IMPLICIT CONTRACTIVE MAPPINGS IN SPHERICALLY COMPLETE ULTRAMETRIC SPACES

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ABSTRACT. In this paper, we apply implicit functions to establish a general fixed point theorem in spherically complete ultrametric spaces which enable us to extend some known results. In particular, we will show that in a spherically complete space $X$ a self-mapping $T$ satisfies

$$d(Tx, Ty) < \max\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx), d(x, y)\}$$

for each $x, y \in X$ with $x \neq y$, then $T$ has a unique fixed point. This improves Gajic’s fixed point theorem in spherically complete ultrametric spaces.

1. Introduction

A metric space $(X, d)$ is said to be an ultrametric space, if it satisfies the strong triangle inequality

$$d(x, y) \leq \max\{d(x, z), d(z, y)\} \quad (x, y \in X).$$

Sometimes the associated metric is also called a non-Archimedean metric or supermetric.

An ultrametric space $(X, d)$ is said to be spherically complete if every shrinking collection of balls in $X$ has a nonempty intersection. Clearly, every spherically complete ultrametric space is complete with respect to the topology induced by its metric. But the converse is not true in general. For example, the completion $\mathbb{C}_p$ of the algebraic closure of the field of rational $p$-adic numbers is complete. However, it is not spherically complete cf. [18], pp. 134-145.

Let $X$ be a nonempty set and $T : X \to X$ be a function. A point $x \in X$ is said to be a fixed point of $T$ provided that $Tx = x$. The function $T$ is called contraction if there exists a constant $r < 1$ such that

$$d(Tx, Ty) \leq rd(x, y) \quad (x, y \in X). \tag{1.1}$$

$T$ is said to be non-expansive if (1.1) holds for $r = 1$. $T$ is called contractive if we replace the inequality [1.1] with strict inequality and $r = 1$. Clearly, every contraction mapping is contractive and every contractive mapping is non-expansive. However, the converse is not true [1].

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In 1922, S. Banach [5] established an important fixed point theorem known as "Banach Contraction Principle" states that every contraction mapping of a complete metric space into itself has a unique fixed point.

The Banach contraction principle has been extended by some mathematicians (see e.g. [1, 7, 9, 10, 11, 15, 17]). In particular, Ćirić proved the following.

Theorem 1.1. [7] Let $(X, d)$ be a complete metric space and $T : X \to X$ be a mapping such that for some $0 \leq r < 1$,

$$d(Tx, Ty) \leq r \max\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx), d(x, y)\} \quad (x, y \in X).$$

Then $T$ has a unique fixed point.

In 1973, Hardy and Rogers gave another remarkable extension of Banach’s fixed point theorem as follows.

Theorem 1.2. [9] Let $X$ be a complete metric space with metric $d$, and let $T : X \to X$ be a function with the following property:

$$d(Tx, Ty) \leq a d(x, Tx) + b d(y, Ty) + c d(x, Ty) + e d(y, Tx) + f d(x, y),$$

where $0 \leq a, b, c, e, f < 1$ and $a + b + c + e + f < 1$. Then $T$ has a unique fixed point.

It is known that a contractive mapping $T : \mathbb{R} \to \mathbb{R}$ need not have a fixed point. For example, let $T : \mathbb{R} \to \mathbb{R}$ be defined by

$$Tx = \ln(1 + e^x) \quad (x \in \mathbb{R}).$$

Then $T'x = \frac{e^x}{1 + e^x} < 1$ for all $x \in \mathbb{R}$. Therefore $T$ is a contractive mapping. It is easy to verify that the equation $Tx = x$ is equivalent to $1 + e^x = e^x$, which is absurd. Therefore $T$ has no fixed point [16]. However, in spherically complete non-Archimedean spaces, we have the following result.

Theorem 1.3. [12] Let $X$ be a non-Archimedean spherically complete normed space. If $T : X \to X$ is a contractive mapping, then $T$ has a unique fixed point.

In 2001, Gajic obtained the following extension of Theorem 1.3.

Theorem 1.4. [8] Let $(X, d)$ be a spherically complete non-Archimedean metric space. If $T : X \to X$ is such that for any $x, y \in X, x \neq y$,

$$d(Tx, Ty) < \max\{d(x, y), d(x, Tx), d(y, Ty)\},$$

then $T$ has a unique fixed point.

In 1997, V. Popa [13] initiated a study of implicit contractive type conditions to give a simple proofs for some classical fixed point theorems. Since then, some authors studied various types of implicit contractions see e.g. [2, 3, 6, 14]. In this paper, we will define a new class of implicit functions to prove a general fixed point theorem in spherically complete ultrametric spaces. This result enable us to generalize Hardy-Rogers and Gajic fixed point theorems in spherically complete ultrametric spaces.
2. Main results

In this section, we define a class of implicit functions to establish a general fixed point theorem on spherically complete ultrametric spaces.

Let $G$ denote the set of all functions $g : [0, \infty)^5 \to [0, \infty)$ with the following properties.

1. $g(1,1,1,1,1) = h \leq 1$,
2. $x_i, y_i \in [0, \infty)$ and $x_i \leq y_i$ for $1 \leq i \leq 5$, implies that
   \[ g(x_1, x_2, x_3, x_4, x_5) \leq g(y_1, y_2, y_3, y_4, y_5), \]
3. $g(\alpha x_1, \alpha x_2, \alpha x_3, \alpha x_4, 0) \leq \alpha g(x_1, x_2, x_3, x_4, x_5)$ whenever $\alpha \geq 0$ and $x_i \in [0, \infty)$ for $1 \leq i \leq 5$.

Here we provide some examples of this class of implicit functions.

Example 2.1. Let $g : [0, \infty)^5 \to [0, \infty)$ be defined by
\[ g(x_1, x_2, x_3, x_4, x_5) = \max\{x_1, x_2, x_3, x_4, x_5\} \]
for all $(x_1, x_2, x_3, x_4, x_5) \in [0, \infty)^5$. Then $g$ satisfies the conditions (1), (2) and (3). Therefore $g \in G$.

Example 2.2. Define $g : [0, \infty)^5 \to [0, \infty)$ by
\[ g(x_1, x_2, x_3, x_4, x_5) = ax_1 + bx_2 + cx_3 + dx_4 + ex_5, \quad (x_1, x_2, x_3, x_4, x_5) \in [0, \infty)^5, \]
where $a, b, c, e, f \in [0, 1]$ and $a + b + c + e + f \leq 1$. One can easily check that $g$ satisfies the conditions (1), (2) and (3). Thus $g \in G$.

In order to state the main result of this section, we need to the following auxiliary result.

Lemma 2.3. If $g \in G$ and $u, v \in [0, \infty)$ are such that
\[ u < \max\{g(v, u, \max(u, v), 0, v), \ g(v, u, v, v, \max(u, v))\} \]
then $u < v$.

Proof. Suppose that $u \geq v$, then
\[ u < g(u, u, u, u, u) \leq ug(1,1,1,1,1) = uh \leq u, \]
which is a contradiction. Thus $u < v$. \qed

Now, we are ready to state the main result of this section.

Theorem 2.4. Let $(X, d)$ be a spherically complete ultrametric space. Let $g \in G$ and $T : X \to X$ satisfy
\[ d(Tx, Ty) < g\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx), d(x, y)\} \quad (2.1) \]
for each $x, y \in X$ with $x \neq y$. Then $T$ has a unique fixed point.

Proof. For each $x \in X$, let $B_x$ be the closed ball centered at $x$ with the radius $d(x, Tx)$. Let $A = \{B_x : x \in X\}$. Define a partial order $\leq$ on $A$ as follows.
\[ B_x \leq B_y \text{ if and only if } B_x \subseteq B_y. \]
Let $A_1$ be a totally ordered subfamily of $A$. Since $(X,d)$ is spherically complete, $B = \bigcap_{B_a \in A_1} B_a \neq \emptyset$. Let $b$ be an arbitrary element of $B$ and $B_a \in A_1$. Then $b \in B_a$. Therefore $d(b,a) \leq d(a,Ta)$. By the strong triangle inequality, we have

$$d(b,Tb) \leq \max\{d(b,a),d(a,Ta),d(Ta,Tb)\} \leq \max\{d(a,Ta),d(Ta,Tb)\}.$$ 

Therefore

$$d(b,Tb) \leq d(a,Ta) \text{ or } d(b,Tb) \leq d(Ta,Tb).$$

We claim that $d(b,Tb) \leq d(a,Ta)$. Suppose that $d(b,Tb) > d(a,Ta)$. By the above inequality, $d(b,Tb) \leq d(Ta,Tb)$. Therefore, we have

$$d(b,Tb) \leq d(Ta,Tb) \leq g(d(a,Ta),d(b,Tb),d(a,Ta),d(b,Ta),d(a,b)) \leq g(d(b,Tb),d(b,Tb),d(b,Tb),d(b,Tb),d(b,Tb)) \leq g(1,1,1,1,1) \leq d(b,Tb).$$

This contradiction shows that $d(b,Tb) \leq d(a,Ta)$. We claim that $B_b \subseteq B_a$. In fact, for every $x \in B_b$, we have

$$d(x,b) \leq d(b,Tb) \leq d(a,Ta).$$

It follows that

$$d(x,a) \leq \max\{d(x,b),d(b,a)\} \leq d(a,Ta).$$

Therefore $x \in B_a$. This proves our claim. Since $B_a$ was an arbitrary element of $A_1$, $B_0$ is a lower bound for $A_1$ in $A$. By Zorn’s lemma, $A$ has a minimal element. Let $B_z$ be a minimal element of $A$ and $w = Tz$. We will show that $w$ is a unique fixed point of $T$. If $Tw \neq w$, we have

$$d(w,Tw) = d(Tz,Tw) < g(d(z,Tz),d(w,Tw),d(z,Tw),d(w,Tw),d(z,w)) \leq g(1,1,1,1,1) \leq d(w,Tw).$$

By Lemma 2.3, $d(w,Tw) < d(z,Tz)$. Hence $Tz \notin B_w$. This means that $d(w,Tw) < d(w,Tz)$ since $Tz = w$, we have $d(w,Tw) < 0$. This contradiction shows that $Tw = w$ is a fixed point of $T$.

Suppose that $w_1$ and $w_2$ are distinct fixed points of $T$. Then

$$d(w_1,w_2) = d(Tw_1,Tw_2) \leq g(0,0,\max\{d(w_1,w_2),d(w_2,Tw_2)\},\max\{d(w_1,w_2),d(w_1,Tw_1)\},d(w_1,w_2)) \leq g(1,1,1,1,1) \leq d(w_1,w_2).$$

It follows that $d(w_1,w_2) < d(w_1,w_2)$, which is a contradiction. This proves the uniqueness of the fixed point of $T$. 

Theorem 2.4 enable us to obtain the following generalization of Theorem 1.2 provided that the metric space is spherically complete non-Archimedean.
Corollary 2.5. Let \((X, d)\) be a spherically complete ultrametric space and \(T\) be a self-mapping on \(X\). Let for each \(x, y \in X\) with \(x \neq y\),
\[
d(Tx, Ty) < a d(x, Tx) + b d(y, Ty) + c d(x, Ty) + e d(y, Tx) + f d(x, y).
\]
Then \(T\) has a unique fixed point provided that \(0 \leq a, b, c, e, f \leq 1\) and \(a + b + c + e + f \leq 1\).

Proof. Apply Theorem 2.4 for the function \(g\) in Example 2.2 \(\Box\)

The following result, which is an extension of Theorem 1.4, improves Ćirić’s fixed point theorem in spherically complete ultrametric spaces.

Corollary 2.6. Let \((X, d)\) be a spherically complete ultrametric space and \(T : X \to X\) be a mapping. Suppose that for all \(x, y \in X\) with \(x \neq y\),
\[
d(Tx, Ty) < \max\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx), d(x, y)\}.
\]
Then \(T\) has a unique fixed point.

Proof. The result follows from Example 2.1 and Theorem 2.4 \(\Box\)

The following result follows immediately from Corollary 2.6 and the ultrametric inequality.

Corollary 2.7. Let \((X, d)\) be a spherically complete ultrametric space. Then a self-mapping \(T : X \to X\) has a unique fixed point provided that one of the following conditions satisfies.

\begin{align*}
\text{(a)} & \quad d(Tx, Ty) \neq \max\{d(Tx, x), d(x, y), d(Ty, y)\} \text{ for each } x, y \in X \text{ with } x \neq y. \\
\text{(b)} & \quad d(Tx, Ty) \neq \max\{d(Tx, y), d(x, y), d(Ty, x)\} \text{ for each } x, y \in X \text{ with } x \neq y. \\
\text{(c)} & \quad d(Tx, Ty) \neq \max\{d(Tx, y), d(Ty, y)\} \text{ for each } x, y \in X \text{ with } x \neq y. \\
\text{(d)} & \quad d(Tx, Ty) \neq \max\{d(Tx, x), d(Ty, x)\} \text{ for each } x, y \in X \text{ with } x \neq y.
\end{align*}

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