula*, Appl. Math. Lett., **11**(5) (1998), 91-95.

[6] S. S. Dragmoir, C. E. M. Pearce, *Selected topics on Hadamard inequalities and applications*, RGMIA Monographs,Victoria University, Math. Sic. Marh. Roum., **47** (2004), 3–14.

[7] G. Farid, M. Marwan, A. U. Rehman, *New mean value theorems and generalization of Hadamard inequality via coordinated m-convex functions*, J. Inequal. Appl., (1) (2015), 283.

[8] P. M. Gill, C. E. M. Pearce, J. Pečarić, *Hadamard's inequality for r-convex functions*, J. Math. Anal. Appl., **215**(2) (1997), 461–470.

[9] İ. İşcan, *Hermite Hadamard Féjer type inequalities for convex functions via fractional integrals*, Stud. Univ. Babes-Bolyai Math., **60**(3) (2007), 26–35.

[10] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier Science Inc. New York, **204** (2006).

[11] S. Miller, B. Ross, *An introduction to fractional calculus and fractional differential equations*, John Wiley, New York, 1993.

[12] M. E. Ozdemir, M Avci, E. Set, *On some inequalities of Hermite-Hadamard type via m-convexity*, Appl. Math. Lett., **23**(9) (2010), 1065-1070.

[13] E. Set, M. E. Ozdemir, S. S. Dragomir, *On the Hermite-Hadamard inequality and other integral inequalities involving two functions*, J. Inequal. Appl., (2010), 9. Art. Id. 148102.

[14] L. Fejér, *Uberdie Fourierreihen, II*, Math. Naturwise. Anz Ungar. Akad., Wiss, 24 (1906).

[15] A. P. Prudnikov, Y. A. Brychkov, O. I. Marichev, *Integral and series. 1. In Elementary Functions*, Nauka, Moscow; 1981.

[16] M. Z. Sarikaya, E. Set, H. Yaldiz, N. Basak, *Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities*, J. Math. Comput. Model., **57**(9) (2013), 2403-2407.

Ghulam Farid
COMSATS Institute of Information Technology, Attock Campus, Pakistan
E-mail address: faridphdsms@hotmail.com, ghlmfarid@ciit-attock.edu.pk

Anum Javed
COMSATS Institute of Information Technology, Attock Campus, Pakistan
E-mail address: javedanum.38@gmail.com

Saira Naqvi
COMSATS Institute of Information Technology, Attock Campus, Pakistan
E-mail address: naqvisaira2013@gmail.com