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EXISTENCE OF POSITIVE SOLUTIONS TO A COUPLED SYSTEM OF NONLINEAR FRACTIONAL ORDER DIFFERENTIAL EQUATIONS WITH M-POINT BOUNDARY CONDITIONS

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Abstract. This article is concerned to the study of existence and uniqueness of positive solutions to a class of coupled system with multi-point boundary conditions of nonlinear fractional order differential equations. By using classical results of nonlinear Leray Schauder type, sufficient conditions are developed for existence of at least one solutions. Further, sufficient conditions for uniqueness of solution is also discussed. To demonstrate the concerned established theory, we provide some appropriate examples.

1. Introduction

The theory of fractional order differential equations is the fastest growing area of research due to its large number of applications in real world problems. Differential equations of fractional order have great importance in many scientific and engineering disciplines such as physics, mechanics, chemistry, biology, viscoelasticity, control theory, signal and image processing phenomenons, economics, optimization theory etc. For detail we refer the reader to study [1, 2, 3, 4, 5] and the references there in. In recent years, many researchers have studied the existence and uniqueness of positive solutions of boundary value problems for fractional order differential equations, for example see [6, 7, 8, 9, 10]. Shah et al. [11], studied the existence and uniqueness of positive solutions for the nonlinear m-point boundary value problem of fractional order differential equation of the following form

$$\begin{cases} -{}^cD^q u(t) = f(t, u(t), {}^cD^{q-1}u(t)); \quad 0 < t < 1, 1 < q \leq 2, \\ u(0) = 0, \quad u(1) = \displaystyle\sum_{i=1}^{m-2} \delta_i u(\eta_i), \end{cases}$$

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where ${}^cD^q$ is the Caputo fractional derivative; $\delta_i, \eta_i \in (0,1)$ with $\sum_{i=1}^{m-2} \delta_i\eta_i < 1$, and $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ explicitly depends on the fractional order derivative. It has been founded that differential equations of fractional order are the best tools to explain many biological and chemical, physical and psychological phenomenon. Differential equations of arbitrary order are used as a best tools for the description of hereditary characteristic of various materials and genetical problems in biological models. It is known fact that those mathematical models involve fractional order differential equations are more realistic and practical as compare to those mathematical models which involve classical differential equations. Due to these importance applications of fractional differential equations researchers are taking interest in the study of frictional order differential equations and their positive solutions as they are meaningful. For more detail see [12, 13, 14, 15, 16] and the reference there in. Recently the area devoted to the study of existence of positive solutions for coupled systems of fractional order boundary/initial value problems has gained considerable attention, because such types of systems often occur in applications in various fields of science and technology. Due to the aforesaid facts, in last few decades this area has gained much attentions from the researchers, we refer [17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27] and the references therein. In [18], Wang et.al, developed sufficient conditions for the existence of positive solutions to the following coupled system of three point boundary value problem

$$\begin{cases} D^{\alpha}_{0+}u(t) = f(t, v(t), D^{\beta}v(t)), \quad D^{\beta}_{0+}v(t) = g(t, u(t), D^{\alpha}u(t)), 0 \le t \le 1, \\ u(0) = v(0) = 0, u(1) = au(\eta), v(1) = bv(\eta), \end{cases}$$

where $1 < \alpha, \beta \le 2$, $0 \le a, b, \le 1$ and $0 < \eta < 1$ and the non-linear functions $f, g : [0,1] \times [0,\infty) \to [0,\infty)$ are assumed to be continuous.

Motivated by the aforementioned applications and importance of fractional order differential equations, we consider the following coupled system of fractional order differential equations with m-point boundary conditions

$$\begin{cases} D^{\alpha}_{0+}x(t) = f(t, x(t), y(t)), \quad D^{\beta}_{0+}y(t) = g(t, x(t), y(t)), \quad 0 \le t \le 1, \\ x(0) = 0, \quad x(1) = \sum_{j=1}^{n-2} \lambda_j x(\eta_j), \quad y(0) = 0, \quad y(1) = \sum_{j=1}^{n-2} \mu_j y(\xi_j), \end{cases} \tag{1.1}$$

where $D^{\alpha}_{0+}, D^{\beta}_{0+}$ are the standard Rieman-Liouvilli fractional order derivative of order α, β respectively, where $\lambda_j, \mu_j \in (0,\infty)$ and $0 < \eta_1 < \eta_2 < \cdots < \eta_{n-2} < 1$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{n-2} < 1$, such that $\sum_{j=1}^{n-2} \lambda_j \eta_j^{\alpha} < 1$, $\sum_{j=1}^{n-2} \mu_j \xi_j^{\beta} < 1$ and $1 < \alpha, \beta \le 2$. The nonlinear functions $f, g : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are assumed to be continuous. We prove our main results by mean of some classical fixed point theorems of nonlinear Laray-Schauder type and Banach contractions principle. We develop necessary and sufficient conditions for the existence of positive solutions and their uniqueness. Some examples are also provided for the illustration of our results.

2. Preliminaries

In this section we recall some basic definitions and lemmas from fractional calculus, fixed point theory and functional analysis [2, 3, 4, 5, 6, 7, 8, 9, 10].

Definition 2.1. The fractional integral of order $\gamma \in \mathbb{R}_+$ of a function $u \in L^1([0,1],\mathbb{R})$ is defined as

$$I^{\gamma}_{0+}u(t) = \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}u(s)ds.$$

Definition 2.2. The Riemann-Liouville fractional order derivative of a function u on the interval $[0,1]$ is defined by

$$D^{\gamma}_{0+}u(t) = \frac{1}{\Gamma(n-\gamma)}\left(\frac{d}{dt}\right)^n\int_0^t (t-s)^{n-\gamma-1}u(s)\,ds,$$

where $n = [\gamma]+1$ and $[\gamma]$ represents the integer part of γ.

Lemma 2.1. [5], *Assume that $u \in C(0,1)\cap L(0,1)$ with fractional derivative of order $\gamma > 0$ that belongs to $C(0,1)\cap L(0,1)$. Then*

$$I^{\gamma}D^{\gamma}u(t) = u(t) + d_1t^{\gamma-1} + d_2t^{\gamma-2} + d_3t^{\gamma-3} + \cdots + d_nt^{\gamma-n},$$

for arbitrary $d_k \in \mathbb{R}$, $\quad k =,1,2,\ldots,n$, where $n = [\gamma]+1$.

Lemma 2.2. *(The nonlinear alternative of Leray -Schauder type* [10]*). Let D be a closed convex subset of a Banach space X. Consider a relative open subset C of D such that $0 \in C$ and let $\mathbb{T} : C \to D$ be continuous and compact mapping. Then either*

(1) *the mapping $\mathbb{T}$ has a fixed point in C; or*
(2) *there exist $x \in \partial C$ and $\delta \in (0,1)$ with $x = \delta\mathbb{T}x$.*

Lemma 2.3. *Let $\Delta_1 = 1 - \sum\limits_{j=1}^{m-2}\lambda_j\eta_j$ and $\Delta_2 = 1 - \sum\limits_{j=1}^{m-2}\mu_j\xi_j$, for a given $x, y \in C(I,\mathbb{R})$ and $\lambda_j, \mu_j \in (0,\infty)$ with $\sum\limits_{j=1}^{m-2}\lambda_j\eta_j^{\alpha} < 1$ and $\sum\limits_{j=1}^{m-2}\mu_j\xi_j^{\beta} < 1$, then BVP (1.1) has a unique positive solution given by*

$$(x,y) = \left(\int\limits_0^1 K_1(t,s)f(s,x(s),y(s))ds, \int\limits_0^1 K_2(t,s)g(s,x(s),y(s))ds\right), \tag{2.1}$$

where $K_1(t,s), K_2(t,s)$ are the Green's functions and given by

$$K_1(t,s) = \begin{cases} \frac{t^{\alpha-1}}{\Delta_1\Gamma(\alpha)}[(1-s)^{\alpha-1} + \sum\limits_{i=j}^{m-2}\lambda_j(\eta_j-s)^{\alpha-1}] - \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}; & s \le t,\ \eta_{j-1} < s \le \eta_j, \\ & j = 1,2,...,m-1, \\ \frac{t^{\alpha-1}}{\Delta_1\Gamma(\alpha)}[(1-s)^{\alpha-1} + \sum\limits_{i=j}^{m-2}\lambda_j(\eta_j-s)^{\alpha-1}]; & t \le s,\ \eta_{j-1} < s \le \eta_j, \\ & j = 1,2,...,m-1. \end{cases} \tag{2.2}$$

$$K_2(t,s) = \begin{cases} \frac{t^{\beta-1}}{\Delta_2\Gamma(\beta)}[(1-s)^{\beta-1} + \sum\limits_{i=j}^{m-2}\mu_j(\xi_j-s)^{\beta-1}] - \frac{1}{\Gamma(\beta)}(t-s)^{\beta-1}; & s \le t,\ \xi_{j-1} < s \le \xi_j, \\ & j = 1,2,...,m-1, \\ \frac{t^{\beta-1}}{\Delta_2\Gamma(\beta)}[(1-s)^{\beta-1} + \sum\limits_{i=j}^{m-2}\mu_j(\xi_j-s)^{\beta-1}]; & t \le s,\ \xi_{j-1} < s \le \xi_j, \\ & j = 1,2,...,m-1. \end{cases} \tag{2.3}$$

Proof. To obtain (2.1), see the proof of Lemma 3.1 in [11]. Therefore, we omit the proof. □

Hence, equivalent system of integral equations of Coupled system (1.1) is given by

$$\begin{cases} x(t) = \dfrac{t^{\alpha-1}}{\Delta_1\Gamma(\alpha)}\left[-\displaystyle\int_0^1 (1-s)^{\alpha-1} f(s,x(s),y(s))\,ds + \sum_{i=j}^{m-2}\lambda_j \int_0^{\eta_j}(\eta_j - s)^{\alpha-1} f(s,x(s),y(s))\,ds\right] \\ \qquad + \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1} f(s,x(s),y(s))\,ds = \int_0^1 K_1(t,s) f(s,x(s),y(s))ds. \\ y(t) = \dfrac{t^{\beta-1}}{\Delta_2\Gamma(\beta)}\left[-\displaystyle\int_0^1 (1-s)^{\beta-1} g(s,x(s),y(s))\,ds + \sum_{i=j}^{m-2}\mu_j \int_0^{\xi_j}(\xi_j - s)^{\beta-1} g(s,x(s),y(s))\,ds\right] \\ \qquad + \dfrac{1}{\Gamma(\beta)}\displaystyle\int_0^t (t-s)^{\beta-1} g(s,x(s),y(s))\,ds = \int_0^1 K_2(t,s) g(s,x(s),y(s))ds, \end{cases}$$

where $K_1(t,s), K_2(t,s)$ can be easily obtain as in (2.2) and (2.3). We call $K(t,s) = (K_1(t,s), K_2(t,s))$ the Green's functions of the BVP(1.1) which satisfies the following lemma

Lemma 2.4. *If* $0 < \Delta_i < 1 (i = 1, 2)$ *then each* $K_i(t,s)(i = 1,2)$ *satisfies the following properties*
(i) $K_i(t,s) \geq 0 (i = 1,2)$ *is continuous* $\forall t, s \in [0,1]$,$K_i(t,s) > 0 (i = 1,2), \forall t, s \in (0,1)$*;*
(ii)$K_i(t,s) \leq K_i(s)(i = 1,2)$, *for each* $t, s \in [0,1]$,*and* $\min_{t\in[\omega,1-\omega]} K_i(t,s) \geq \gamma_i K_i(s)(i = 1,2)$, *where* $0 < \omega < 0.5$.

Proof. (i) Obviously $K_1(t,s)$ is continuous on $[0,1]\times[0,1]$ and by easy calculations one can show that $K_1(t,s) \geq 0, \forall t, s \in [0,1]$. Similarly for $K_2(t,s)$ one can do the same. Also it is obvious that $K_i(t,s)(i = 1,2) > 0,\ \forall t, s \in (0,1)$.
(ii) One can easily check that $K_i(t,s)(i = 1,2)$ is decreasing with respect to t for $s \leq t$ and increasing with respect to t for $s \geq t$. Thus by the monotonicity of $K_i(t,s)(i = 1,2)$ we obtain

$$\begin{aligned} &\max_{t\in[0,1]} K_1(t,s) = K_1(s,s) = \frac{t^{\alpha-1}}{\Delta_1\Gamma(\alpha)}[(1-s)^{\alpha-1} + \sum_{i=j}^{m-2}\lambda_j(\eta_j - s)^{\alpha-1}], \\ &\text{where } s \in (0,1),\ \eta_{j-1} < s \leq \eta_j,\ \text{for } j = 1,2,...,m-1. \\ &\text{Similarly} \max_{t\in[0,1]} K_2(t,s) = K_1(s,s) = \frac{t^{\beta-1}}{\Delta_2\Gamma(\beta)}[(1-s)^{\beta-1} + \sum_{i=j}^{m-2}\mu_j(\xi_j - s)^{\beta-1}], \\ &\text{where } s \in (0,1),\ \xi_{j-1} < s \leq \xi_j,\ \text{for } j = 1,2,...,m-1. \end{aligned} \tag{2.4}$$

Hence the proof is completed. □

Let us introduce the space $X = \{x(t)|x(t) \in C^1([0,1])\}$ whose norm is defined by $\|x\| = \sup\{|x(t)|, t \in [0,1]\}$. Then obviously $\bigg(X, \|x\|\bigg)$ is Banach space. Similarly $Y = \{y(t)|y(t) \in C^1([0,1])\}$ whose norm is defined by $\|y\| = \sup\{|y(t)|, t \in [0,1]\}$.

Then obviously $\Big(Y, \|y\|\Big)$ is a Banach space. The product space $X \times Y$ is also a Banach space under the norm defined by $\|(x,y)\| = \max\{\|x\|, \|y\|\}$.
Define a cone $C = \{(x,y) \in X \times Y : x(t) \geq 0,\ y(t) \geq 0\}$, then the cone $C \subset X \times Y$.

3. Main Work

Theorem 3.1. *Assume that $f(t,x,y),\ g(t,x,y)$ are continuous, then $(x,y) \in X \times Y$ is a solution of BVP (1.1) if and only if $(x,y) \in X \times Y$ is a solution of the following system of integral equations*

$$x(t) = \int_0^1 K_1(t,s) f(s,x(s),y(s))ds,\ \ y(t) = \int_0^1 K_2(t,s) g(s,x(s),y(s))ds \qquad (3.1)$$

Define the operator $T : X \times Y \to X \times Y$ by

$$\begin{aligned} T(x,y)(t) &= \left(\int_0^1 K_1(t,s) f(s,x(s),y(s))ds, \int_0^1 K_2(t,s) g(s,x(s),y(s))ds \right) \\ &= (T_1(x,y)(t), T_2(x,y)(t)). \end{aligned} \qquad (3.2)$$

Then by Theorem (3.1), the fixed points of the operator T are the solutions of the integral equations (3.1).

Theorem 3.2. *Assume that $f(t,x,y), g(t,x,y)$ are continuous on $[0,1] \times \mathbb{R}^2 \to \mathbb{R}$, then the operator defined in (3.2) $\ T : C \to C$ is completely continuous.*

Proof. We first prove that $T : C \to C$ defined in (3.2) is completely continuous. Due to continuity and nonnegativity of $K_i(t,s)$ for $(i = 1,2)$, f and g the operator T is continuous for all $(x,y) \in C$. Let $\Omega \subset C$ be bounded then there exist a positive constant K and L such that $|f(t,x,y)| \leq K$ and $|g(t,x,y)| \leq L$ for all $(x,y) \in \Omega$. Then for every $(x,y) \in \Omega$ and using $|t| \leq 1$ we have

$$\begin{aligned} |T_1(x,y)(t)| &= \Bigg| \frac{t^{\alpha-1}}{\Delta_1 \Gamma(\alpha)} \sum_{j=1}^{m-2} \lambda_j \int_0^{\eta_j} (\eta_j - s)^{\alpha-1} f(s,x(s),y(s))ds - \frac{t^{\alpha-1}}{\Delta_1 \Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s,x(s),y(s))ds \\ &\quad + \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,x(s),y(s))ds \Bigg| \\ &\leq \Bigg| \frac{t^{\alpha-1}}{\Delta_1 \Gamma(\alpha)} \sum_{j=1}^{m-2} \lambda_j \int_0^{\eta_j} (\eta_j - s)^{\alpha-1} f(s,x(s),y(s))ds - \frac{t^{\alpha-1}}{\Delta_1 \Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s,x(s),y(s))ds \Bigg| \\ &\quad + \Bigg| \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,x(s),y(s))ds \Bigg| \\ &\leq \frac{|t^{\alpha-1}|}{\Delta_1 \Gamma(\alpha)} \sum_{j=1}^{m-2} \lambda_j \int_0^{\eta_j} (\eta_j - s)^{\alpha-1} |f(s,x(s),y(s))|ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s,x(s),y(s))|ds \\ &\Rightarrow |T_1(x,y)(t)| \leq \frac{2K}{\Delta_1 \Gamma(\alpha+1)} \end{aligned}$$

similarly $\ |T_2(x,y)(t)| \leq \dfrac{2L}{\Delta_2 \Gamma(\beta+1)}$.

Thus from the above inequality it is follow that the operator T is uniformly bounded. Next we show that T is equi-continuous for this let $0 \le t \le \tau \le 1$ we have

$$\begin{aligned}
|T_1(x,y)(\tau) - T_1(x,y)(t)| &= \bigg| \frac{\tau^{\alpha-1}}{\Delta_1\Gamma(\alpha)} \sum_{j=1}^{m-2} \lambda_j \int_0^{\eta_j} (\eta_j - s)^{\alpha-1} f(s,x(s),y(s))ds \\
&+ \frac{\tau^{\alpha-1}}{\Gamma(\alpha)} \int_0^{\tau} (\tau - s)^{\alpha-1} f(s,x(s),y(s))ds \\
&- \frac{t^{\alpha-1}}{\Delta_1\Gamma(\alpha)} \sum_{j=1}^{m-2} \int_0^{\eta_j} (\eta_j - s)^{\alpha-1} f(s,x(s),y(s))ds \\
&- \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^{t} (t - s)^{\alpha-1} f(s,x(s),y(s))ds \bigg| \\
&\le \frac{K}{\Delta_1\Gamma(\alpha)} |\tau^{\alpha-1} - t^{\alpha-1}| \sum_{j=1}^{m-2} \lambda_j \bigg| \int_0^{\eta_j} (\eta_j - s)^{\alpha-1} \bigg| ds \\
&+ \frac{K}{\Gamma(\alpha)} \bigg(\int_0^{t} [(\tau - s)^{\alpha-1} - (t-s)^{\alpha-1}]ds - \int_0^{t} (t-s)^{\alpha-1} ds \bigg).
\end{aligned}$$

Thus after simplification, we get

$$|T_1(x,y)(\tau) - T_1(x,y)(t)| \le \frac{K}{\Delta_1\Gamma(\alpha+1)} \bigg([\tau^{\alpha-1} - t^{\alpha-1}] \sum_{j=1}^{m-2} \lambda_j \eta_j^{\alpha} + [\tau^{\alpha} - t^{\alpha}] \bigg)$$

implies that $$|T_1(x,y)(\tau) - T_1(x,y)(t)| \le \frac{K}{\Delta_1\Gamma(\alpha+1)} \bigg(\tau^{\alpha-1} - t^{\alpha-1} + \tau^{\alpha} - t^{\alpha} \bigg), \tag{3.3}$$

similarly we can obtain

$$|T_2(x,y)(\tau) - T_2(x,y)(t)| \le \frac{L}{\Delta_2\Gamma(\beta+1)} \bigg(\tau^{\beta-1} - t^{\beta-1} + \tau^{\beta} - t^{\beta} \bigg). \tag{3.4}$$

As the functions on the right hands of (3.3) and (3.4) are uniformly continuous on $[0,1]$. Therefore the operator $T(x,y)$ is equi-continuous and thus the $T(\Omega)$ is equi-continuous set and uniformly bounded as $T(\Omega) \subset \Omega$. Thus T is completely continuous. Hence proof is completed. $\square$

For existence of at least one positive solution, we prove the following lemma.

Theorem 3.3. *Assume that $f(t,x,y), g(t,x,y)$ are continuous on $[0,1] \times \mathbb{R}^2 \to \mathbb{R}$ and satisfy*

(A_1) $|f(t,x(t),y(t))| \le a_0(t) + a(t)[|x(t)| + |y(t)|]$,

(A_2) $|g(t,x(t),y(t))| \le b_0(t) + b(t)[|x(t)| + |y(t)|]$,

(A_3) $0 < \Upsilon_1 = \int_0^1 K_1(t,s)a_0(s)ds < \infty,\ \Upsilon_2 = \int_0^1 K_1(t,s)a(s)ds < 1$,

(A_4) $0 < \Upsilon_3 = \int_0^1 K_2(t,s)b_0(s)ds < \infty,\ \Upsilon_4 = \int_0^1 K_2(t,s)b(s)ds < 1$.

Then the system (1.1) *has at least one positive solution in*

$$\Omega = \left\{ (x, y) \in C : \|(x, y)\| < \min\left(\frac{\Upsilon_1}{1 - \Upsilon_2}, \frac{\Upsilon_3}{1 - \Upsilon_4} \right) \right\}. \tag{3.5}$$

Proof. Let $\Omega = \{(x, y) \in X \times Y | \|(x, y)\| < \epsilon\}$, where $\epsilon = \min\left\{ \frac{\Upsilon_1}{1-\Upsilon_2}, \frac{\Upsilon_3}{1-\Upsilon_4} \right\}$.
Let us define$T : \Omega \to C$ as in (3.2). Let $(x, y) \in \Omega$, so $\|(x, y)\| < \epsilon$. Then

$$\begin{aligned} \|T_1(x, y)\| &= \max_{t \in [0,1]} |\int_0^1 K_1(t, s) f(s, x(s), y(s)) ds| \\ &\le \int_0^1 K_1(t, s) a_0(s) ds + \int_0^1 K_1(t, s) a(s)[|x(s)| + |y(s)|] ds \\ &\le \int_0^1 K_1(t, s) a_0(s) ds + \int_0^1 K_1(t, s) a(s) \|(x, y)\| ds \\ &= \Upsilon_1 + \Upsilon_2 \|(x, y)\| \le \epsilon. \end{aligned} \tag{3.6}$$

Similarly, $\|T_2(x, y)\| \le \epsilon$, so $\|T(x, y)\| \le \epsilon$.

Hence $T(x, y) \subset \bar{\Omega}$. Therefore, inview of Theorem 3.2 $T : \bar{\Omega} \to \bar{\Omega}$ is completely continuous.
Now let us call the eigen value problem

$$(x, y) = \rho T(x, y), \ \rho \in (0, 1). \tag{3.7}$$

Then under the assumption that (x, y) is a solution of (3.7) for $\rho \in (0, 1)$, we obtain

$$\begin{aligned} \|x\| = \|\rho T_1(x, y)\| &= \rho \max_{t \in [0,1]} \left\| \int_0^1 K_1(t, s) f(s, x(s), y(s) \right\| \\ &< \int_0^1 K_1(s, s)[a_0(s) + a(s)(\|x\| + \|y\|) ds \\ &= \int_0^1 K_1(s, s) a_0(s) ds + \int_0^1 K_1(s, s) a(s) \|(x, y)\| ds \\ &= \Upsilon_1 + \Upsilon_2 \|(x, y)\| \le \epsilon. \end{aligned} \tag{3.8}$$

Similarly $\|y\| = \|\rho T_2(x, y)\| \le \epsilon$, so $\|(x, y)\| < \epsilon$,

which show that (x, y) is not in $\partial\Omega$. Hence by Lemma 2.2, T has a fixed point in $\bar{\Omega}$. So BVP(1.1) has at least one positive solution. Thus proof is completed. □

Theorem 3.4. *Assume that* $f, g : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *are continuous functions and there exist two positive functions* $\phi(t)$, $\psi(t)$ *such that for all* $t \in [0, 1]$ *and* $x_i, y_i \in \mathbb{R}, i = 1, 2$,

(A_5) $|f(t, x_1, y_1) - f(t, x_2, y_2)| \le \phi(t)[|x_1 - x_2| + |y_1 - y_2|]$,
(A_6) $|g(t, x_1, y_1) - g(t, x_2, y_2)| \le \psi(t)[|x_1 - x_2| + |y_1 - y_2|]$.

Then the system(1.1) *has a unique positive solutions if*

$$\lambda = \int_0^1 K_1(s, s)\phi(s) ds < 1, \ \mu = \int_0^1 K_2(s, s)\phi(s) ds < 1. \tag{3.9}$$

Proof. Since $K_i(t,s), i=1,2$ and $f(t,x,y),\ g(t,x,y)$, are nonnegative, so for any $(x,y)\in C$ we have $T(x,y)\geq 0$, also $T(C)\subset C$.

$$\begin{aligned}\|T_1(x_2,y_2)-T_1(x_1,y_1)\| &= \max_{t\in[0,1]}|T_1(x_2,y_2)-T_1(x_1,y_1)|\\ &= \max_{t\in[0,1]}\left|\int_0^1 K_1(t,s)[f(s,x_2(s),y_2(s))-f(s,x_1(s),y_1(s))]ds\right|\\ &\leq \int_0^1 K_1(s,s)\phi(s)ds[\|x_2-x_1\|+\|y_2-y_1\|]\\ &\leq \lambda[\|x_2-x_1\|+\|y_2-y_1\|].\end{aligned} \tag{3.10}$$

Similarly, we can get

$$\|T_2(x_2,y_2)-T_2(x_1,y_1)\|\leq \mu[\|x_2-x_1\|+\|y_2-y_1\|]. \tag{3.11}$$

Thus from (3.10),(3.11) we have

$$\|T(x_2,y_2)-T(x_1,y_1)\|\leq \max(\lambda,\mu)\|(x_2,y_2)-(x_1,y_1)\|. \tag{3.12}$$

Therefore under the condition(3.9) T is a contraction operator. Also T is completely continuous operator by Lemma 3.2, so by Banach's fixed-point theorem the operator T has a unique fixed point, which is the the unique positive solution of BVP(1.1). □

4. Example

Example 4.1. Consider the following problem

$$\begin{cases} D^{\frac{7}{4}}x(t)=1+\dfrac{t^2}{4}+\dfrac{\sin x(t)}{4}+\dfrac{t\ln(1+y(t))}{t^2+3},\ t\in(0,1),\\ D^{\frac{3}{2}}y(t)=10+e^{-2\pi t}+\dfrac{\cos 2x(t)}{16}+\dfrac{y(t)}{15+t}, t\in(0,1),\\ x(0)=0,\quad x(1)=\displaystyle\sum_{j=1}^{10}\frac{1}{3^j}x\left(\frac{1}{2^j}\right), y(0)=0,\quad y(1)=\sum_{j=1}^{10}\frac{1}{2^j}y\left(\frac{1}{3^j}\right).\end{cases} \tag{4.1}$$

We have

$$\begin{aligned}\Big|f(t,x(t),y(t)\Big| &= \left|1+\frac{t^2}{4}+\frac{\sin(x(t))}{4}+\frac{t\ln(1+y(t))}{t^2+3}\right|\leq\left(1+\frac{t^2}{4}\right)+\frac{|x(t)|}{4}+\frac{t}{t^2+3}|y(t)|,\\ \Big|g(t,x(t),y(t)\Big| &= \left|10+e^{-2\pi t}+\frac{\cos 2(x(t))}{16}+\frac{y(t)}{15+t}\right|\leq\left(10+e^{-2\pi t}\right)+\frac{|x(t)|}{16}+\frac{1}{15+t}|y(t)|.\end{aligned} \tag{4.2}$$

Hence

$$\begin{aligned}\Upsilon_1 &= \int_0^1 K_1(s,s)a_0(s)ds=\int_0^1 K_1(s,s)(1+\frac{s^2}{4})ds<\infty,\\ \Upsilon_2 &= \int_0^1 K_1(s,s)a(s)ds\leq\int_0^1 K_1(s,s)ds\approx 0.99984<1,\\ \Upsilon_3 &= \int_0^1 K_2(s,s)b_0(s)ds=\int_0^1 K_2(s,s)(10+e^{-2\pi s})ds<\infty,\\ \Upsilon_4 &= \int_0^1 K_2(s,s)b(s)ds\leq\int_0^1 K_2(s,s)ds\approx 0.76889<1.\end{aligned} \tag{4.3}$$

by Theorem (3.3), BVP (4.1) has at least one positive solution in

$$\Omega = \left\{ (x,y) \in C : \|(x,y)\| < \min\left(\frac{\Upsilon_2}{1-\Upsilon_1}, \frac{\Upsilon_4}{1-\Upsilon_3} \right) \right\}.$$

Example 4.2. Consider the following multi-point BVP

$$\begin{cases} D^{\frac{3}{2}} x(t) = \dfrac{1}{t^2+1} + \dfrac{\cos^2(x(t))}{4} + \dfrac{\sin^2(y(t))}{8}, t \in [0,1], \\ D^{\frac{3}{2}} y(t) = e^{-t^2} + \dfrac{\cos 2(x(t))}{16} + \dfrac{1}{15+y(t)}, t \in [0,1], \\ x(0) = 0, \quad x(1) = \displaystyle\sum_{j=1}^{100} \frac{1}{2^j} x\left(\frac{1}{4^j}\right), \ y(0) = 0, \quad y(1) = \sum_{j=1}^{100} \frac{1}{3^j} y\left(\frac{1}{2^j}\right). \end{cases} \tag{4.4}$$

Here

$$\alpha = \frac{3}{2}, \beta = \frac{3}{2}, \lambda_j = \frac{1}{2^j}, \mu_j = \frac{1}{3^j}, \eta_j = \frac{1}{4^j}, \xi_j = \frac{1}{2^j}, j = 1,2,3,\ldots,100$$

$$\begin{aligned} f(t,x,y) &= \frac{1}{t^2+1} + \frac{\cos^2(x(t))}{4} + \frac{\sin^2(y(t))}{8}, \\ g(t,x,y) &= e^{-t^2} + \frac{\cos(2x(t))}{16} + \frac{1}{15+y(t)}. \end{aligned}$$

Set $x_i(t), y_i(t), (i=1,2) \in [0,\infty)$ and $t \in [0,1]$, then we have

$$\begin{aligned} &|f(t,x_2,y_2) - f(t,x_1,y_1)| \le \frac{1}{2}[|x_2 - x_1| + |y_2 - y_1|], \\ &|g(t,x_2,y_2) - g(t,x_1,y_1)| \le \frac{1}{8}[|x_2 - x_1| + |y_2 - y_1|] \\ &\phi(s) = \frac{1}{2}, \ \psi(s) = \frac{1}{8}. \end{aligned} \tag{4.5}$$

$$\begin{aligned} \lambda &= \int_0^1 K_1(s,s)\phi(s)ds \le \int_0^1 K_1(s,s)ds = 0.99984 < 1, \\ \mu &= \int_0^1 K_2(s,s)\psi(s)ds \le \int_0^1 K_1(s,s)ds = 0.76889 < 1. \end{aligned} \tag{4.6}$$

Thus all the conditions of Theorem (3.4) are satisfied, so the given BVP(4.4) has a unique positive solution.

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