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A NEW CLASS OF HARMONIC MULTIVALENT MEROMORPHIC FUNCTIONS

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ABSTRACT. In the present paper, we introduce some new subclasses of harmonic multivalent meromorphic functions defined by generalized Liu-Srivastava operator. Sufficient coefficient conditions, distortion bounds and extreme points for functions of these classes are obtained.

1. INTRODUCTION AND PRELIMINARIES

Let f_1 and f_2 be two analytic functions in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. We say that the function f_1 is subordinate to f_2 in $\mathbb{U}$, and write $f_1(z) \prec f_2(z)$ $(z \in \mathbb{U})$, if there exists a Schwarz function ω, which is analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ $(z \in \mathbb{U})$, such that $f_1(z) = f_2(\omega(z))$ $(z \in \mathbb{U})$ [3].

A continuous function $f = u + iv$ is a complex valued harmonic function in a complex domain D if both u and v are real harmonic in D. In any simply connected domain $D \subset \mathbb{C}$, we can write $f = h + \overline{g}$, where h and g are analytic in D. we call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense preserving in D is that $|h'(z)| > |g'(z)|$ in D (see [2]).

Denote by $\Sigma_H(p)$ the class of p-valent harmonic functions f that are sense preserving in $\mathbb{U}^* = \{z \in \mathbb{C} : 0 < |z| < 1\} = \mathbb{U}\backslash\{0\}$ and f of the form

$$f = h + \overline{g}, \tag{1.1}$$

where

$$h(z) = z^{-p} + \sum_{k=p+1}^{\infty} a_k z^k \quad \text{and} \quad g(z) = \sum_{k=p+1}^{\infty} b_k z^k. \tag{1.2}$$

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Also, we denote by $\Sigma_{\overline{H}}(p)$ the class of p-valent harmonic functions $f \in \Sigma_H(p)$ and

$$h(z) = z^{-p} - \sum_{k=p+1}^{\infty} |a_k| z^k \quad \text{and} \quad g(z) = -\sum_{k=p+1}^{\infty} |b_k| z^k. \tag{1.3}$$

Let F be fixed multivalent harmonic function given by

$$F = H(z) + \overline{G(z)} = z^{-p} + \sum_{k=p+1}^{\infty} A_k z^k \quad + \overline{\sum_{k=p+1}^{\infty} B_k z^k}. \tag{1.4}$$

We define the Hadamard product (or convolution) of F and f by

$$(F * f)(z) := z^{-p} + \sum_{k=p+1}^{\infty} a_k A_k z^k \quad + \overline{\sum_{k=p+1}^{\infty} b_k B_k z^k} = (f * F)(z). \tag{1.5}$$

For positive real values of α_i $(i = 1, \cdots, l)$ and β_j $(j = 1, \cdots, m)$, the generalized hypergeometric function ${}_lF_m$ (with l numerator and m denominator parameters) is defined by

$${}_lF_m(\alpha_1, \cdots, \alpha_l; \beta_1, \cdots, \beta_m)(z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_l)_k}{(\beta_1)_k \cdots (\beta_m)_k} \cdot \frac{z^k}{k!},$$

where $l \leq m+1; l, m \in \mathbb{N}_0 := \{0, 1, 2 \cdots\} = \mathbb{N} \cup \{0\}$, and $(\lambda)_n$ is the Pochhammer symbol (or the shifted factorial) defined (in terms of the Gamma function) by

$$(\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} 1, & n = 0, \\ \lambda(\lambda+1)\cdots(\lambda+n-1), & n \in \mathbb{N}. \end{cases}$$

Corresponding to the function

$$h_p(\alpha_1, \cdots, \alpha_l; \beta_1, \cdots, \beta_m; z) = z^{-p} {}_lF_m(\alpha_1, \cdots, \alpha_l; \beta_1, \cdots, \beta_m)(z),$$

the linear operator

$$H_p(\alpha_1, \cdots, \alpha_l; \beta_1, \cdots, \beta_m) : \Sigma_H(p) \longrightarrow \Sigma_H(p)$$

is defined by using the following Hadamard product (or convolution):

$$H_p(\alpha_1, \cdots, \alpha_l; \beta_1, \cdots, \beta_m) f(z) = h_p(\alpha_1, \cdots, \alpha_l; \beta_1, \cdots, \beta_m; z) * f(z).$$

For a function f of the form (1.1), we have

$$\begin{aligned} &H_p(\alpha_1, \cdots, \alpha_l; \beta_1, \cdots, \beta_m) f(z) \\ &= z^{-p} + \sum_{k=p+1}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_l)_k}{k!(\beta_1)_k \cdots (\beta_m)_k} a_k z^k + \overline{\sum_{k=p+1}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_l)_k}{k!(\beta_1)_k \cdots (\beta_m)_k} b_k z^k} \\ &:= H_{p,l,m}[\alpha_1] f(z). \end{aligned} \tag{1.6}$$

The above-defined operator $H_{p,l,m}[\alpha_1]$ $(b_k = 0)$ was introduced by Liu and Srivastava [7] and it was the development of the Dziok-Srivastava operator (see [4, 5]).

Using the same methods of [10] and [11], we introduce the generalized Liu-Srivastava operator in $\Sigma_H(p)$ as follows:

$$\begin{aligned} L^{1,\alpha_1}_{\lambda,l,m} f(z) &= (1-\lambda) H_{p,l,m}[\alpha_1] f(z) - \frac{\lambda}{p} z (H_{p,l,m}[\alpha_1] f(z))' \\ &:= L^{\alpha_1}_{\lambda,l,m} f(z) \quad (\lambda \geq 0), \end{aligned}$$

where

$$z(H_{p,l,m}[\alpha_1]f(z))' = z(H_{p,l,m}[\alpha_1]h(z))' - \overline{z(H_{p,l,m}[\alpha_1]g(z))'}.$$

In general,

$$L^{\tau,\alpha_1}_{\lambda,l,m}f(z) = L^{\alpha_1}_{\lambda,l,m}(L^{\tau-1,\alpha_1}_{\lambda,l,m}f(z)) \quad (l \le m+1; l, m \in \mathbb{N}_0, \tau \in \mathbb{N}), \tag{1.7}$$

where

$$L^{\tau,\alpha_1}_{\lambda,l,m}f(z) = z^{-p} + \sum_{k=p+1}^{\infty}\left(\frac{(1-\frac{k\lambda}{p})(\alpha_1)_k\cdots(\alpha_l)_k}{k!(\beta_1)_k\cdots(\beta_m)_k}\right)^{\tau} a_k z^k + \overline{\sum_{k=p+1}^{\infty}\left(\frac{(1-\frac{k\lambda}{p})(\alpha_1)_k\cdots(\alpha_l)_k}{k!(\beta_1)_k\cdots(\beta_m)_k}\right)^{\tau} a_k z^k} \tag{1.8}$$

and $\lambda \ge 0, \tau \in \mathbb{N}$.

For $\mu > 0$ and $\tau \in \mathbb{N}$, we introduce the following linear operator $\mathcal{J}^{\mu}_{\tau} : \Sigma_H(p) \longrightarrow \Sigma_H(p)$, defined by

$$\mathcal{J}^{\mu}_{\tau}f(z) = \mathcal{J}^{\mu}_{\tau}(z) * f(z) = \mathcal{J}^{\mu}_{\tau}(z) * h(z) + \overline{\mathcal{J}^{\mu}_{\tau}(z) * g(z)} \quad (z \in \mathbb{U}^*), \tag{1.9}$$

where $\mathcal{J}^{\mu}_{\tau}(z)$ is the function defined as follows:

$$L^{\tau,\alpha_1}_{\lambda,l,m}(z) * \mathcal{J}^{\mu}_{\tau}(z) = \frac{1}{z^p(1-z)^{\mu}} \quad (\mu > 0, k\lambda \ne p, z \in \mathbb{U}^*), \tag{1.10}$$

and

$$L^{\tau,\alpha_1}_{\lambda,l,m}(z) = z^{-p} + \sum_{k=p+1}^{\infty}\left(\frac{(1-\frac{k\lambda}{p})(\alpha_1)_k\cdots(\alpha_l)_k}{k!(\beta_1)_k\cdots(\beta_m)_k}\right)^{\tau} z^k. \tag{1.11}$$

Since

$$\frac{1}{z^p(1-z)^{\mu}} = z^{-p} + \sum_{k=1}^{\infty}\frac{(\mu)_k}{k!}z^{k-p} \quad (\mu > 0, z \in \mathbb{U}^*), \tag{1.12}$$

combining (1.9)–(1.12), we obtain

$$\mathcal{J}^{\mu}_{\tau}(z) = z^{-p} + \sum_{k=p+1}^{\infty}\left(\frac{k!(\beta_1)_k\cdots(\beta_m)_k}{(1-\frac{k\lambda}{p})(\alpha_1)_k\cdots(\alpha_l)_k}\right)^{\tau}\frac{(\mu)_k}{k!}z^k \quad (\mu > 0, k\lambda \ne p, z \in \mathbb{U}^*). \tag{1.13}$$

If f is given by (1.1), then we find from (1.9) and (1.13) that

$$\mathcal{J}^{\mu}_{\tau}f(z) = \mathcal{J}^{\mu}_{\tau}h(z) + \overline{\mathcal{J}^{\mu}_{\tau}g(z)} = z^{-p} + \sum_{k=p+1}^{\infty}\Phi^{\mu}_k a_k z^k + \overline{\sum_{k=p+1}^{\infty}\Phi^{\mu}_k b_k z^k}, \tag{1.14}$$

where

$$\Phi^{\mu}_k = \left(\frac{k!(\beta_1)_k\cdots(\beta_m)_k}{(1-\frac{k\lambda}{p})(\alpha_1)_k\cdots(\alpha_l)_k}\right)^{\tau}\frac{(\mu)_k}{k!} \ (\mu > 0, k\lambda \ne p). \tag{1.15}$$

Also, from (1.14) and (1.15), we easily get

$$z(\mathcal{J}^{\mu}_{\tau}h(z))' = \mu\mathcal{J}^{\mu+1}_{\tau}h(z) - (p+\mu)\mathcal{J}^{\mu}_{\tau}h(z)$$

and

$$z(\mathcal{J}^{\mu}_{\tau}g(z))' = \mu\mathcal{J}^{\mu+1}_{\tau}g(z) - (p+\mu)\mathcal{J}^{\mu}_{\tau}g(z).$$

By making use of the principle of subordination between analytic functions, we introduce the class $L_p(A, B; \mu, \tau, \alpha, \delta)$.

Dedinition 1.1. *A function $f \in \Sigma_H(p)$ of the form (1.1) is said to be in the class $L_p(A, B; \mu, \tau, \alpha, \delta)$ if and only if*

$$\chi_{\delta,\mu}(f) - \alpha|\chi_{\delta,\mu}(f) - 1| \prec \frac{1 + Az}{1 + Bz}, \tag{1.16}$$

where

$$\chi_{\delta,\mu}(f) = (1 - \delta)z^p \cdot \mathcal{J}_\tau^\mu f(z) - \frac{\delta}{p} z^{p+1} \cdot (\mathcal{J}_\tau^\mu f(z))' \tag{1.17}$$

and $\mathcal{J}_\tau^\mu f(z)$ is defined by (1.14) and $p \in \mathbb{N}$; $A, B \in \mathbb{R}, A \neq B, |B| \leq 1$; $\tau \in \mathbb{N}, \mu > 0, \alpha \geq 0, \delta \geq 0, k\lambda \neq p$.

For $\delta = 0$, we obtain the following new subclass:

A function $f \in \Sigma_H(p)$ of the form (1.1) is said to be in the class $\Sigma_H(A, B; \mu, \tau, \alpha)$ if and only if

$$z^p \cdot \mathcal{J}_\tau^\mu f(z) - \alpha|z^p \cdot \mathcal{J}_\tau^\mu f(z) - 1| \prec \frac{1 + Az}{1 + Bz}, \tag{1.18}$$

where $\mathcal{J}_\tau^\mu f(z)$ is defined by (1.14) and $p \in \mathbb{N}$; $A, B \in \mathbb{R}, A \neq B, |B| \leq 1$; $\tau \in \mathbb{N}, \mu > 0, \alpha \geq 0, k\lambda \neq p$.

We also let

$$\overline{L}_p(A, B; \mu, \tau, \alpha, \delta) = \Sigma_{\overline{H}}(p) \bigcap L_p(A, B; \mu, \tau, \alpha, \delta)$$

and

$$\Sigma_{\overline{H}}(A, B; \mu, \tau, \alpha) = \Sigma_{\overline{H}}(p) \bigcap \Sigma_H(A, B; \mu, \tau, \alpha).$$

Recently, Jahangiri [6], Ahuja and Jahangiri [1] and Murugusundaramoorthy [9] have introduced and studied some classes of meromorphic harmonic functions. In this paper, we aim to introduce some new subclasses of harmonic multivalent meromorphic functions defined by generalized Liu-Srivastava operator and obtain some results including sufficient coefficient conditions, distortion bounds and extreme points for functions of these classes.

2. Main Results

Lemma 2.1. *(see [8]) Let $\alpha \geq 0, A, B \in \mathbb{R}, A \neq B$ and $|B| \leq 1$. If $\omega(z)$ is an analytic function with $\omega(0) = 1$, then we have*

$$\omega - \alpha|\omega - 1| \prec \frac{1 + Az}{1 + Bz} \Longleftrightarrow \omega(1 - \alpha e^{-i\phi}) + \alpha e^{-i\phi} \prec \frac{1 + Az}{1 + Bz} \quad (\phi \in \mathbb{R}). \tag{2.1}$$

Proof. Suppose $\omega - 1 = |\omega - 1|e^{i\phi}, \phi \in \mathbb{R}$, so we have $|\omega - 1| = (\omega - 1)e^{-i\phi}$. Therefore,

$$\omega - \alpha|\omega - 1| \prec \frac{1 + Az}{1 + Bz} \Longleftrightarrow \omega(1 - \alpha e^{-i\phi}) + \alpha e^{-i\phi} \prec \frac{1 + Az}{1 + Bz} \quad (\phi \in \mathbb{R}).$$

Using Lemma 2.1 and (1.18), we get that $f \in \Sigma_H(p; A, B; \mu, \tau, \alpha)$ if and only if

$$\chi_{\delta,\mu}(f)(1 - \alpha e^{-i\phi}) + \alpha e^{-i\phi} \prec \frac{1 + Az}{1 + Bz}, \tag{2.2}$$

where $\chi_{\delta,\mu}(f)$ is given by (1.17). □

Theorem 2.2. *Let $f = h + \overline{g}$ be such that h and g are given by (1.2). Also, suppose that $p \in \mathbb{N}, A, B \in \mathbb{R}$ and $k\lambda \neq p, A \neq B, |B| \leq 1$. If*

$$\sum_{k=p+1}^{\infty} (1 + |B|)(1 + \alpha)(|\xi_k^\mu||a_k| + |\eta_k^\mu||b_k|) \leq |A - B|, \tag{2.3}$$

where

$$\xi_k^\mu = (1-\delta-\frac{\delta k}{p})\Phi_k^\mu, \quad \eta_k^\mu = (1-\delta+\frac{\delta k}{p})\Phi_k^\mu \tag{2.4}$$

and Φ_k^μ *is given by (1.15), then* $f \in L_p(A,B;\mu,\tau,\alpha,\delta)$.

Proof. We first show that if the inequality (2.3) holds for the coefficients of $f = h+\overline{g}$, then the required condition (2.2) is satisfied. In view of (2.2), we need to prove that $p(z) \prec \frac{1+Az}{1+Bz}$, where

$$p(z) = \chi_{\delta,\mu}(f)(1-\alpha e^{-i\phi}) + \alpha e^{-i\phi}. \tag{2.5}$$

Using the fact that $p(z) \prec \frac{1+Az}{1+Bz} \iff |1-p(z)| \le |Bp(z)-A|$, it suffices to show that

$$|1-p(z)| - |Bp(z)-A| \le 0. \tag{2.6}$$

Therefore, we get

$$\begin{aligned}
&\left| B - B(1-\alpha e^{-i\phi}) \sum_{k=p+1}^{\infty} [\xi_k^\mu a_k z^{k+p} + \eta_k^\mu b_k z^p \overline{z^k}] - A \right| \\
&\le |(1+\alpha) \sum_{k=p+1}^{\infty} [|\xi_k^\mu||a_k||z|^{k+p} + |\eta_k^\mu||b_k||z|^{k+p}] - \\
&\quad (|A-B| - |B|(1+\alpha) \sum_{k=p+1}^{\infty} [|\xi_k^\mu||a_k||z|^{k+p} + |\eta_k^\mu||b_k||z|^{k+p}] \\
&= \sum_{k=p+1}^{\infty} (1+|B|)(1+\alpha)[|\xi_k^\mu||a_k||z|^{k+p} + |\eta_k^\mu||b_k||z|^{k+p}] - |A-B| \\
&\le \sum_{k=p+1}^{\infty} (1+|B|)(1+\alpha)[|\xi_k^\mu||a_k| + |\eta_k^\mu||b_k|] - |A-B| \\
&\le 0.
\end{aligned}$$

By hypothesis the last expression is non-positive. Thus the proof is completed. The coefficient bound (2.3) is sharp for the function

$$f(z) = z^{-p} + \sum_{k=p+1}^{\infty} \frac{|A-B|}{(1+|B|)(1+\alpha)} \left(\frac{1}{|\xi_k^\mu|} X_k z^k + \frac{1}{|\eta_k^\mu|} \overline{Y_k} \overline{z^k} \right), \tag{2.7}$$

where $\sum_{k=p+1}^{\infty} (|X_k| + |Y_k|) = 1$.

□

Corollary 2.3. *Let* $f = h+\overline{g}$ *be such that* h *and* g *are given by (1.2),* ξ_k^μ *and* η_k^μ *are given by (2.4). Also, suppose that* $p \in \mathbb{N}$ *and* $A, B \in \mathbb{R}$. *Then,*

(i) for $-1 \le B < A \le 1, B < 0$, *if*

$$\sum_{k=p+1}^{\infty} (1-B)(1+\alpha)(|\xi_k^\mu||a_k| + |\eta_k^\mu||b_k|) \le A-B,$$

then $f \in L_p(A,B;\mu,\tau,\alpha,\delta)$.

(ii) for $-1 \leq A < B \leq 1, B > 0$, *if*

$$\sum_{k=p+1}^{\infty}(1+B)(1+\alpha)(|\xi_k^{\mu}||a_k| + |\eta_k^{\mu}||b_k|) \leq B - A,$$

then $f \in L_p(A, B; \mu, \tau, \alpha, \delta)$.

Corollary 2.4. *Let* $f = h + \overline{g}$ *be such that* h *and* g *are given by (1.2). Also, suppose that* $p \in \mathbb{N}, A, B \in \mathbb{R}, k\lambda \neq p, A \neq B$ *and* $|B| \leq 1$. *If*

$$\sum_{k=p+1}^{\infty}(1+|B|)(1+\alpha)|\Phi_k^{\mu}|(|a_k| + |b_k|) \leq |A - B|,$$

where Φ_k^{μ} *is given by (1.15), then* $f \in \Sigma_H(A, B; \mu, \tau, \alpha)$.

Theorem 2.5. *Let* $f = h + \overline{g}$ *be such that* h *and* g *are given by (1.2),* ξ_k^{μ} *and* η_k^{μ} *are given by (2.4). Also, suppose that* $p \in \mathbb{N}, A, B \in \mathbb{R}, A \neq B, |B| \leq 1, k\lambda < p$ *and* $0 \leq \delta < \frac{p}{2p+1}$. *Then,*

(i) for $-1 \leq B < A \leq 1,\ B < 0,\ f \in \overline{L}_p(A, B; \mu, \tau, \alpha, \delta)$ *if and only if*

$$\sum_{k=p+1}^{\infty}(1-B)(1+\alpha)(\xi_k^{\mu}|a_k| + \eta_k^{\mu}|b_k|) \leq A - B. \tag{2.8}$$

(ii) for $-1 \leq A < B \leq 1,\ B > 0,\ f \in \overline{L}_p(A, B; \mu, \tau, \alpha, \delta)$ *if and only if*

$$\sum_{k=p+1}^{\infty}(1+B)(1+\alpha)(\xi_k^{\mu}|a_k| + \eta_k^{\mu}|b_k|) \leq B - A. \tag{2.9}$$

Proof. Since $\overline{L}_p(A, B; \mu, \tau, \alpha, \delta) \subset L_p(A, B; \mu, \tau, \alpha, \delta)$. According to Corollary 2.3, we only need to prove the "only if" part of the theorem.

(i) Let $f \in \overline{L}_p(A, B; \mu, \tau, \alpha, \delta), -1 \leq B < A \leq 1, B < 0$. Then

$$\left|\frac{1 - p(z)}{Bp(z) - A}\right| < 1, \tag{2.10}$$

where $p(z)$ is defined by (2.5). Clearly, (2.10) is equivalent to

$$\left|\frac{(1-\alpha e^{-i\phi})\sum\limits_{k=p+1}^{\infty}(\xi_k^{\mu}|a_k|z^{k+p} + \eta_k^{\mu}|b_k|z^p\overline{z^k})}{B - B(1-\alpha e^{-i\phi})\sum\limits_{k=p+1}^{\infty}(\xi_k^{\mu}|a_k|z^{k+p} + \eta_k^{\mu}|b_k|z^p\overline{z^k}) - A}\right| < 1. \tag{2.11}$$

From (2.11), we have

$$\Re\left\{\frac{(1-\alpha e^{-i\phi})\sum\limits_{k=p+1}^{\infty}(\xi_k^{\mu}|a_k|z^{k+p} + \eta_k^{\mu}|b_k|z^p\overline{z^k})}{A - B + B(1-\alpha e^{-i\phi})\sum\limits_{k=p+1}^{\infty}(\xi_k^{\mu}|a_k|z^{k+p} + \eta_k^{\mu}|b_k|z^p\overline{z^k})}\right\} < 1. \tag{2.12}$$

Taking $z = r$ $(0 < r < 1)$ and $\phi = \pi$, then (2.12) gives

$$\sum_{k=p+1}^{\infty}(1-B)(1+\alpha)(\xi_k^{\mu}|a_k| + \eta_k^{\mu}|b_k|)r^{k+p} \leq A - B. \tag{2.13}$$

Letting $r \to 1^-$ in (2.13), we will get (2.8).

(ii) Similar to the proof of (2.8), we can prove (2.9). □

Corollary 2.6. *Let $f = h + \overline{g}$ be such that h and g are given by (1.2), Φ_k^{μ} is given by (1.15). Also, suppose that $p \in \mathbb{N}, A, B \in \mathbb{R}, A \neq B, |B| \leq 1$ and $k\lambda < p$. Then,*

(i) for $-1 \leq B < A \leq 1, B < 0$, $f \in \Sigma_{\overline{H}}(A, B; \mu, \tau, \alpha)$ if and only if

$$\sum_{k=p+1}^{\infty} (1-B)(1+\alpha)\Phi_k^{\mu}(|a_k| + |b_k|) \leq A - B.$$

(ii) for $-1 \leq A < B \leq 1, B > 0$, $f \in \Sigma_{\overline{H}}(A, B; \mu, \tau, \alpha)$ if and only if

$$\sum_{k=p+1}^{\infty} (1+B)(1+\alpha)\Phi_k^{\mu}(|a_k| + |b_k|) \leq B - A.$$

Theorem 2.7. *Let $f = h + \overline{g}$ be such that h and g are given by (1.3), ξ_k^{μ} and η_k^{μ} are given by (2.4). Also, suppose that $k\lambda < p, \mu > 1$ and $0 \leq \delta < \frac{p}{2p+1}$. Then,*

(i) for $-1 \leq B < A \leq 1,\ B < 0$, if $f \in \overline{L}_p(A, B; \mu, \tau, \alpha, \delta)$, then

$$r^{-p} - \frac{A-B}{(1-B)(1+\alpha)\xi_{p+1}^{\mu}} r^{p+1} \leq |f(z)| \leq r^{-p} + \frac{A-B}{(1-B)(1+\alpha)\xi_{p+1}^{\mu}} r^{p+1} \quad (2.14)$$

(ii) for $-1 \leq A < B \leq 1,\ B > 0$, if $f \in \overline{L}_p(A, B; \mu, \tau, \alpha, \delta)$, then

$$r^{-p} - \frac{B-A}{(1+B)(1+\alpha)\xi_{p+1}^{\mu}} r^{p+1} \leq |f(z)| \leq r^{-p} + \frac{B-A}{(1+B)(1+\alpha)\xi_{p+1}^{\mu}} r^{p+1}. \quad (2.15)$$

Proof. Since $f \in \overline{L}_p(A, B; \mu, \tau, \alpha, \delta)$, then by using Theorem 2.5, we have

$$(1-B)(1+\alpha)\xi_{p+1}^{\mu} \sum_{k=p+1}^{\infty} (|a_k|+|b_k|) \leq \sum_{k=p+1}^{\infty} (1-B)(1+\alpha)(\xi_k^{\mu}|a_k|+\eta_k^{\mu}|b_k|) \leq A-B, \quad (2.16)$$

which implies that

(i) if $-1 \leq B < A \leq 1$ and $B < 0$, then from (2.16),we have

$$\sum_{k=p+1}^{\infty} (|a_k| + |b_k|) \leq \frac{A-B}{(1-B)(1+\alpha)\xi_{p+1}^{\mu}}. \quad (2.17)$$

On the other word,

$$\begin{aligned} |f(z)| &\leq r^{-p} + \sum_{k=p+1}^{\infty} (|a_k| + |b_k|) r^k \\ &\leq r^{-p} + r^{p+1} \sum_{k=p+1}^{\infty} (|a_k| + |b_k|) \\ &\leq r^{-p} + \frac{A-B}{(1-B)(1+\alpha)\xi_{p+1}^{\mu}} r^{p+1} \end{aligned}$$

and

$$|f(z)| \geq r^{-p} - \frac{A-B}{(1-B)(1+\alpha)\xi_{p+1}^{\mu}} r^{p+1}.$$

Hence (2.14) follows. The case for (ii) $-1 \leq A < B \leq 1$ and $B > 0$ can be proved in the same manner and hence we omit it. □

Corollary 2.8. *Let $f = h + \overline{g}$ be such that h and g are given by (1.3), Φ_k^μ is given by (1.15). Also, suppose that $k\lambda < p, |z| = r < 1$ and $\mu > 1$. Then,*

(i) for $-1 \leq B < A \leq 1,\ B < 0$, if $f \in \Sigma_{\overline{H}}(A, B; \mu, \tau, \alpha)$, then

$$r^{-p} - \frac{A-B}{(1-B)(1+\alpha)\Phi_{p+1}^\mu} r^{p+1} \leq |f(z)| \leq r^{-p} + \frac{A-B}{(1-B)(1+\alpha)\Phi_{p+1}^\mu} r^{p+1}.$$

(ii) for $-1 \leq A < B \leq 1,\ B > 0$, if $f \in \Sigma_{\overline{H}}(A, B; \mu, \tau, \alpha)$, then

$$r^{-p} - \frac{B-A}{(1+B)(1+\alpha)\Phi_{p+1}^\mu} r^{p+1} \leq |f(z)| \leq r^{-p} + \frac{B-A}{(1+B)(1+\alpha)\Phi_{p+1}^\mu} r^{p+1}.$$

Theorem 2.9. *Let $f = h + \overline{g}$ be such that h and g are given by (1.2), ξ_k^μ and η_k^μ are given by (2.4). Also, suppose that $p \in \mathbb{N}, A, B \in \mathbb{R}, A \neq B, |B| \leq 1, k\lambda < p$ and $0 \leq \delta < \frac{p}{2p+1}$. Then $f \in clco\overline{L}_p(A, B; \mu, \tau, \alpha, \delta)$ if and only if*

$$f(z) = \sum_{k=p}^{\infty} X_k h_k + \sum_{k=p+1}^{\infty} Y_k (h_p + g_k),\ z \in \mathbb{U}^*, \tag{2.18}$$

where

$$h_p = z^{-p},$$

$$h_k = \begin{cases} z^{-p} - \frac{A-B}{(1-B)(1+\alpha)\xi_k^\mu} z^k, & k \geq p+1, -1 \leq B < A \leq 1, B < 0, \\ z^{-p} - \frac{B-A}{(1+B)(1+\alpha)\xi_k^\mu} z^k, & k \geq p+1, -1 \leq A < B \leq 1, B > 0, \end{cases}$$

$$g_k = \begin{cases} -\frac{A-B}{(1-B)(1+\alpha)\eta_k^\mu} \overline{z^k}, & k \geq p+1, -1 \leq B < A \leq 1, B < 0, \\ -\frac{B-A}{(1+B)(1+\alpha)\eta_k^\mu} \overline{z^k}, & k \geq p+1, -1 \leq A < B \leq 1, B > 0, \end{cases}$$

and

$$X_p \equiv 1 - \sum_{k=p+1}^{\infty} (X_k + Y_k)\ (X_k \geq 0, Y_k \geq 0).$$

In particular, the extreme points of $\overline{L}_p(A, B; \mu, \tau, \alpha, \delta)$ are h_k and g_k.

Proof. Let $-1 \leq B < A \leq 1, B < 0$ and $k\lambda < p$, we get

$$f(z) = z^p - \sum_{k=p+1}^{\infty} \frac{A-B}{(1-B)(1+\alpha)} \left(\frac{1}{\xi_k^\mu} X_k z^k + \frac{1}{\eta_k^\mu} Y_k \overline{z^k} \right). \tag{2.19}$$

Since, $0 \leq X_k \leq 1\ (k = p+1, \cdots)$, we obtain

$$\sum_{k=p+1}^{\infty} \left(\frac{(1-B)(1+\alpha)\xi_k^\mu}{A-B} \frac{A-B}{(1-B)(1+\alpha)\xi_k^\mu} X_k + \frac{(1-B)(1+\alpha)\eta_k^\mu}{A-B} \frac{A-B}{(1-B)(1+\alpha)\eta_k^\mu} Y_k \right)$$
$$= \sum_{k=p+1}^{\infty} (X_k + Y_k)$$
$$= 1 - X_p$$
$$\leq 1.$$

Consequently, using Theorem 2.5, we have $f \in \overline{L}_p(A, B; \mu, \tau, \alpha, \delta)$.

Conversely, if $f \in \overline{L}_p(A,B;\mu,\tau,\alpha,\delta)$, then

$$|a_{_k}| \le \frac{A-B}{(1-B)(1+\alpha)\xi_k^\mu},\ |b_{_k}| \le \frac{A-B}{(1-B)(1+\alpha)\eta_k^\mu}. \tag{2.20}$$

Putting

$$X_k = \frac{(1-B)(1+\alpha)\xi_k^\mu |a_{_k}|}{A-B},\ Y_k = \frac{(1-B)(1+\alpha)\eta_k^\mu |b_{_k}|}{A-B} \tag{2.21}$$

and

$$X_{_p} = 1 - \sum_{k=p+1}^{\infty}(X_k+Y_k) \ge 0,$$

we obtain

$$\begin{aligned} f(z) &= z^{-p} - \sum_{k=p+1}^{\infty}|a_k|z^k - \sum_{k=p+1}^{\infty}|b_k|\overline{z}^k \\ &= (X_p + \sum_{k=p+1}^{\infty}(X_k+Y_k))z^{-p} - \sum_{k=p+1}^{\infty}\frac{A-B}{(1-B)(1+\alpha)\xi_k^\mu}X_k z^k - \sum_{k=p+1}^{\infty}\frac{A-B}{(1-B)(1+\alpha)\eta_k^\mu}Y_k\overline{z}^k \\ &= X_k z^{-p} + \sum_{k=p+1}^{\infty}h_k(z)X_k + \sum_{k=p+1}^{\infty}(z^{-p}+g_k(z))Y_k \\ &= X_p h_p + \sum_{k=p+1}^{\infty}h_k X_k + \sum_{k=p+1}^{\infty}(h_p+g_k)Y_k \\ &= \sum_{k=p}^{\infty}h_k X_k + \sum_{k=p+1}^{\infty}(h_p+g_k)Y_k. \end{aligned}$$

Thus f can be expressed in the form (2.18). The case for $-1 \le A < B \le 1, B > 0$ can be proved in the same manner and hence we omit it. □

Corollary 2.10. *Let $f = h + \overline{g}$ be such that h and g are given by (1.2), Φ_k^μ is given by (1.15). Also, suppose that $p \in \mathbb{N}, A, B \in \mathbb{R}, A \ne B, |B| \le 1$ and $k\lambda < p$. Then $f \in clco\Sigma_{\overline{H}}(A,B;\mu,\tau,\alpha)$ if and only if*

$$f(z) = \sum_{k=p}^{\infty}X_k h_k + \sum_{k=p+1}^{\infty}Y_k(h_p+g_k),\ z \in \mathbb{U}^*,$$

where

$$h_p = z^{-p},$$

$$h_k = \begin{cases} z^{-p} - \frac{A-B}{(1-B)(1+\alpha)\Phi_k^\mu}z^k, & k \ge p+1, -1 \le B < A \le 1, B < 0, \\ z^{-p} - \frac{B-A}{(1+B)(1+\alpha)\Phi_k^\mu}z^k, & k \ge p+1, -1 \le A < B \le 1, B > 0, \end{cases}$$

$$g_k = \begin{cases} -\frac{A-B}{(1-B)(1+\alpha)\Phi_k^\mu}\overline{z^k}, & k \ge p+1, -1 \le B < A \le 1, B < 0, \\ -\frac{B-A}{(1+B)(1+\alpha)\Phi_k^\mu}\overline{z^k}, & k \ge p+1, -1 \le A < B \le 1, B > 0, \end{cases}$$

and

$$X_p \equiv 1 - \sum_{k=p+1}^{\infty}(X_k+Y_k).$$

In particular, the extreme points of $\Sigma_{\overline{H}}(A,B;\mu,\tau,\alpha)$ *are* h_k *and* g_k.

Theorem 2.11. *The class* $\overline{L}_p(A,B;\mu,\tau,\alpha,\delta)(0 \le \delta < \frac{p}{2p+1})$ *is closed under convex combinations.*

Proof. For $j=1,2$, let the functions f_j given by

$$f_j(z) = z^{-p} - \sum_{k=p+1}^{\infty} |a_{jk}|z^k - \sum_{k=p+1}^{\infty} |b_{jk}|\overline{z}^k \tag{2.22}$$

be in the class $\overline{L}_p(A,B;\mu,\tau,\alpha,\delta)$.

For λ_j, $\sum\limits_{j=1}^{\infty} \lambda_j = 1$, the convex combinations can be expressed in the form

$$\sum_{j=1}^{\infty} \lambda_j f_j = z^p - \sum_{k=p+1}^{\infty} \left(\sum_{j=1}^{\infty} \lambda_j |a_{jk}| \right) z^k - \sum_{k=p+1}^{\infty} \left(\sum_{j=1}^{\infty} \lambda_j |b_{jk}| \right) \overline{z}^k \tag{2.23}$$

(i) For $k\lambda < p, -1 \le B < A \le 1,\ B < 0$, from (2.8), (2.22) and (2.23), we get

$$\begin{aligned}
&\sum_{k=p+1}^{\infty} (1-B)(1+\alpha)(\sum_{j=1}^{\infty} \lambda_j(\xi_k^{\mu}|a_{jk}| + \eta_k^{\mu}|b_{jk}|)) \\
&= \sum_{j=1}^{\infty} \lambda_j [\sum_{k=p+1}^{\infty} (1-B)(1+\alpha)(\xi_k^{\mu}|a_{jk}| + \eta_k^{\mu}|b_{jk}|)] \\
&\le \sum_{j=1}^{\infty} \lambda_j (A-B) \\
&= A - B.
\end{aligned}$$

That is, $\sum\limits_{j=1}^{\infty} \lambda_j f_j \in \overline{L}_p(A,B;\mu,\tau,\alpha,\delta)$. The case for (ii) $k\lambda < p,\ -1 \le A < B \le 1,\ B > 0$ can be proved in the same manner and hence we omit it. □

Corollary 2.12. *The class* $\Sigma_{\overline{H}}(A,B;\mu,\tau,\alpha)$ *is closed under convex combinations.*

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