NEW FIXED-CIRCLE RESULTS ON S-METRIC SPACES

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Abstract. In this paper our aim is to study some fixed-circle theorems on S-metric spaces. For this purpose we give new examples of S-metric spaces and investigate some relationships between circles on metric and S-metric spaces. Then we investigate some existence and uniqueness conditions for fixed circles of self-mappings on S-metric spaces.

1. Introduction

Recently Sedghi, Shobe and Aliouche introduced the concept of an S-metric space as a generalization of a metric space as follows:

Definition 1.1. [8] Let $X$ be a nonempty set and $S : X \times X \times X \rightarrow [0, \infty)$ be a function satisfying the following conditions for all $x, y, z, a \in X$:

1. $S(x, y, z) = 0$ if and only if $x = y = z$,
2. $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

Then $S$ is called an S-metric on $X$ and the pair $(X, S)$ is called an S-metric space.

For example, let $\mathbb{R}$ be the real line. If we consider the following function

$S(x, y, z) = |x - z| + |y - z|$ for all $x, y, z \in \mathbb{R}$, then this function defines an S-metric on $\mathbb{R}$ and it is called the usual S-metric [9].

Sedghi, Shobe and Aliouche investigated some fixed-point results on an S-metric space in [8]. Then Özgür and Taş studied some generalizations of the Banach’s contraction principle on S-metric spaces in [7]. Also they introduced new fixed-point theorems for the Rhoades’ contractive condition on S-metric spaces in [3]. After, it was generalized these fixed-point theorems for generalized Rhoades’ contractive conditions in [4].

More recently, the notion of a fixed circle have been defined on metric and S-metric spaces in [5] and [6], respectively. It is important to investigate some fixed-circle theorems on various metric spaces to obtain new generalizations of known fixed-point results. Some interesting fixed-circle theorems were studied on metric spaces and S-metric spaces by Özgür and Taş (see [5] and [6] for more details).
They studied some existence and uniqueness conditions for the fixed circles of self-mappings.

Our aim in this paper is to obtain new fixed-circle theorems for self-mappings on $S$-metric spaces. In Section 2 we recall some basic facts and give new examples of $S$-metric spaces. We draw some circles on these new $S$-metric spaces [10]. Also we investigate some relationships between circles on various metric spaces. In Section 3 we study some existence and uniqueness theorems for fixed circles. Some illustrative examples of self-mappings with a fixed circle are also given.

2. Comparisons of Circles on Metric and $S$-Metric Spaces

In this section we give new examples of $S$-metric spaces to determine some comparisons of circles on metric and $S$-metric spaces.

We recall the notion of a circle on an $S$-metric space.

Definition 2.1. [6] Let $(X, S)$ be an $S$-metric space and $x_0 \in X$, $r \in (0, \infty)$. We define the circle centered at $x_0$ with radius $r$ as

$$C^S_{x_0, r} = \{ x \in X : S(x, x, x_0) = r \}.$$  

Now we recall the following basic lemmas.

Lemma 2.2. [8] Let $(X, S)$ be an $S$-metric space. Then we get

$$S(x, x, y) = S(y, y, x).$$

Lemma 2.2 can be considered as the symmetry condition on an $S$-metric space. In the following lemma, we see the relationships between a metric and an $S$-metric.

Lemma 2.3. [2] Let $(X, d)$ be a metric space. Then the following properties are satisfied:

1. $S_d(x, y, z) = d(x, z) + d(y, z)$ for all $x, y, z \in X$ is an $S$-metric on $X$.
2. $x_n \rightarrow x$ in $(X, d)$ if and only if $x_n \rightarrow x$ in $(X, S_d)$.
3. $\{ x_n \}$ is Cauchy in $(X, d)$ if and only if $\{ x_n \}$ is Cauchy in $(X, S_d)$.
4. $(X, d)$ is complete if and only if $(X, S_d)$ is complete.

The metric $S_d$ was called as the $S$-metric generated by $d$ [4].

Now we give new examples of $S$-metric spaces and draw some circles.

Example 2.4. Let $X = \mathbb{R}^+$ and the function $S_1 : X \times X \times X \rightarrow [0, \infty)$ be defined by

$$S_1(x, y, z) = |x^2 - y^2| + |x^2 + y^2 - 2z^2|,$$

for all $x, y, z \in \mathbb{R}^+$. Then $S_1$ is an $S$-metric on $\mathbb{R}^+$ which is not generated by any metric and the pair $(\mathbb{R}^+, S_1)$ is an $S$-metric space.

Conversely, assume that there exists a metric $d$ such that

$$S_1(x, y, z) = d(x, z) + d(y, z),$$

for all $x, y, z \in \mathbb{R}^+$. Then we obtain

$$S_1(x, x, z) = 2d(x, z)$$

and

$$S_1(y, y, z) = 2d(y, z),$$

for all $x, y, z \in \mathbb{R}^+$. So we get

$$|x^2 - y^2| + |x^2 + y^2 - 2z^2| = |x^2 - z^2| + |y^2 - z^2|.$$
which is a contradiction. Hence $S_1$ is not generated by any metric.

In the following example we extend the $S$-metric $S_1$ defined in Example 2.4 to the three dimensional case.

**Example 2.5.** Let us consider the set $X^* = \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$ and the function $S_1^* : X^* \times X^* \times X^* \to [0, \infty)$ be defined as

$$S_1^*(x, y, z) = \sum_{i=1}^{3} \left( |x_i^2 - y_i^2| + |x_i^2 + y_i^2 - 2z_i^2| \right),$$

for all $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$ and $z = (z_1, z_2, z_3)$ on $X^*$. Then $S_1^*$ is an $S$-metric on $X^*$ and the pair $(X^*, S_1^*)$ is an $S$-metric space.

If we choose $x_0 = 0 = (0, 0, 0)$ and $r = 12$, then we get

$$C_{0, 12}^{S_1^*} = \{ x \in X^* : S_1^*(x, x, 0) = 12 \} = \{ x \in X^* : x_1^2 + x_2^2 + x_3^2 = 6 \},$$

as shown in Figure 1.

If we choose $x_0 = (2, 1, 1)$ and $r = 12$, then we get

$$C_{x_0, 12}^{S_1^*} = \{ x \in X^* : S_1^*(x, x, x_0) = 12 \} = \{ x \in X^* : |x_1^2 - 4| + |x_2^2 - 1| + |x_3^2 - 1| = 6 \},$$

as shown in Figure 2. Notice that the shape of the circles can be changed according to the center.

**Example 2.6.** Let $X = \mathbb{R}^+$ and the function $S_2 : X \times X \times X \to [0, \infty)$ be defined by

$$S_2(x, y, z) = \left| \ln \frac{x}{y} \right| + \left| \ln \frac{xy}{z^2} \right|,$$
Figure 2. The circle $C_{x_0,12}^S$ on $(X^*, S_1^*)$.

for all $x, y, z \in \mathbb{R}^+$. Then $S_2$ is an $S$-metric on $\mathbb{R}^+$ which is not generated by any metric and the pair $(\mathbb{R}^+, S_2)$ is an $S$-metric space.

Conversely, suppose that there exists a metric $d$ such that

$$S_2(x, y, z) = d(x, z) + d(y, z),$$

for all $x, y, z \in \mathbb{R}^+$. Then we obtain

$$S_2(x, x, z) = 2d(x, z)$$

and

$$S_2(y, y, z) = 2d(y, z)$$

for all $x, y, z \in \mathbb{R}^+$. So we get

$$\left| \ln \frac{x}{y} \right| + \left| \ln \frac{xy}{z^2} \right| = \left| \ln \frac{x}{z} \right| + \left| \ln \frac{y}{z} \right|,$$

which is a contradiction. Hence $S_2$ is not generated by any metric.

Now we consider $X^* = \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$ and the function $S_2^* : X^* \times X^* \times X^* \to [0, \infty)$ be defined by

$$S_2^*(x, y, z) = \sum_{i=1}^{3} \left( |\ln \frac{x_i}{y_i}| + \left| \ln \frac{x_i y_i}{z_i^2} \right| \right),$$

for all $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3)$ and $z = (z_1, z_2, z_3)$ in $X^*$. Then $S_2^*$ is an $S$-metric on $X^*$ and the pair $(X^*, S_2^*)$ is an $S$-metric space.

If we choose $x_0 = (1, 1, 1)$ and $r = 1$, then we get

$$C_{x_0,1}^{S_2^*} = \{ x \in X^* : S_2^*(x, x, x_0) = 1 \}$$

$$= \{ x \in X^* : |\ln x_1^2| + |\ln x_2^2| + |\ln x_3^2| = 1 \},$$

as shown in Figure 3.
Using Lemma 2.3, we obtain the following proposition for the comparison of the circles on a metric space and the corresponding $S$-metric space generated by the metric.

**Proposition 2.7.** Let $(X, S)$ be an $S$-metric space such that $S$ is generated by a metric $d$. Then any circle $C_{x_0, r}^S$ on the $S$-metric space is the circle $C_{x_0, 2r}$ on the metric space $(X, d)$.

**Proof.** By Definition 2.1 and Lemma 2.2 we have

$$S(x, x_0, x_0) = d(x, x_0) + d(x, x_0) = 2d(x, x_0) = 2r.$$ Then the proof follows easily. \qed

**Corollary 2.8.** The circle $C_{x_0, r}$ on a metric space $(X, d)$ is the circle $C_{x_0, 2r}$ on the $S$-metric space which is generated by $d$.

We give an example to show that a circle $C_{x_0, r}$ in a metric space can be a circle with the same center and same radius in an $S$-metric space which can not be generated by $d$.

**Example 2.9.** Let $X = \mathbb{R}$, $(X, S)$ be the usual $S$-metric space and the function $d : X \times X \to [0, \infty)$ be defined by

$$d(x, y) = 2|x - y|,$$

for all $x, y \in X$. Then $(X, d)$ is a metric space and the usual $S$-metric is not generated by $d$. Conversely, assume that $S$ is generated by $d$ such that

$$S(x, y, z) = d(x, z) + d(y, z),$$

for all $x, y, z \in X$. Then we obtain

$$|x - z| + |y - z| = 2|x - z| + 2|y - z|.$$
which is a contradiction. Therefore the usual $S$-metric is not generated by $d$. If we consider the unit circles on the metric space $(X,d)$ and the usual $S$-metric space, respectively, then we get

$$C_{0,1} = \{x \in X : d(x,0) = 1\} = \left\{-\frac{1}{2}, \frac{1}{2}\right\}$$

and

$$C_{0,1}^S = \{x \in X : S(x,x,0) = 1\} = \left\{-\frac{1}{2}, \frac{1}{2}\right\}.$$

Consequently, we have $C_{0,1} = C_{0,1}^S$.

Let $(X,S)$ be any $S$-metric space. In [1], it was shown that every $S$-metric on $X$ defines a metric $d_S$ on $X$ as follows:

$$d_S(x,y) = S(x,x,y) + S(y,y,x), \quad (2.1)$$

for all $x, y \in X$. However Özgür and Taş showed that the function $d_S(x,y)$ defined in (2.1) does not always define a metric because of the reason that the triangle inequality does not satisfied for all elements of $X$ everywhen [4].

If the $S$-metric is generated by a metric $d$ on $X$ then it can be easily seen that the function $d_S$ is explicitly a metric on $X$, especially we have

$$d_S(x,y) = 4d(x,y).$$

But, if we consider an $S$-metric which is not generated by any metric then $d_S$ can be or cannot be a metric on $X$. This metric $d_S$ is called as the metric generated by $S$ in the case $d_S$ is a metric.

Example 2.10. Let $X = \{a,b,c\}$ and the function $S : X \times X \times X \to [0,\infty)$ be defined as:

$$S(x,y,z) = \begin{cases} 7 & x = y = a, z = b \text{ or } x = y = b, z = a \\ 3 & x = y = a, z = c \text{ or } x = y = c, z = a \text{ or } x = y = b, z = c \text{ or } x = y = c, z = b \\ 0 & x = y = z \\ 1 & \text{otherwise} \end{cases},$$

for all $x, y, z \in X$. Then the function $S$ is an $S$-metric which is not generated by any metric and the pair $(X,S)$ is an $S$-metric space. But the function $d_S$ defined in (2.1) is not a metric on $X$. Indeed, for $x = a, y = b, z = c$ we get

$$d_S(a,b) = 14 \not\leq d_S(a,c) + d_S(c,b) = 12.$$

We give the following proposition for a circle.

Proposition 2.11. Let $(X,d_S)$ be a metric space such that $d_S$ is generated by an $S$-metric $S$. Then any circle $C_{x_0,r}$ on the metric space $(X,d_S)$ is the circle $C_{S_{x_0,r}}$ on the $S$-metric space $(X,S)$.

Proof. By the Definition 2.1, the equality (2.1) and Lemma 2.2 we have

$$d_S(x,x_0) = S(x,x,x_0) + S(x_0,x,x_0) = 2S(x,x_0)$$

and

$$S(x,x_0) = \frac{r}{2}.$$

Then the proof follows easily. \qed
Corollary 2.12. The circle $C^S_{x_0, r}$ on an $S$-metric space $(X, S)$ is the circle $C_{x_0, 2r}$ on the metric space $(X, d_S)$ where $d_S$ is generated by $S$.

3. SOME EXISTENCE AND UNIQUENESS CONDITIONS FOR FIXED CIRCLES ON $S$-METRIC SPACES

In this section we recall the notion of a fixed circle on an $S$-metric space and present some fixed-circle theorems.

**Definition 3.1.** [6] Let $(X, S)$ be an $S$-metric space, $C^S_{x_0, r}$ be a circle on $X$ and $T : X \to X$ be a self-mapping. If $Tx = x$ for all $x \in C^S_{x_0, r}$ then we call the circle $C^S_{x_0, r}$ as the fixed circle of $T$.

We give the following existence theorem for fixed circles on an $S$-metric space.

**Theorem 3.2.** Let $(X, S)$ be an $S$-metric space and $C^S_{x_0, r}$ be any circle on $X$. Let us define the mapping

$$\varphi : X \to [0, \infty), \quad \varphi(x) = S(x, x, x_0),$$

for all $x \in X$. If there exists a self-mapping $T : X \to X$ satisfying

(i) $S(x, x, Tx) \leq \varphi(x) - \varphi(Tx)$

and

(ii) $S(Tx, Tx, x_0) \geq r$, for all $x \in C^S_{x_0, r}$, then $C^S_{x_0, r}$ is a fixed circle of $T$.

**Proof.** Let $x \in C^S_{x_0, r}$. Using the condition (SC1) we obtain

$$S(x, x, Tx) \leq \varphi(x) - \varphi(Tx)$$

$$= S(x, x, x_0) - S(Tx, Tx, x_0)$$

$$= r - S(Tx, Tx, x_0).$$

Because of the condition (SC2), the point $Tx$ should be lie on or exterior of the circle $C^S_{x_0, r}$. If $S(Tx, Tx, x_0) > r$ then using the inequality (3.2) we have a contradiction. Therefore it should be $S(Tx, Tx, x_0) = r$. In this case, using the inequality (3.2) we get

$$S(x, x, Tx) \leq r - S(Tx, Tx, x_0) = r - r = 0$$

and so $Tx = x$.

Hence we obtain $Tx = x$ for all $x \in C^S_{x_0, r}$. Consequently, the self-mapping $T$ fixes the circle $C^S_{x_0, r}$.

**Figure 4.** The geometric description of the condition (SC1).
Remark. Notice that the condition (SC1) guarantees that $Tx$ is not in the exterior of the circle $C^{S}_{x_0, r}$ for each $x \in C^{S}_{x_0, r}$. Similarly, the condition (SC2) guarantees that $Tx$ is not in the interior of the circle $C^{S}_{x_0, r}$ for each $x \in C^{S}_{x_0, r}$. Consequently, $Tx \in C^{S}_{x_0, r}$ for each $x \in C^{S}_{x_0, r}$, and so we have $T(C^{S}_{x_0, r}) \subset C^{S}_{x_0, r}$ (see Figures 4, 5 and 6).

Now we give an example of a self-mapping which has a fixed circle on an $S$-metric space.

Example 3.3. Let $(X, S)$ be an $S$-metric space, $C^{S}_{x_0, r}$ be a circle on $X$ and $\alpha$ be a constant such that

$$S(\alpha, \alpha, x_0) \neq r.$$ 

If we define the self-mapping $T : X \rightarrow X$ as

$$Tx = \begin{cases} x &; x \in C^{S}_{x_0, r}, \\ \alpha &; \text{otherwise}, \end{cases}$$

for all $x \in X$, then it can be easily checked that the conditions (SC1) and (SC2) are satisfied. Consequently, $C^{S}_{x_0, r}$ is the fixed circle of $T$.

We give another example of a self-mapping which has a fixed circle as follows:

Example 3.4. Let $X = \mathbb{R}$ and the function $S : X \times X \times X \rightarrow [0, \infty)$ be defined by

$$S(x, y, z) = \alpha |x - z| + \beta |x + z - 2y|,$$
for all $x, y, z \in \mathbb{R}$ and $\alpha, \beta > 0$ with $\alpha \leq \beta$. Then $S$ is an $S$-metric on $\mathbb{R}$ which is not generated by any metric and the pair $(\mathbb{R}, S)$ is an $S$-metric space.

Let us consider the circle $C^S_{10, \alpha + \beta}$ and define the self-mapping $T : \mathbb{R} \to \mathbb{R}$ as

$$T x = \begin{cases} 
  x & : x \in C^S_{10, \alpha + \beta} \\
  12 & : \text{otherwise}
\end{cases},$$

for all $x \in \mathbb{R}$. Then the self-mapping $T$ satisfies the conditions (SC1) and (SC2). Hence $C^S_{10, \alpha + \beta}$ is a fixed circle of $T$.

**Example 3.5.** Let $(X, d)$ be a metric space and $(X, S)$ be an $S$-metric space. Let us consider a circle $C^S_{x_0, r}$ satisfying

$$d(x, x_0) \neq S(x, x, x_0)$$

and define the self-mapping $T : X \to X$ as

$$T x = x - S(x, x, x_0) + r,$$

for all $x \in X$. Then the self-mapping $T$ satisfies the conditions (SC1) and (SC2). Therefore $C^S_{x_0, r}$ is a fixed circle of $T$. But $T$ does not fix a circle $C^S_{x_0, r}$ on the metric space $(X, d)$.

Now, in the following example, we give an example of a self-mapping which satisfies the condition (SC1) and does not satisfy the condition (SC2).

**Example 3.6.** Let $X = \mathbb{R}^+$ and the function $S : X \times X \times X \to [0, \infty)$ be defined in Example 2.6. Let us consider a circle $C^S_{x_0, r}$ and define the self-mapping $T : X \to X$ as

$$T x = \begin{cases} 
  x_0 & : x \in C^S_{x_0, r} \\
  \beta & : \text{otherwise}
\end{cases},$$

for all $x \in X$ where $S(\beta, \beta, x_0) < r$. Then the self-mapping $T$ satisfies the condition (SC1) but does not satisfy the condition (SC2). Clearly $T$ does not fix the circle $C^S_{x_0, r}$.

In the following examples, we give some examples of self-mappings which satisfy the condition (SC2) and do not satisfy the condition (SC1).

**Example 3.7.** Let $(X, S)$ be any $S$-metric space and $C^S_{x_0, r}$ be any circle on $X$. Let $k$ be chosen such that $S(k, k, x_0) = m > r$ and consider the self-mapping $T : X \to X$ defined by

$$T x = k,$$

for all $x \in X$. Then the self-mapping $T$ satisfies the condition (SC2) but does not satisfy the condition (SC1). Clearly $T$ does not fix the circle $C^S_{x_0, r}$.

**Example 3.8.** Let $X = \mathbb{R}$ and the function $S : X \times X \times X \to [0, \infty)$ be defined by

$$S(x, y, z) = \alpha |x - z| + \beta |x + z - 2y|,$$

for all $x, y, z \in \mathbb{R}$ and some $\alpha, \beta \in \mathbb{R}$ with $\alpha + \beta > 0$. Then $S$ is an $S$-metric on $\mathbb{R}$ which is not generated by any metric and the pair $(\mathbb{R}, S)$ is an $S$-metric space.

Let us consider a circle $C^S_{x_0, r}$ and define the self-mapping $T : \mathbb{R} \to \mathbb{R}$ as

$$T x = \begin{cases} 
  k_1 & : x \in C^S_{x_0, r} \\
  k_2 & : \text{otherwise}
\end{cases},$$
for all $x \in \mathbb{R}$, where $S(k_1, k_1, x_0) = 2\varepsilon$ and $k_2$ is a constant such that $k_2 \neq k_1$. Then the self-mapping $T$ satisfies the condition (SC2) but does not satisfy the condition (SC1). Clearly $T$ does not fix the circle $C_{x_0, r}$.

**Remark.** Let $(X, S)$ be an S-metric space and $C_{x_0, r}^S, C_{x_1, \rho}^S$ be two circles on $X$. There exists at least one self-mapping $T : X \to X$ which fixes both of the circles $C_{x_0, r}^S$ and $C_{x_1, \rho}^S$. Indeed, let us define the mappings $\varphi_1, \varphi_2 : X \to [0, \infty)$ as

$$\varphi_1(x) = S(x, x, x_0)$$

and

$$\varphi_2(x) = S(x, x, x_1),$$

for all $x \in X$. Let us consider the self-mapping $T : X \to X$ defined as

$$T_x = \begin{cases} x & \text{if } x \in C_{x_0, r}^S \cup C_{x_1, \rho}^S, \\ k & \text{otherwise}, \end{cases}$$

for all $x \in X$, where $k$ is a constant satisfying $S(k, k, x_0) \neq r$ and $S(k, k, x_1) \neq \rho$. It can be easily verified that the self-mapping $T$ satisfies the conditions (SC1) and (SC2) in Theorem 3.2 for the circles $r_{x_0, r}$ and $C_{x_1, \rho}^S$ with the mappings $\varphi_1$ and $\varphi_2$, respectively. Clearly $T$ fixes both of the circles $C_{x_0, r}^S$ and $C_{x_1, \rho}^S$. The number of fixed circles can be extended to any positive integer $n$ using the same arguments.

In the following theorem, we give a uniqueness condition for the fixed circles in Theorem 3.2 using Rhoades’ contractive condition on an S-metric space.

We recall the definition of Rhoades’ contractive condition.

**Definition 3.9.** [3] Let $(X, S)$ be an S-metric space and $T$ be a self-mapping of $X$. Then

$$(S25) \quad S(Tx, Tx, Ty) < \max\{S(x, x, y), S(Tx, Tx, x), S(Ty, Ty, y), S(Ty, Ty, x), S(Tx, Tx, y)\},$$

for each $x, y \in X, x \neq y$.

**Theorem 3.10.** Let $(X, S)$ be an S-metric space and $C_{x_0, r}^S$ be any circle on $X$. Let $T : X \to X$ be a self-mapping satisfying the conditions (SC1) and (SC2) given in Theorem 3.2. If the contractive condition (S25) is satisfied for all $x \in C_{x_0, r}^S$, $y \in X \setminus C_{x_0, r}^S$ by $T$, then $C_{x_0, r}^S$ is the unique fixed circle of $T$.

**Proof.** Suppose that there exist two fixed circles $C_{x_0, r}^S$ and $C_{x_1, \rho}^S$ of the self-mapping $T$, that is, $T$ satisfies the conditions (SC1) and (SC2) for each circles $C_{x_0, r}^S$ and $C_{x_1, \rho}^S$. Let $x \in C_{x_0, r}^S$ and $y \in C_{x_1, \rho}^S$ be arbitrary points with $x \neq y$. Using the contractive condition (S25) we find

$$S(x, y) = S(Tx, Tx, Ty) < \max\{S(x, x, y), S(Tx, Tx, x), S(Ty,Ty,y), S(Ty, Ty, x), S(Tx, Tx, y)\} = S(x, x, y),$$

which is a contradiction. Therefore it should be $x = y$. Consequently, $C_{x_0, r}^S$ is the unique fixed circle of $T$.  \[\square\]
Notice that the contractive condition in Theorem 3.10 is not to be unique. For example, if we consider the Banach’s contractive condition given in [8]
\[ S(Tx, Tx, Ty) \leq \alpha S(x, x, y), \]
for some \( 0 \leq \alpha < 1 \) and all \( x, y \in X \) in Theorem 3.10 then the fixed circle \( C_{x_0, r}^S \) is unique.

Now we give another existence theorem.

**Theorem 3.11.** Let \((X, S)\) be an \(S\)-metric space and \(C_{x_0, r}^S\) be any circle on \(X\). Let the mapping \(\varphi\) be defined as (3.1). If there exists a self-mapping \(T : X \to X\) satisfying
\[ (SC1)^* \quad S(x, x, Tx) \leq \varphi(x) + \varphi(Tx) - 2r \]
and
\[ (SC2)^* \quad S(Tx, Tx, x_0) \leq r, \]
for each \(x \in C_{x_0, r}^S\), then \(C_{x_0, r}^S\) is a fixed circle of \(T\).

**Proof.** Let \(x \in C_{x_0, r}^S\) be any arbitrary point. Using the condition \((SC1)^*\) we obtain
\[
S(x, x, Tx) \leq \varphi(x) + \varphi(Tx) - 2r \quad (3.3)
\leq S(x, x, x_0) + S(Tx, Tx, x_0) - 2r
= S(Tx, Tx, x_0) - r.
\]

![Figure 7](image-url)  
**Figure 7.** The geometric description of the condition \((SC1)^*\).

Because of the condition \((SC2)^*\) the point \(Tx\) should be lie on or interior of the circle \(C_{x_0, r}^S\). If \(S(Tx, Tx, x_0) < r\) then we have a contradiction using the inequality (3.3).

![Figure 8](image-url)  
**Figure 8.** The geometric description of the condition \((SC2)^*\).
Therefore it should be $S(Tx, Tx, x_0) = r$. If $S(Tx, Tx, x_0) = r$ then using the inequality (3.3) we get
\[ S(x, x, Tx) \leq S(Tx, Tx, x_0) - r = r - r = 0 \]
and so we find $Tx = x$. Consequently, $C_{x_0, r}^S$ is a fixed circle of $T$.

\[ \text{Figure 9. The geometric description of the condition (SC1)$^*$ $\cap$ (SC2)$^*$.} \]

Remark. Notice that the condition (SC1)$^*$ guarantees that $Tx$ is not in the interior of the circle $C_{x_0, r}^S$ for each $x \in C_{x_0, r}^S$. Similarly the condition (SC2)$^*$ guarantees that $Tx$ is not in the exterior of the circle $C_{x_0, r}^S$ for each $x \in C_{x_0, r}^S$. Consequently, $Tx \in C_{x_0, r}^S$ for each $x \in C_{x_0, r}^S$ and so we have $T(C_{x_0, r}^S) \subset C_{x_0, r}^S$ (see Figures 7, 8 and 9).

Now we give the following example.

**Example 3.12.** Let $X = \mathbb{R}$ and the mapping $S : X \times X \times X \to [0, \infty)$ be defined as
\[ S(x, y, z) = |x^3 - z^3| + |y^3 - z^3|, \]
for all $x, y, z \in X$. Then $(X, S)$ is an $S$-metric space. Let us consider the circle $C_{0, 16}^S$ and define the self-mapping $T : \mathbb{R} \to \mathbb{R}$
\[ Tx = \frac{3x + 4\sqrt{2}}{\sqrt{2}x + 3}, \]
for all $x \in \mathbb{R}$. Then it can be easily checked that the conditions (SC1)$^*$ and (SC2)$^*$ are satisfied. Therefore the circle $C_{0, 16}^S$ is a fixed circle of $T$.

In the following example, we give an example of a self-mapping which satisfies the condition (SC1)$^*$ and does not satisfy the condition (SC2)$^*$.

**Example 3.13.** Let $X = \mathbb{R}$ and $(X, S)$ be the $S$-metric space defined in Example 3.12. Let us consider the circle $C_{-1, 18}^S$ and define the self-mapping $T : \mathbb{R} \to \mathbb{R}$ as
\[ Tx = \begin{cases} 
-3 & \text{if } x = -2 \\
3 & \text{if } x = 2 \\
10 & \text{otherwise}
\end{cases} \]
for all $x \in \mathbb{R}$. Then the self-mapping $T$ satisfies the condition (SC1)$^*$ but does not satisfy the condition (SC2)$^*$. Clearly $T$ does not fix the circle $C_{-1, 18}^S$. 

In the following example, we give an example of a self-mapping which satisfies the condition \((SC2)^*\) and does not satisfy the condition \((SC1)^*\).

**Example 3.14.** Let \(X = \mathbb{C}\) and the mapping \(S : X \times X \times X \to [0, \infty)\) be defined as
\[
S(z_1, z_2, z_3) = |z_1 - z_3| + |z_1 + z_3 - 2z_2|,
\]
for all \(z_1, z_2, z_3 \in \mathbb{C}\) [4]. Then \((C, S)\) is an \(S\)-metric space. Let us consider the circle \(C_{0, 1}^S\) and define the self-mapping \(T_1 : \mathbb{C} \to \mathbb{C}\)
\[
T_1 z = \begin{cases} \frac{1}{z} & ; \quad z \neq 0 \\ 0 & ; \quad z = 0 \end{cases},
\]
for all \(z \in \mathbb{C}\), where \(\overline{z}\) is the complex conjugate of \(z\). Then it can be easily checked that the conditions \((SC1)^*\) and \((SC2)^*\) are satisfied. Therefore the circle \(C_{0, 1}^S\) is a fixed circle of \(T_1\). But if we define the self-mapping \(T_2 : \mathbb{C} \to \mathbb{C}\)
\[
T_2 z = \begin{cases} \frac{1}{z} & ; \quad z \neq 0 \\ 0 & ; \quad z = 0 \end{cases},
\]
for all \(z \in \mathbb{C}\). Then the self-mapping \(T_2\) satisfies the condition \((SC2)^*\) but does not satisfy the condition \((SC1)^*\). Clearly \(T_2\) does not fix the circle \(C_{0, 1}^S\). Especially, \(T_2\) maps the circle \(C_{0, 1}^S\) onto itself while fixes the points \(z_1 = \frac{1}{2}\) and \(z_2 = -\frac{1}{2}\) only.

Now we determine a uniqueness condition for the fixed circles in Theorem 3.11. We recall the following definition.

**Definition 3.15.** [7] Let \((X, S)\) be a complete \(S\)-metric space and \(T\) be a self-mapping of \(X\). There exist real numbers \(a, b\) satisfying \(a + 3b < 1\) with \(a, b \geq 0\) such that
\[
S(Tx, Tx, Ty) \leq aS(x, x, y) + b \max\{S(Tx, Tx, x), S(Tx, Tx, y), \min\{S(Tx, Ty, y), S(Ty, Ty, x)\}\},
\]
for all \(x, y \in X\).

We give the following theorem.

**Theorem 3.16.** Let \((X, S)\) be an \(S\)-metric space and \(C_{x_0, r}^S\) be any circle on \(X\). Let \(T : X \to X\) be a self-mapping satisfying the conditions \((SC1)^*\) and \((SC2)^*\) given in Theorem 3.11. If the contractive condition (3.4) is satisfied for all \(x \in C_{x_0, r}^S\), \(y \in X \setminus C_{x_0, r}^S\) by \(T\) then \(C_{x_o, r}^S\) is the unique fixed circle of \(T\).

**Proof.** Assume that there exist two fixed circles \(C_{x_0, r}^S\) and \(C_{x_1, r}^S\) of the self-mapping \(T\), that is, \(T\) satisfies the conditions \((SC1)^*\) and \((SC2)^*\) for each circle \(C_{x_0, r}^S\) and \(C_{x_1, r}^S\). Let \(x \in C_{x_0, r}^S\) and \(y \in C_{x_1, r}^S\) be arbitrary points with \(x \neq y\). Using the contractive condition (3.4) we obtain
\[
S(x, y) = S(Tx, Tx, Ty) \leq aS(x, x, y) + b \max\{S(Tx, Tx, x), S(Tx, Ty, y), S(Ty, Ty, y), S(Ty, Ty, x)\},
\]
which is a contradiction since \(a + b < 1\). Hence it should be \(x = y\). Consequently, \(C_{x_0, r}^S\) is the unique fixed circle of \(T\). \(\square\)
Notice that the contractive condition in Theorem 3.16 is not to be unique. For example, in Theorem 3.16, if we consider the contractive condition given in [7]

\[ S(Tx, Tx, Ty) \leq aS(x, x, y) + bS(Tx, Tx, x) + cS(Ty, Ty, y) + d \max\{S(Tx, Tx, y), S(Ty, Ty, x)\} \]

where the real numbers \(a, b, c, d\) satisfying \(\max\{a + b + c + 3d, 2b + d\} < 1\) with \(a, b, c, d \geq 0\), for all \(x, y \in X\) then the fixed circle \(C_{n, r}^S\) is unique.

Finally we note that the identity mapping \(I_X\) defined as \(I_X(x) = x\) for all \(x \in X\) satisfies the conditions \((SC1)\) and \((SC2)\) (resp. \((SC1)^*\) and \((SC2)^*\)) in Theorem 3.2 (resp. Theorem 3.11). If a self-mapping \(T\), which has a fixed circle, satisfies the conditions \((SC1)\) and \((SC2)\) (resp. \((SC1)^*\) and \((SC2)^*\)) in Theorem 3.2 (resp. Theorem 3.11) but does not satisfy the condition \((IS)\) in the following theorem given in [6] then the self-mapping \(T\) can not be identity map.

**Theorem 3.17.** [6] Let \((X, S)\) be an \(S\)-metric space and \(C_{n, r}^S\) be any circle on \(X\). Let the mapping \(\varphi\) be defined as (3.1). If there exists a self-mapping \(T : X \to X\) satisfying the condition

\[ (IS) \quad S(x, x, Tx) \leq \frac{\varphi(x) - \varphi(Tx)}{h}, \]

for all \(x \in X\) and some \(h > 2\), then \(C_{n, r}^S\) is a fixed circle of \(T\) and \(T = I_X\).

**References**


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