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# A NEW SUMMABILITY FACTOR THEOREM FOR TRIGONOMETRIC FOURIER SERIES

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Abstract. In this paper, a known theorem dealing with $\mid \bar{N}, p_n \mid_k$ summability factors of trigonometric Fourier series has been generalized to $\mid \bar{N}, p_n, \theta_n \mid_k$ summability. Some new results have also been obtained.

## 1. Introduction

Let $\sum a_n$ be a given infinite series with partial sums $(s_n)$. We denote by $u_n^\alpha$ the $n$th Cesàro mean of order $\alpha$, with $\alpha > -1$, of the sequence $(s_n)$, that is ( see [5]),

$$u_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} s_v \tag{1.1}$$

where

$$A_n^\alpha = \frac{(\alpha+1)(\alpha+2)....(\alpha+n)}{n!} = O(n^\alpha), \quad A_{-n}^\alpha = 0 \quad \texttt{for} \quad n > 0. \tag{1.2}$$

A series $\sum a_n$ is said to be summable $\mid C, \alpha \mid_k$, $k \geq 1$, if (see [6])

$$\sum_{n=1}^{\infty} n^{k-1} \mid u_n^\alpha - u_{n-1}^\alpha \mid^k < \infty. \tag{1.3}$$

If we take $\alpha$=1, then we obtain $\mid C, 1 \mid_k$ summability. Let $(p_n)$ be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, i \geq 1). \tag{1.4}$$

The sequence-to-sequence transformation

$$t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{1.5}$$

defines the sequence $(t_n)$ of the Riesz mean or simply the $(\bar{N}, p_n)$ mean of the sequence $(s_n)$ generated by the sequence of coefficients $(p_n)$ (see [7]). Let $(\theta_n)$

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be any sequence of positive constants. The series $\sum a_n$ is said to be summable $|\bar{N}, p_n; \theta_n|_k$, $k \geq 1$, if (see [8])

$$\sum_{n=1}^{\infty} \theta_n^{k-1} \mid t_n - t_{n-1} \mid^k < \infty. \tag{1.6}$$

If we take $\theta_n = \frac{P_n}{p_n}$, then $\mid \bar{N}, p_n, \theta_n \mid_k$ summability reduces to $\mid \bar{N}, p_n \mid_k$ summability (see [1]). Also, if we take $\theta_n = n$ and $p_n = 1$ for all values of $n$, then we get $\mid C, 1 \mid_k$ summability. Furthermore, if we take $\theta_n = n$, then $\mid \bar{N}, p_n, \theta_n \mid_k$ summability reduces to $\mid R, p_n \mid_k$ summability (see [2]). Finally, if we take $k = 1$ (resp. $p_n = 1/n+1$), then $\mid \bar{N}, p_n, \theta_n \mid_k$ summability is the same as $\mid \bar{N}, p_n \mid$ (resp. $\mid R, logn, 1 \mid$ ) summability. For any sequence $(\lambda_n)$ we write that $\Delta^2 \lambda_n = \Delta\lambda_n - \Delta\lambda_{n+1}$ and $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$. The sequence $(\lambda_n)$ is said to be convex if $\Delta^2\lambda_n \geq 0$ for every positive integer $n$ (see [9]).

Let $f(x)$ be a periodic function with period $2\pi$ and Lebesgue integrable over $(-\pi, \pi)$. Without loss of generality we may assume that the constant term in the Fourier series of $f(x)$ is zero, so that

$$\int_{-\pi}^{\pi} f(x)dx = 0 \tag{1.7}$$

and

$$f(x) \sim \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \sum_{n=1}^{\infty} A_n(x). \tag{1.8}$$

## 2. Known results

The following two theorems concerning the $\mid \bar{N}, p_n \mid_k$ summability factors of trigonometric Fourier series are known.

**Theorem 2.1 ([3])**. If $(\lambda_n)$ is a convex sequence such that $\sum p_n\lambda_n < \infty$, where $(p_n)$ is a sequence of positive numbers such that $P_n \to \infty$ *as* $n \to \infty$, and $\sum_{v=1}^{n} P_v A_v(x) = O(P_n)$ *as* $n \to \infty$, then the series $\sum A_n(x) P_n \lambda_n$ is summable $\mid \bar{N}, p_n \mid_k, k \geq 1$.

**Theorem 2.2 ([4])**. If $(\lambda_n)$ is a non-negative and non-increasing sequence such that $\sum p_n\lambda_n < \infty$, where $(p_n)$ is a sequence of positive numbers such that $P_n \to \infty$ *as* $n \to \infty$, and $\sum_{v=1}^{n} P_v A_v(x) = O(P_n)$ *as* $n \to \infty$, then the series $\sum A_n(x) P_n \lambda_n$ is summable $\mid \bar{N}, p_n \mid_k$, $k \geq 1$.

It should be noted that the conditions on the sequence $(\lambda_n)$ in Theorem 2.2, are more general than in Theorem 2.1.

## 3. Main result

The aim of this paper is to generalize Theorem 2.2 in the following form.

**Theorem 3.1**. Let $(\frac{\theta_n p_n}{P_n})$ be a non-increasing sequence. If $(\lambda_n)$ is a non-negative and non-increasing sequence such that $\sum p_n\lambda_n < \infty$, where $(p_n)$ is a sequence of positive numbers such that $P_n \to \infty$ *as* $n \to \infty$, and $\sum_{v=1}^{n} P_v A_v(x) = O(P_n)$ *as* $n \to \infty$, then the series $\sum A_n(x) P_n \lambda_n$ is summable $\mid \bar{N}, p_n, \theta_n \mid_k, k \geq 1$.

In the proof of Theorem 3.1 , we will use the following lemma from [4].

**Lemma 3.2**. If $(\lambda_n)$ is a non-negative and non-increasing sequence such that $\sum p_n\lambda_n$ is convergent, where $(p_n)$ is a sequence of positive numbers such that

$P_n \to \infty$ *as* n$\to \infty$, then $P_n \lambda_n = O(1)$ as $n \to \infty$ and $\sum P_n \Delta \lambda_n < \infty$.
**Remark**. It should be noted that, since

$$\sum_{v=1}^{n-1} P_v P_v \Delta \lambda_v \leq P_{n-1} \sum_{v=1}^{n-1} P_v \Delta \lambda_v$$

it follows by Lemma 3.2 that

$$\frac{1}{P_{n-1}} \sum_{v=1}^{n-1} P_v P_v \Delta \lambda_v \leq \sum_{v=1}^{n-1} P_v \Delta \lambda_v = O(1) \quad as \quad m \to \infty. \tag{3.1}$$

## 4. Proof of Theorem 3.1

Let $T_n(x)$ denote the $(\bar{N}, p_n)$ mean of the series $\sum A_n(x) P_n \lambda_n$. Then, by definition, we have

$$T_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v \sum_{r=0}^{v} A_r(x) P_r \lambda_r = \frac{1}{P_n} \sum_{v=0}^{n} (P_n - P_{v-1}) A_v(x) \lambda_v P_v.$$

Then, for $n \geq 1$, we have

$$T_n(x) - T_{n-1}(x) = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} P_v A_v(x) \lambda_v.$$

By Abel's transformation, we have

$$\begin{aligned} T_n(x) - T_{n-1}(x) &= \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \Delta(P_{v-1}\lambda_v) \sum_{r=1}^{v} P_r A_r(x) + \frac{p_n}{P_n} \lambda_n \sum_{v=1}^{n} P_v A_v(x) \\ &= O(1)\{\frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} (P_v \lambda_v - p_v \lambda_v - P_v \lambda_{v+1}) P_v\} + O(1) p_n \lambda_n \\ &= O(1)\{\frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v P_v \Delta \lambda_v - \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v p_v \lambda_v + p_n \lambda_n\} \\ &= O(1)\{T_{n,1}(x) + T_{n,2}(x) + T_{n,3}(x)\}. \end{aligned}$$

To complete the proof of Theorem 3.1, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} \theta_n^{k-1} \mid T_{n,r}(x) \mid^k < \infty, \quad for \quad r = 1, 2, 3. \tag{4.1}$$

Now, when $k>1$, applying Hölder's inequality with indices $k$ and $k'$, where $\frac{1}{k}+\frac{1}{k'}=1$, we get that

$$\begin{aligned}
\sum_{n=1}^{m}\theta_n^{k-1}\mid T_{n,1}(x)\mid^k &\leq \sum_{n=2}^{m+1}\theta_n^{k-1}\frac{{p_n}^k}{{P_n}^k P_{n-1}}\left(\sum_{v=1}^{n-1}P_vP_v\Delta\lambda_v\right)\times\left(\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}P_vP_v\Delta\lambda_v\right)^{k-1}\\
&= O(1)\sum_{n=2}^{m+1}\theta_n^{k-1}\frac{{p_n}^k}{{P_n}^k P_{n-1}}\sum_{v=1}^{n-1}P_vP_v\Delta\lambda_v\\
&= O(1)\sum_{v=1}^{m}P_vP_v\Delta\lambda_v\sum_{n=v+1}^{m+1}\left(\frac{\theta_n p_n}{P_n}\right)^{k-1}\frac{p_n}{P_nP_{n-1}}\\
&= O(1)\sum_{v=1}^{m}P_vP_v\Delta\lambda_v\left(\frac{\theta_v p_v}{P_v}\right)^{k-1}\sum_{n=v+1}^{m+1}\frac{p_n}{P_nP_{n-1}}\\
&= O(1)\left(\frac{\theta_1 p_1}{P_1}\right)^{k-1}\sum_{v=1}^{m}P_v\Delta\lambda_v = O(1)\sum_{v=1}^{m}P_v\Delta\lambda_v = O(1)\quad as\quad m\to\infty,
\end{aligned}$$

by Lemma 3. 2. Again we have that

$$\begin{aligned}
\sum_{n=2}^{m+1}\theta_n^{k-1}\mid T_{n,2}(x)\mid^k &\leq \sum_{n=2}^{m+1}\theta_n^{k-1}\frac{{p_n}^k}{{P_n}^k P_{n-1}}\left(\sum_{v=1}^{n-1}(P_v\lambda_v)^k p_v\right)\times\left(\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v\right)^{k-1}\\
&= O(1)\sum_{v=2}^{m+1}\theta_n^{k-1}\frac{{p_n}^k}{{P_n}^k P_{n-1}}\sum_{v=1}^{n-1}(P_v\lambda_v)^k p_v\\
&= O(1)\sum_{v=1}^{m}(P_v\lambda_v)^k p_v\sum_{n=v+1}^{m+1}\left(\frac{\theta_n p_n}{P_n}\right)^{k-1}\frac{p_n}{P_nP_{n-1}}\\
&= O(1)\sum_{v=1}^{m}\left(\frac{\theta_v p_v}{P_v}\right)^{k-1}(P_v\lambda_v)^k p_v\sum_{n=v+1}^{m+1}\frac{p_n}{P_nP_{n-1}}\\
&= O(1)\left(\frac{\theta_1 p_1}{P_1}\right)^{k-1}\sum_{v=1}^{m}(P_v\lambda_v)^k\frac{p_v}{P_v}\\
&= O(1)\sum_{v=1}^{m}(P_v\lambda_v)^{k-1}p_v\lambda_v = O(1)\sum_{v=1}^{m}p_v\lambda_v = O(1)\quad as\quad m\to\infty,
\end{aligned}$$

by virtue of the hypotheses of Theorem 3.1 and the Lemma 3.2. Finally, as in $T_{n,1}(x)$, we have that

$$\begin{aligned}
\sum_{n=1}^{m}\theta_n^{k-1}\mid T_{n,3}(x)\mid^k &= \sum_{n=1}^{m}\left(\frac{\theta_n p_n}{P_n}\right)^{k-1}\left(\frac{P_n}{p_n}\right)^{k-1}(p_n\lambda_n)^{k-1}p_n\lambda_n\\
&= \left(\frac{\theta_1 p_1}{P_1}\right)^{k-1}\sum_{n=1}^{m}(P_n\lambda_n)^{k-1}p_n\lambda_n = O(1)\sum_{n=1}^{m}p_n\lambda_n = O(1)\quad as\quad m\to\infty.
\end{aligned}$$

This completes the proof of Theorem 3.1.

## 5. Conclusions

**1.** If we take $\theta_n = \frac{P_n}{p_n}$, then Theorem 3.1 reduces to Theorem 2.2. In this case the condition "$(\frac{\theta_n p_n}{P_n})$ is a non-increasing sequence" is trivial. Similarly by assigning specific values to parameters in Theorem 3.1 we obtain several interesting results about trigonometric Fourier series. For example;

**2.** If in Theorem 3.1 we put $\theta_n = n$ and $p_n = 1$, then we get a new result about $\mid C, 1 \mid_k$ summability factors of trigonometric Fourier series.

**3.** If in Theorem 3.1 we take $k = 1$ and $p_n = 1/(n+1)$, then we get another new result related to $\mid R, logn, 1 \mid$ summability factors of trigonometric Fourier series.

**4.** If in Theorem 3.1 we set $\theta_n = n$ , then we get a new result about $\mid R, p_n \mid_k$ summability factors of trigonometric Fourier series.

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