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f-HARMONIC MAPS FROM FINSLER MANIFOLDS

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Abstract. In this paper, the first and second variation formulas of the f-energy functional for a smooth map from a Finsler manifold to a Riemannian manifold are obtained. As an application, it is proved that there exists no non-constant stable f-harmonic map from a Finsler manifold to the standard unit sphere $S^n (n > 2)$.

1. Introduction

f-harmonic maps as a generalization of harmonic maps, geodesics and minimal surfaces were first studied by A. Lichnerowicz [9] in 1970. Recently, N. Course [6] studied the f-harmonic flow on surfaces. Y. Ou [14] analysed the f-harmonic morphisms as a subclass of harmonic maps which pull back harmonic functions to f-harmonic functions. In [4], the researchers studied the stability of harmonic and f-harmonic maps on spheres. Many scholars have studied and done researches on the f-harmonic maps, see for instance, [3, 4, 5, 9, 10, 14, 15].

f-harmonic maps are applied in many branches of geometry and mathematical physics. In view of Physics, f-harmonic maps could be considered as the stationary solutions of inhomogeneous Heisenberg spin system, see for instance [5, 14]. Furthermore, the intersection of f-harmonicity with curvature conditions justifies their application for gleaning valuable information on weighted manifolds and gradient Ricci solitons, see [10, 15].

Let $\phi : (M, g) \longrightarrow (N, h)$ be a smooth map between Riemannian manifolds and $f \in C^\infty(M)$ a positive smooth function on M. The map ϕ is called f-harmonic if $\phi \mid_\Omega$ is a critical point of the f-energy functional

$$E_f(\phi) := \frac{1}{2} \int_\Omega f \mid d\phi \mid^2 dv_g,$$

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for any compact sub-domain $\Omega \subseteq M$. Here dv_g is the volume element of M and $|d\phi|$ denotes the Hilbert-Schmidt norm of the differential $d\phi \in \Gamma(T^*M \otimes \phi^{-1}TN)$.

Let $\phi : (M, F) \longrightarrow (N, h)$ be a smooth map from a Finsler manifold (M, F) to a Riemannian manifold (N, h) and $f : SM \longrightarrow (0, \infty)$ be a smooth positive function on the projective sphere bundle of M. In this paper, the f-energy functional of ϕ is introduced and the corresponding variation formulas are obtained. It can be seen that the first and second variation formulas of the f-energy functional is consistent to that of Riemannian case if M is Riemannian and f is defined on M, see [4].

The concept of harmonic maps from a Finsler manifold to a Riemannian manifold was first introduced by X. Mo, see [11]. On the workshop of Finsler Geometry in 2000, Professor S. S. Chern conjectured that the fundamental existence theorem of harmonic maps on Finsler spaces is true. In [13], the researchers have proved this conjecture and shown that any smooth map from a compact Finsler manifold to a compact Riemannian manifold of non-positive sectional curvature could be deformed into a harmonic map which has minimum energy in its homotopy class. Y. Shen and Y. Zhang [16] extended Mo's work to Finsler target manifold and obtained the first and second variation formulas.

As an application, Q. He and Y. Shen [7] proved that any harmonic map from an Einstein Riemannian manifold to a Finsler manifold with certain conditions is totally geodesic and there is no stable harmonic map from an Euclidean unit sphere S^n to any Finsler manifolds. Harmonic maps between Finsler manifolds have been studied extensively by various researchers, see for instance, [7, 8, 11, 12, 13, 16].

The current paper is organized as follows:
In the second section, a few concepts of Finsler geometry are reviewed. In section 3, the f-energy functional of a smooth map from a Finsler manifold to a Riemannian manifold is introduced and the corresponding Euler-Lagrange equation is obtained via calculating the first variation formula of the f-energy functional. In section 4, the second variation formula of the f-energy functional for an f-harmonic map is derived. Finally, it is shown that there exists no non-constant stable f-harmonic map from a Finsler manifold to the standard sphere $S^n (n > 2)$.

2. Preliminaries and Notations

In this section, a few basic notions of Finsler geometry are provided which will be used later. For more details see ([1, 11, 12, 16]). Throughout this paper, it is assumed that M is an m-dimensional connected compact oriented manifold without boundary and $\pi : TM \longrightarrow M$ be its tangent bundle. Let (x^i) be a local coordinates system with the domain $U \subseteq M$ and (x^i, y^i) the induced standard local coordinates system on $\pi^{-1}(U)$. A Finsler manifold is a pair (M, F) includes a smooth manifold M and a Finsler metric $F : TM \longrightarrow [0, \infty)$ satisfies the following properties: i) F is smooth on $TM \setminus 0$, ii) $F(x, \lambda y) = \lambda F(x, y)$ for $\lambda > 0$, iii) The fundamental quadratic form

$$g := g_{ij}(x, y)\, dx^i \otimes dx^j, \qquad g_{ij} := \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j}, \tag{2.1}$$

is positive definite at every point $(x, y) \in TM \setminus 0$. The Riemannian manifolds and locally Minkowski manifolds are important examples of Finsler manifolds. In the sequel, the following convention of index ranges are used

$$1 \leqslant i, j, k, ... \leqslant m, \qquad 1 \leqslant a, b, c, ... \leqslant m-1, \qquad 1 \leqslant A, B, C, ... \leqslant 2m-1.$$

The Finsler structure F induces two more significant quantities as follows

$$\begin{aligned} A &:= A_{ijk}\, dx^i \otimes dx^j \otimes dx^k, & A_{ijk} &:= \frac{F}{4}[F^2]_{y^i y^j y^k}, \\ \eta &:= \eta_i dx^i, & \eta_i &:= g^{jk} A_{ijk}, \end{aligned}$$

called Cartan tensor and Cartan form, respectively.

Let us denote the projective sphere bundle of M by SM, where $SM := \cup_x S_x M$. Almost every geometric quantities constructed by Finsler structure are invariant under rescaling $y \longrightarrow ty$ for $t > 0$, thus make sense on SM. The canonical projection $p : SM \longrightarrow M$ defined by $(x, y) \longrightarrow x$ pulls back the tangent bundle TM to the m-dimensional vector bundle p^*TM over $(2m-1)-$dimensional manifold SM. The bundle p^*TM and its dual p^*T^*M are said to be the *Finsler bundle* and *dual Finsler bundle*, respectively.

At each point $(x, y) \in SM$, the fibre of p^*TM has a local basis $\{\frac{\partial}{\partial x^k}\}$ and a metric g defined by (2.1). Here $\frac{\partial}{\partial x^k}$ and its dual dx^k stand for the sections $(x, y, \frac{\partial}{\partial x^k}) \in \Gamma(p^*TM)$ and $(x, y, dx^k) \in \Gamma(p^*T^*M)$, respectively. The bundle p^*TM has a global section $l(x, y) := \frac{y^i}{F}\frac{\partial}{\partial x^i}$ which is called the *distinguished section.* The dual of the former section $\omega = [F]_{y^i} dx^i$ is called *Hilbert form.* Furthermore, each fibre of the Riemannian vector bundle (p^*TM, g) has an *adapted frame* $\{e_i := u_i^j \frac{\partial}{\partial x^j}\}$, i.e. $g(e_i, e_j) = \delta_{ij}$ and $e_m := l$. Denote its dual by $\{\omega^i := v_j^i dx^j\}$, $\omega^i(e_j) = \delta_j^i$. It is clear that $\omega^m = \omega$. In the rest of this paper, these abbreviations will be used. According to the notations above, it can be seen that $\frac{\partial}{\partial x^i} = v_i^k e_k$ and $dx^i = u_k^i \omega^k$, where (u_j^i) and (v_j^i) are related by $u_i^k v_k^j = \delta_i^j$. More relations among (u_j^i)'s, (v_j^i)'s and the quadratic form of F can be found in [1].

Let $N_j^i := \frac{1}{2}\frac{\partial G^i}{\partial y^j}$ be the coefficients of non-linear connection on TM, where $G^i := \frac{1}{4} g^{ih}(\frac{\partial^2 F^2}{\partial y^h \partial x^j} y^j - \frac{\partial F^2}{\partial x^h})$. Consider the local orthogonal basis $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\}$ on $T_z TM$, where $\frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - N_i^j \frac{\partial}{\partial y^j}$ and dual basis as $\{dx^i, \delta y^i\}$, where $\delta y^i := dy^i + N_j^i dx^j$. It can be shown that $\{\omega^i := v_j^i dx^j,\ \omega^{m+a} := v_j^a \frac{\delta y^j}{F}\}$ is a local basis for the tangent bundle T^*SM. Consider $\omega^{2m} = [F]_{y^i}\frac{\delta y^i}{F}$ as dual to the vector $y^i \frac{\partial}{\partial y^i}$. Therefore, ω^{2m} vanishes on SM. Based on the above notations, the Sasaki-type metric, the volume element, the horizontal sub-bundle and the vertical sub-bundle of SM are defined by

$$\begin{aligned} G &:= \delta_{ij}\omega^i \otimes \omega^j + \delta_{ab}\omega^{m+a} \otimes \omega^{m+b}, & dV_{SM} &:= \omega^1 \wedge \omega^2 \wedge \cdots \wedge \omega^{2m-1}, \\ HSM &:= \{v \in TSM, \quad \omega^{m+a}(v) = 0\}, & VSM &:= \cup_{x \in M} TS_x M, \end{aligned} \tag{2.2}$$

respectively, see [2]. Due to the fact that HSM is isomorph with p^*TM, HSM is also called the Finsler bundle. In the sequel, for any $X \in \Gamma(p^*TM)$ the corresponding horizontal lift of X is denoted by X^H.

As well-known, there exists a linear connection on p^*TM called the Chern connection and denoted by ${}^c\nabla$. Its connection forms are characterized by the following equations

$$d(dx^i) - dx^k \wedge \omega_k^i = 0, \tag{2.3}$$

and

$$dg_{ij} - g_{ik}\omega_j^k - g_{jk}\omega_i^k = 2A_{ijk}\frac{\delta y^k}{F}. \tag{2.4}$$

By taking the exterior derivative of (2.3), the curvature $2 - forms$ of the Chern connection, $\Omega_j^i := d\omega_j^i - \omega_j^k \wedge \omega_k^i$, have the following structure

$$\Omega_j^i = \frac{1}{2}R_{j\ kl}^{\ i}dx^k \wedge dx^l + P_{j\ kl}^{\ i}dx^k \wedge \frac{\delta y^l}{F}. \tag{2.5}$$

By (2.5), the Landsberg curvature is defined as follows

$$L := L_{ijk}dx^i \otimes dx^j \otimes dx^k, \qquad L_{ijk} := g_{il}\frac{y^m}{F}P_{m\ jk}^{\ l}.$$

It can be seen that $L_{ijk} = -\dot{A}_{ijk}$, where dot denotes the covariant derivative along the Hilbert form, (see [16], p. 41).
Let D denotes the Levi-Civita connection on (SM, G). The divergence of a form $\psi = \psi_i\omega^i \in \Gamma(p^*T^*M)$ is

$$div_G\psi := Tr_G D\psi.$$

Note that the bundle p^*T^*M is isomorph with the horizontal sub-bundle of T^*SM. It can be shown that

$$div_G\psi = \sum_i \psi_{i|i} + \sum_{a,b}\psi_a L_{bba} = \sum_i ({}^c\nabla_{e_i^H}\psi)(e_i) + \sum_{a,b}\psi_a L_{bba}, \tag{2.6}$$

where " | " denotes the horizontal covariant differential with respect to the Chern connection, $\{e_i\}$ be the adapted frame with respect to g and $L_{abc} = L(e_a, e_b, e_c)$, (see [8], Lemma 2.1).

3. The first variation formula

Let $\phi : (M^m, F) \longrightarrow (N^n, h)$ be a smooth map from an m-dimensional Finsler manifold (M, F) to an n-dimensional Riemannian manifold (N, h). Henceforth, the Chern connection on p^*TM, the Levi-Civita connection on (N, h) and the pull-back connection on $p^*(\phi^{-1}TN)$ are denoted by ${}^c\nabla, {}^N\nabla$ and ∇, respectively.

Let $f \in C^\infty(SM)$ be a smooth positive function on SM. The *f-energy density* of ϕ is a function $e_f(\phi) : SM \longrightarrow \mathbb{R}$ defined by

$$e_f(\phi)(x, y) := \frac{1}{2}f(x, y)Tr_g h(d\phi, d\phi), \tag{3.1}$$

where Tr_g stands for taking the trace with respect to g (the fundamental quadratic form of F) at $(x, y) \in SM$. In the local coordinates (x^i) on M and $(\tilde{x}^\alpha)$ on N, the f-energy density of ϕ can be written as follows

$$e_f(\phi)(x, y) = \frac{1}{2}f(x, y)\delta^{ij}h(d\phi(e_i), d\phi(e_j)) = \frac{1}{2}f(x, y)\delta^{ij}\phi_i^\alpha\phi_j^\beta h_{\alpha\beta}(\tilde{x}), \tag{3.2}$$

where $\{e_i = u_i^k \frac{\partial}{\partial x^k}\}$ is the adapted frame with respect to g at $(x, y) \in SM$, $\tilde{x} = \phi(x)$ and $d\phi(e_i) = \phi_i^\alpha \frac{\partial}{\partial \tilde{x}^\alpha} \circ \phi$.

Definition 3.1. *A map $\phi : (M, F) \longrightarrow (N, h)$ is said to be f-harmonic, if it is a critical point of the f-energy functional*

$$E_f(\phi) := \frac{1}{c_{m-1}} \int_{SM} e_f(\phi) dV_{SM}, \tag{3.3}$$

where c_{m-1} denotes the volume of the standard $(m-1)-$dimensional sphere and dV_{SM} is the canonical volume element of SM defined by (2.2).

Let $\phi_t : M \longrightarrow N$ $(-\varepsilon < t < \varepsilon)$ be a smooth variation of ϕ such that $\phi_0 = \phi$ and set

$$V = \frac{\partial \phi_t}{\partial t}\Big|_{t=0} := V^\alpha \frac{\partial}{\partial \tilde{x}^\alpha} \circ \phi.$$

By (3.2), the f-energy density of ϕ_t can be written as follows

$$e_f(\phi_t)(x, y) = \frac{1}{2} f(x, y) \delta^{ij} \phi^\alpha_{t|i} \phi^\beta_{t|j} h_{\alpha\beta}(\tilde{x}), \tag{3.4}$$

where $\tilde{x} = \phi_t(x)$, $d\phi_t(e_i) = u_i^k \frac{\partial \phi_t^\alpha}{\partial x^k} \frac{\partial}{\partial \tilde{x}^\alpha} \circ \phi := \phi^\alpha_{t|i} \frac{\partial}{\partial \tilde{x}^\alpha} \circ \phi$. Due to the fact that $\{V^\alpha\}$ is independent of y and using (3.4), it is obtained that

$$\begin{aligned}
\frac{\partial}{\partial t} e_f(\phi_t)\Big|_{t=0} &= \frac{1}{2} \frac{\partial}{\partial t} (\delta^{ij} f \phi^\alpha_{t|i} \phi^\beta_{t|j} h_{\alpha\beta})\Big|_{t=0} \\
&= \delta^{ij} \{ f u_i^k \frac{\partial V^\alpha}{\partial x^k} \phi_j^\beta h_{\alpha\beta} + \frac{1}{2} f \phi_i^\alpha \phi_j^\beta \frac{\partial h_{\alpha\beta}}{\partial \tilde{x}^\gamma} V^\gamma \} \\
&= \sum_i \{ f u_i^k \frac{\delta V^\alpha}{\delta x^k} \phi_i^\beta h_{\alpha\beta} + f \phi_i^\alpha \phi_i^\beta \; {}^N\Gamma^\sigma_{\beta\gamma} h_{\alpha\sigma} V^\gamma \} \\
&= \sum_i h(\nabla_{e_i^H} V, f d\phi(e_i)),
\end{aligned} \tag{3.5}$$

where $\{{}^N\Gamma^\alpha_{\beta\gamma}\}$ denotes the coefficients of the Levi-Civita connections on (N,h). Let $\psi := h(V, f d\phi(e_i))\omega^i \in \Gamma(p^*T^*M)$. Using the fact that $L_{bba} = -\dot{A}_{bba}$ and equation (2.6), it follows that

$$\begin{aligned}
div_G \psi &= \sum_i ({}^c\nabla_{e_i^H} \psi)(e_i) + \sum_{a,b} h(V, f d\phi(e_a)) L_{bba} \\
&= \sum_i \{ h(\nabla_{e_i^H} V, f d\phi(e_i)) + h(V, (\nabla_{e_i^H} f d\phi)(e_i)) \} - \sum_{a,b} h(V, f d\phi(e_a)) \dot{A}_{bba} \\
&= h\Big(V, f Tr_g \nabla d\phi + d\phi \circ p(grad^H f) - f d\phi \circ p(K^H) \Big) \\
&+ \sum_i h(\nabla_{e_i^H} V, f d\phi(e_i)).
\end{aligned} \tag{3.6}$$

where $Tr_g \nabla d\phi = g^{ij} (\nabla_{\frac{\partial}{\partial x^i}} d\phi(\frac{\partial}{\partial x^j}) - d\phi({}^c\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}))$, $A_{bba} = A(e_b, e_b, e_a)$ and K is defined as follows

$$K := \sum_{a,b} \dot{A}_{bba} e_a \in \Gamma(p^*TM). \tag{3.7}$$

Combining (3.5) and (3.6) and considering the Green's theorem, it can be concluded that

$$\frac{d}{dt}E_f(\phi_t)\big|_{t=0} = -\frac{1}{c_{m-1}}\int_{SM} h(\tau_f(\phi), V)dV_{SM},$$

where

$$\tau_f(\phi) := fTr_g\nabla d\phi + d\phi\circ p(grad^H f) - fd\phi\circ p(K^H)\in\Gamma((\phi\circ p)^*TN), \quad (3.8)$$

here $p : SM \longrightarrow M$ is the canonical projection on SM, $grad^H f$ denotes the horizontal part of $grad\, f \in \Gamma(TSM)$ and K is defined by (3.7). The field $\tau_f(\phi)$ is said to be the *f-tension field* of ϕ.

Theorem 3.2. *Let $\phi : (M, F) \longrightarrow (N, h)$ be a smooth map from a Finsler manifold to a Riemannian manifold and $f \in C^\infty(SM)$ a smooth positive function on SM. Then, ϕ is f-harmonic if and only if $\tau_f(\phi) \equiv 0$*

Due to the fact that the Landsberg curvature of locally Minkowski manifold vanishes and considering Theorem 3.2 and equation (3.8), the following result is obtained immediately

Corollary 3.3. *Let $\phi : (M, F) \longrightarrow (N, h)$ be an immersion harmonic map from a locally Minkowski manifold (M, F) to an arbitrary Riemannian manifold (N, h) and $f \in C^\infty(SM)$ a smooth positive function on SM. Then, ϕ is f-harmonic if and only if $f(x, y) = f(y)$ for any $(x, y) \in SM$.*

Example 3.4. *Assume that $(\mathbb{R}^2, F)$ be a locally Minkowski manifold and $(\mathbb{R}^3, \langle,\rangle)$ be the three-dimensional Euclidean space. Let $\phi : (\mathbb{R}^2, F) \longrightarrow (\mathbb{R}^3, \langle,\rangle)$ is defined by $\phi(x) := (x^2, x^1 + 2x^2, 3x^1 - x^2)$, where $x = (x^1, x^2) \in \mathbb{R}^2$. Let $f(x, y) := exp(\frac{y^1(y^1-2y^2)}{(y^1)^2+(y^2)^2})$ be a positive smooth map on $S\mathbb{R}^2$. By (3.8), it can be seen that ϕ is f-harmonic.*

Remark 3.5. *Let (M, F) be a locally Minkowski manifold and (M, h) be a flat Riemannian manifold. It is conspicuous that the identity map $Id : (M, F) \longrightarrow (M, h)$ is harmonic, (see [12], Proposition 9.5.1). By Corollary 3.3, it can be concluded that Id is f-harmonic if and only if $f(x, y) = f(y)$ for all $(x, y) \in SM$.*

Before proceeding, it is worth noting that f-harmonic maps shouldn't be confused with $\mathcal{F}$-harmonic maps and p-harmonic maps from a Finsler manifolds to a Riemannian manifolds. Let $\mathcal{F} : [0,\infty) \longrightarrow [0,\infty)$ be a C^2 strictly increasing function on $(0,\infty)$. The smooth map $\phi : (M, F) \longrightarrow (N, h)$ from a Finsler manifold (M, F) to a Riemannian manifold (N, h) is called $\mathcal{F}$-harmonic if it is a critical points of the $\mathcal{F}$-energy functional

$$E_{\mathcal{F}}(\phi) := \int_{SM}\mathcal{F}(\frac{|d\phi|^2}{2})dV_{SM}. \quad (3.9)$$

The notion of $\mathcal{F}-$harmonic maps was first introduced by J. Li [8]. $\mathcal{F}$-energy functional can be categorized as energy , p-energy and exponential energy when $\mathcal{F}(t)$ is equal to t, $(2t)^{\frac{p}{2}}\setminus p$ $(p \geq 4)$ and e^t, respectively. In terms of the Euler-Lagrange equation, ϕ is $\mathcal{F}$-harmonic if it satisfies the following equation

$$\tau_{\mathcal{F}}(\phi) := Tr_g\nabla(\mathcal{F}'(\frac{|\, d\phi\,|^2}{2})d\phi) - \mathcal{F}'(\frac{|\, d\phi\,|^2}{2})d\phi(K) = 0. \quad (3.10)$$

For more details, see [8]. The field $\tau_{\mathcal{F}}(\phi)$ is called the $\mathcal{F}-$tension field of ϕ. Let $\phi : (M, F) \longrightarrow (N, h)$ be a non-degenerate smooth map (i.e. $d\phi_x \neq 0$ for all $x \in M$) from a Finsler manifold to a Riemannian manifold. By (3.8) and (3.10), the following proposition is obtained immediately.

Proposition 3.6. *Let $\phi : (M, F) \longrightarrow (N, h)$ be a non-degenerate $\mathcal{F}$-harmonic map from a Finsler manifold to a Riemannian manifold. Then, ϕ is an f-harmonic map with $f = \mathcal{F}'(\frac{|d\phi|^2}{2})$. Particularly, any non-degenerate p-harmonic map is an f-harmonic map with $f =| d\phi |^{p-2}$.*

Remark 3.7. *This result was obtained by Y. Chiang* [5] *in the Riemannian case.*

4. The second variation formula

In this section, the second variation formula of the f-energy functional for an f-harmonic map from a Finsler manifold to a Riemannian manifold is obtained. As an application, it is shown that any stable f-harmonic map ϕ from a Finsler manifold to the standard sphere $\mathbb{S}^n(n > 2)$ is constant.

Theorem 4.1. *(The second variation formula). Let $\phi : (M, F) \longrightarrow (N, h)$ be an f-harmonic map from a Finsler manifold (M, F) to a Riemannian manifold (N, h). Let $\phi_t : M \longrightarrow N$ $(-\varepsilon < t < \varepsilon)$ be a smooth variation such that $\phi_0 = \phi$ and set $V = \frac{\partial \phi_t}{\partial t}|_{t=0}$. Then*

$$\frac{d^2}{dt^2}E_f(\phi_t)\big|_{t=0} = -\frac{1}{c_{m-1}}\int_{SM} h\big(V, fTr_g(\nabla^2 V) + fTr_g R^N(V, d\phi)d\phi + \nabla_{grad^H f}V - f\nabla_{K^H}V\big)dV_{SM}, \tag{4.1}$$

where K is defined by (3.7), R^N is the curvature tensor on (N, h) and $Tr_g(\nabla^2 V) = g^{ij}(\nabla_{\frac{\partial}{\partial x^i}}\nabla_{\frac{\partial}{\partial x^j}}V - \nabla_{^c\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}}V)$.

Set

$$Q_f^{\phi}(V) := \frac{d^2}{dt^2}E_f(\phi_t)\big|_{t=0}.$$

An f-harmonic map ϕ is said to be *stable f-harmonic* if $Q_f^{\phi}(V) \geq 0$ for any vector field V along ϕ.

Proof. Let $\tilde{M}$ denotes the product manifold $(-\varepsilon, \varepsilon) \times M$, $\Phi : \tilde{M} \longrightarrow N$ is defined by $\Phi(t, x) := \phi_t(x)$ and $\tilde{p} : S\tilde{M} \longrightarrow \tilde{M}$ be the natural projection on the sphere bundle $S\tilde{M}$. Denote the same notations of ${}^c\nabla$ and ∇ for the Chern connection on $\tilde{p}^*T\tilde{M}$ and the pull-back connection on $\tilde{p}^*(\Phi^{-1}TN)$, respectively. By (3.1), it can be shown

$$\begin{aligned}\frac{\partial^2}{\partial t^2}e_f(\phi_t) &= \frac{\partial}{\partial t}h(\nabla_{\frac{\partial}{\partial t}}d\Phi(e_i), fd\Phi(e_i))\\ &= \frac{\partial}{\partial t}h(\nabla_{e_i^H}d\Phi(\frac{\partial}{\partial t}), fd\Phi(e_i))\\ &= h(\nabla_{\frac{\partial}{\partial t}}\nabla_{e_i^H}d\Phi(\frac{\partial}{\partial t}), fd\Phi(e_i)) + fh(\nabla_{e_i^H}d\Phi(\frac{\partial}{\partial t}), \nabla_{e_i^H}d\Phi(\frac{\partial}{\partial t}))\\ &= h(\nabla_{e_i^H}\nabla_{\frac{\partial}{\partial t}}d\Phi(\frac{\partial}{\partial t}), fd\Phi(e_i)) + fh(\nabla_{e_i^H}d\Phi(\frac{\partial}{\partial t}), \nabla_{e_i^H}d\Phi(\frac{\partial}{\partial t}))\\ &+ fh(R^N(d\Phi(\frac{\partial}{\partial t}), d\Phi(e_i))d\Phi(\frac{\partial}{\partial t}), d\Phi(e_i)), \end{aligned} \tag{4.2}$$

where it is used

$$\nabla_{\frac{\partial}{\partial t}}d\Phi(e_i) - \nabla_{e_i^H}d\Phi(\frac{\partial}{\partial t}) = d(\Phi\circ\tilde{p})[\frac{\partial}{\partial t}, e_i^H] = 0,$$

for the third and fourth equalities in (4.2). By (4.2), it can be seen that

$$\begin{aligned}\frac{\partial^2}{\partial t^2}E_f(\phi_t)\big|_{t=0} &= \frac{1}{c_{m-1}}\sum_i\Big\{\int_{SM} fh(\nabla_{e_i^H}d\Phi(\frac{\partial}{\partial t}), \nabla_{e_i^H}d\Phi(\frac{\partial}{\partial t}))\big|_{t=0}dV_{SM}\\ &+ \int_{SM} h(\nabla_{e_i^H}\nabla_{\frac{\partial}{\partial t}}d\Phi(\frac{\partial}{\partial t}), fd\Phi(e_i))\big|_{t=0}dV_{SM}\\ &+ \int_{SM} fh(R^N(d\Phi(\frac{\partial}{\partial t}), d\Phi(e_i))d\Phi(\frac{\partial}{\partial t}), d\Phi(e_i))\big|_{t=0}dV_{SM}\Big\}\\ &= I_1 + I_2 + I_3\end{aligned} \tag{4.3}$$

Now each term of the right hand side(RHS) of the above equation is calculated. First, let
$\psi := fh(\nabla_{e_i^H}d\Phi(\frac{\partial}{\partial t}), d\Phi(\frac{\partial}{\partial t}))w^i \in \Gamma(p^*T^*M)$. By (2.6), it follows that

$$\begin{aligned}div_G\psi &= \sum_i({}^c\nabla_{e_i^H}\psi)(e_i) + \sum_{a,b} fh(\nabla_{e_a^H}d\Phi(\frac{\partial}{\partial t}), d\Phi(\frac{\partial}{\partial t}))L_{bba}\\ &= \sum_i\Big\{e_i^H(f)h(\nabla_{e_i^H}d\Phi(\frac{\partial}{\partial t}), d\Phi(\frac{\partial}{\partial t})) + fh(\nabla_{e_i^H}\nabla_{e_i^H}d\Phi(\frac{\partial}{\partial t}), d\Phi(\frac{\partial}{\partial t}))\\ &+ fh(\nabla_{e_i^H}d\Phi(\frac{\partial}{\partial t}), \nabla_{e_i^H}d\Phi(\frac{\partial}{\partial t})) + fh(\nabla_{e_j^H}d\Phi(\frac{\partial}{\partial t}), d\Phi(\frac{\partial}{\partial t}))({}^c\nabla_{e_i^H}w^j)(e_i)\Big\}\\ &- \sum_{a,b} fh(\nabla_{e_a^H}d\Phi(\frac{\partial}{\partial t}), d\Phi(\frac{\partial}{\partial t}))\dot{A}_{bba}.\end{aligned} \tag{4.4}$$

By (4.4) and Green's theorem, I_1 can be obtained as follows

$$I_1 = -\frac{1}{c_{m-1}}\int_{SM} h\Big(fTr_g(\nabla^2 V) + \nabla_{grad^H f}V - f\nabla_{K^H}V, V\Big)dV_{SM}. \tag{4.5}$$

Similarly, let $\Psi := h(\nabla_{\frac{\partial}{\partial t}} d\phi(\frac{\partial}{\partial t}), f d\phi(e_i))w^i \in \Gamma(p^*T^*M)$. It can be seen that

$$div_G\Psi = \sum_i \Big\{ h(\nabla_{e_i^H}\nabla_{\frac{\partial}{\partial t}} d\Phi(\frac{\partial}{\partial t}), f d\Phi(e_i)) + h\big(\nabla_{\frac{\partial}{\partial t}} d\Phi(\frac{\partial}{\partial t}), (\nabla_{e_i^H} f d\Phi)(e_i)\big)\Big\} - f\sum_{a,b} h(\nabla_{\frac{\partial}{\partial t}} d\Phi(\frac{\partial}{\partial t}), \dot{A}_{bba} d\phi(e_a)). \tag{4.6}$$

By (4.6) and considering the Green's Theorem and the f-harmonicity condition of ϕ, I_2 is given by

$$I_2 = -\frac{1}{c_{m-1}}\int_{SM} h\big(\nabla_{\frac{\partial}{\partial t}} d\Phi(\frac{\partial}{\partial t}) \mid_{t=0}, \tau_f(\phi)\big) dV_{SM} = 0. \tag{4.7}$$

Substituting the formulas (4.5) and (4.7) into equation (4.3) yields the formula (4.1) and hence completes the proof. □

5. Stability of f-harmonic maps to $\mathbb{S}^n$

Consider the unit sphere $\mathbb{S}^n$ as a submanifold of the Euclidean space $(\mathbb{R}^{n+1}, \langle , \rangle)$. At each point $x \in \mathbb{S}^n$ any vector field V in $\mathbb{R}^{n+1}$ can be decomposed as $V = V^\top + V^\perp$, where $V^\top$ is the component of V tangent to $\mathbb{S}^n$ and $V^\perp = \langle V, x\rangle x$ is the component of V normal to $\mathbb{S}^n$. Let ${}^R\nabla$ be the Levi-Civita connection on $\mathbb{R}^{n+1}$, ${}^S\nabla$ be the Levi-Civita connection on $\mathbb{S}^n$ and B be the second fundamental form of $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$. We have the following relation

$${}^R\nabla_X Y = {}^S\nabla_X Y + B(X,Y), \tag{5.1}$$

where X and Y are smooth vector fields on $\mathbb{S}^n$. The *shape operator* with respect to any normal vector field W on $\mathbb{S}^n$ is defined by

$$A^W(X) := -({}^R\nabla_X W)^\top, \tag{5.2}$$

for any smooth vector field X on $\mathbb{S}^n$. At any point of $x \in \mathbb{S}^n$, the tensors A and B are related by

$$\langle A^W(X), Y\rangle = \langle B(X,Y), W\rangle = -\langle X, Y\rangle\langle x, W\rangle, \tag{5.3}$$

where X and Y are vector fields on $\mathbb{S}^n$ and W is a normal vector field on $\mathbb{S}^n$.

Theorem 5.1. *Any stable f-harmonic map $\phi : (M,F) \longrightarrow \mathbb{S}^n$ from a Finsler manifold (M,F) to the standard sphere $\mathbb{S}^n (n > 2)$ is constant.*

Proof. The above notations are used to prove this theorem. Choose an arbitrary point $z \in SM$. Then, set $\bar{\phi} = \phi \circ p$ and $\bar{x} = \bar{\phi}(z)$, where p is the canonical projection on SM. Let R^S denotes the curvature tensor of $\mathbb{S}^n$ and $\{\Lambda_1, \cdots, \Lambda_{n+1}\}$ a constant orthonormal basis in $\mathbb{R}^{n+1}$. By (4.1), it follows that

$$\sum_{\alpha=1}^{n+1} Q_f^\phi(\Lambda_\alpha^\top) = -\frac{1}{c_{m-1}}\sum_{\alpha=1}^{n+1}\int_{SM} h\Big(\nabla_{grad^H f}\Lambda_\alpha^\top - f\nabla_{K^H}\Lambda_\alpha^\top + fTr_g(\nabla^2\Lambda_\alpha^\top) + fTr_g R^S(\Lambda_\alpha^\top, d\phi)d\phi, \Lambda_\alpha^\top\Big) dV_{SM}. \tag{5.4}$$

Since Λ_α is parallel in $\mathbb{R}^{n+1}$ and considering (5.2), we obtain

$$\begin{aligned}\nabla_{grad^H f}\Lambda_\alpha^\top &= {}^S\nabla_{d\bar\phi(grad^H f)}\Lambda_\alpha^\top = ({}^R\nabla_{d\bar\phi(grad^H f)}\Lambda_\alpha^\top)^\top \\ &= ({}^R\nabla_{d\bar\phi(grad^H f)}(\Lambda_\alpha - \Lambda_\alpha^\perp))^\top = -({}^R\nabla_{d\bar\phi(grad^H f)}\Lambda_\alpha^\perp)^\top \\ &= A^{\Lambda_\alpha^\perp}(d\bar\phi(grad^H f)). \end{aligned} \tag{5.5}$$

Let $\lambda_\alpha : \mathbb{S}^n \longrightarrow \mathbb{R}$ defined by $\lambda_\alpha(x) := \langle \Lambda_\alpha, x\rangle$ for all $x \in \mathbb{S}^n$. One can easily check that

$$A^{\Lambda_\alpha^\perp}(X) = -\lambda_\alpha X, \tag{5.6}$$

for every vector field X on $\mathbb{S}^n$. By means of (5.3) and (5.5) at $\bar x$, it follows that

$$\begin{aligned}-\sum_\alpha \big\langle \nabla_{grad^H f}\Lambda_\alpha^\top, \Lambda_\alpha^\top\big\rangle &= \sum_\alpha \big\langle -A^{\Lambda_\alpha^\perp}(d\bar\phi(grad^H f)), \Lambda_\alpha^\top\big\rangle \\ &= \sum_\alpha \big\langle d\bar\phi(grad^H f), \Lambda_\alpha^\top\big\rangle\langle \bar x, \Lambda_\alpha^\perp\rangle \\ &= \sum_\alpha \big\langle d\bar\phi(grad^H f), \Lambda_\alpha^\top\big\rangle\langle \bar x, \Lambda_\alpha\rangle \\ &= \sum_\alpha \lambda_\alpha(\bar x)\,\big\langle d\bar\phi(grad^H f), \Lambda_\alpha^\top\big\rangle. \end{aligned} \tag{5.7}$$

Thus, the first term of RHS of (5.4) is obtained as follows

$$\sum_\alpha \big\langle \nabla_{grad^H f}\Lambda_\alpha^\top, \Lambda_\alpha^\top\big\rangle = -\sum_\alpha \lambda_\alpha \circ \bar\phi\,\big\langle d\bar\phi(grad^H f), \Lambda_\alpha^\top\big\rangle. \tag{5.8}$$

Similarly, the second term of RHS of (5.4) is given by

$$-\sum_\alpha f\big\langle \nabla_{K^H}\Lambda_\alpha^\top, \Lambda_\alpha^\top\big\rangle = \sum_\alpha f\lambda_\alpha \circ \bar\phi\,\big\langle d\bar\phi(K^H), \Lambda_\alpha^\top\big\rangle. \tag{5.9}$$

Due to the fact that $\nabla_{e_i^H}\Lambda_\alpha^\top = A^{\Lambda_\alpha^\perp}(d\bar\phi(e_i^H))$ from (5.5) and considering (5.6), it can be concluded that

$$\begin{aligned}\sum_i \nabla_{e_i^H}\nabla_{e_i^H}\Lambda_\alpha^\top &= \sum_i \nabla_{e_i^H} A^{\Lambda_\alpha^\perp}(d\bar\phi(e_i^H)) \\ &= -\sum_i \nabla_{e_i^H}(\lambda_\alpha \circ \bar\phi\;\, d\bar\phi(e_i^H)) \\ &= -d\bar\phi\,(grad\,\lambda_\alpha \circ \bar\phi) - \lambda_\alpha \circ \bar\phi\quad \sum_i \nabla_{e_i^H} d\phi(e_i). \end{aligned} \tag{5.10}$$

Since $grad\,\lambda_\alpha = \Lambda_\alpha^\top$ and using definition of gradient operator, it can be seen that

$$\begin{aligned} d\bar\phi(grad\,\lambda_\alpha \circ \bar\phi) &= \sum_i \big\langle d\bar\phi(e_i^H), (grad\,\lambda_\alpha)\circ\bar\phi\big\rangle\, d\bar\phi(e_i^H) \\ &= \sum_i \big\langle d\bar\phi(e_i^H), \Lambda_\alpha^\top \circ \bar\phi\big\rangle\, d\bar\phi(e_i^H). \end{aligned} \tag{5.11}$$

By means of (5.10) and (5.11), the third term of RHS of (5.4) has the following expression

$$\sum_\alpha f\big\langle Tr_g(\nabla^2\Lambda_\alpha^\top), \Lambda_\alpha^\top\big\rangle = -\sum_\alpha \lambda_\alpha \circ \bar\phi\;\, \langle f Tr_g \nabla d\phi, \Lambda_\alpha^\top\rangle - f \mid d\phi \mid^2. \tag{5.12}$$

Finally, since the sphere $\mathbb{S}^n$ has constant curvature, it can be shown that

$$\sum_{\alpha} f\langle Tr_g\ R^S(\Lambda_\alpha^\top, d\phi)d\phi, \Lambda_\alpha^\top\rangle = (n-1)f \mid d\phi \mid^2 . \tag{5.13}$$

Replacing (5.8), (5.9), (5.12) and (5.13) in (5.4) and using the f-harmonicity condition of ϕ, it follows

$$\begin{aligned}\sum_{\alpha} Q_f^\phi(\Lambda_\alpha^\top) &= \frac{2-n}{c_{m-1}} \int_{SM} f \mid d\phi \mid^2 dV_{SM} \\ &+ \frac{1}{c_{m-1}} \sum_{\alpha} \int_{SM} \lambda_\alpha \circ \bar{\phi}\ \langle \tau_f(\phi), \Lambda_\alpha^\top\rangle dV_{SM} \\ &= \frac{2-n}{c_{m-1}} \int_{SM} f \mid d\phi \mid^2 dV_{SM} \leq 0,\end{aligned} \tag{5.14}$$

by means of (5.14) and the stable f-harmonicity condition of ϕ, it can be concluded that ϕ is constant. This completes the proof. □

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