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ON WEAK SYMMETRIES OF δ- LORENTZIAN β- KENMOTSU MANIFOLD

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Abstract. The purpose of this paper is to study weakly symmetric and weakly Ricci symmetric δ- Lorentzian β- Kenmotsu Manifolds. We prove that the sum of the associated 1- forms of weakly symmetric δ- Lorentzian β- Kenmotsu Manifold and weakly Ricci symmetric δ- Lorentzian β- Kenmotsu Manifold is nonzero everywhere provided that nonvanishing ξ-sectional curvature. The existence of δ- Lorentzian β- Kenmotsu Manifold is ensured by an example.

1. Introduction

In the year 1987, Chaki [4] establish the proper generalization of pseudosymmetric manifolds. Furthermore, in 1989, Tamassy and Binh [11] introduced the notion of weakly symmetric manifolds. A non-flat Riemannian manifold $(M^n, g)(n > 2)$ is called weakly symmetric if its curvature tensor $\bar{R}$ of the type $(0, 4)$ satisfies the condition

$$\begin{aligned} \nabla_X \bar{R}(Y,Z,U,V) = A(X)\bar{R}(Y,Z,U,V) + B(Y)\bar{R}(X,Z,U,V) + C(Z)\bar{R}(Y,X,U,V) \\ + D(U)\bar{R}(Y,Z,X,V) + E(V)\bar{R}(Y,Z,U,X) \end{aligned} \tag{1.1}$$

for all vector fields $X, Y, Z, U, V \in X(M^n)$, A, B, , C, , D and E are 1-forms (not simultaneously zero) and ∇ denotes the operator of covariant differentiation with respect to the Riemannian metric g. The 1-Forms are called the associated 1-forms of the manifold and n- dimensional manifold of this kind is denoted by $(WS)_n$. If in (1.1) 1-form A is replaced by $2A$ and E is replaced by A, then a $(WS)_n$ reduces to the notion of generalized pseudosymmetric manifold by Chaki [5]. Furthermore, in 1999, De and Bandyopadhyay [7] studied a $(WS)_n$ and provided that in such manifold the associated 1- form $B = C$ and $D = E$ and hence the equation (1.1)

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reduces as follows

$$\nabla_X \bar{R}(Y,Z,U,V) = A(X)\bar{R}(Y,Z,U,V) + B(Y)\bar{R}(X,Z,U,V) + B(Z)\bar{R}(Y,X,U,V) + D(U)\bar{R}(Y,Z,X,V) + D(V)\bar{R}(Y,Z,U,X) \quad (1.2)$$

Thereafter, in the year 1993, Tamassy and Binh [12] introduced the notion of weakly Ricci symmetric manifolds. A Riemannian manifold $(M^n, g)(n > 2)$ is called weakly symmetric if its curvature tensor $\bar{R}$ of the type $(0,2)$ is not identically zero satisfies the condition

$$\nabla_X S(Y,Z) = A(X)S(Y,Z) + B(Y)S(X,Z) + C(Z)S(Y,X) \quad (1.3)$$

where $A,\ B,,\ C$, are three nonzero 1- forms called the associated 1- forms of the manifold and ∇ denotes the operator of covariant differentiation with respect to the metric g and this type of n- dimensional manifold is denoted by $(WRS)_n$. As an equivalent notion of $(WRS)_n$, Chaki and Koley [6] introduce the notion of generalized pseudo Ricci symmetric manifold. If in the equation (1.3) the 1-form A is replaced by $2A$, then a $(WRS)_n$ reduces to the notion of generalized pseudo Ricci symmetric manifold by Chaki and Koley. Now, if $A = B = C = 0$ then $(WRS)_n$ reduces to Ricci symmetric manifold and if $B = C = 0$ then it reduces to Recci recurrent manifold.

At the same time, in the year 1969, Takahashi [13] has introduced the Sasakian manifolds with Pseudo-Riemannian metric and prove that one can study the Lorentzian Sasakian structure with an indefinite metric. Furthermore, in 1990, K. L. Duggal [8] has initiated the space time manifolds with contact structure and analyzed the paper of Takahashi. T. [13]. In 2009, S. Y. Perktas, E. Kilie, M. M. Tripathi [15] have studied the various properties of Lorentzian β-Kenmotsu manifolds and S.S. Pujar [10] have introduced the notion of δ Lorentzian β-Kenmotsu manifolds and studied basic results in δ Lorentzian β-Kenmotsu manifolds and its properties. Inspired by these papers and some other papers (see the exhaustive list [1, 9, 11, 14]) we have studied on weak symmetres of δ Lorentzian β-Kenmotsu manifolds. In section 2, we consider the $(2n + 1)$ dimensional differentiale manifold M with Lorentzian almost contact metric structure with indefinite metric g. This section deals with preliminaries of δ Lorentzian β-Kenmotsu manifolds. In section 3 of the paper it is proved that the sum of the associated 1-forms of a weakly symmetric δ Lorentzian β-Kenmotsu manifolds of non-vanishing ξ-sectional curvature is nonzero everywhere and hence such a structure exists. In section 4 we study weakly Ricci symmetric δ -Lorentzian β-Kenmotsu manifolds and prove that in such astructure, with non-vanishing ξ-sectional curvature, the sum of the associated 1-forms is non-vanishing everywhere and consequently such a structure exists. Finally section 5 deals with a concrete example of δ Lorentzian β-Kenmotsu manifolds.

2. δ-Lorentzian β-Kenmotsu manifold

In this section we study δ-Lorentzian - β-Kenmotsu manifold. For the manifold almost-Lorentzian contact, we have

$$\phi^2 X = X + \eta(X)\xi,\ \eta(\xi) = -1,\ \eta(X) = g(X,\xi)$$

where ϕ is a tensor field of type $(1,1)$ and ξ is a characteristic vector field and η is the 1-form. Therefore, from these conditions one can reduce that $\phi(\xi) = 0,\ \eta(\phi(X)) =$

0 for any vector field X on M. It is well known that the Lorentzian contact metric structure [2] or Lorentzian Kenmotsu structure [11] satisfies

$$(\nabla_X \phi)Y = g(\phi(X), Y) + \eta(Y)\phi(X)$$

for any C^∞ vector field X and Y on M. More generally, one has the notion of Lorentzian - β-Kenmotsu structure [9] which may be defined by the requirement

$$(\nabla_X \phi)Y = \beta[g(\phi(X), Y) + \eta(Y)\phi(X)] \tag{2.1}$$

for any C^∞ vector field X and Y on M and β is a nonzero constant on M. Using the equation (2.1), one can reduce the Lorentzian - β-Kenmotsu manifold.

$$(\nabla_X \xi) = \beta[X + \eta(X)\xi] \text{ and}, (\nabla_X \eta)Y = \beta[g(X, Y) + \eta(X)\eta(Y)].$$

At this stage, S.S Pujar [10] introducing the notion of δ- Lorentzian β-Kenmotsu manifold in the following definition.

Definition 2.1. A differentiable manifold M of dimension $(2n + 1)$ is called a δ-Lorentzian manifold, if it admits as a one-one tensor field ϕ a contravariant vector field ξ, a covariant vector field η and an indefinite metric g which satisfy

(i) $\phi^2 X = X + \eta(X)\xi,\ \eta(\xi) = -1,\ \eta(\phi(X)) = 0$

(ii) $g(\xi, \xi) = -\delta,\ \eta(X) = \delta g(X, \xi)$

(iii) $g(\phi X, \phi Y) = g(x, Y) + \delta\eta(X)\eta(Y)$

where δ is such that $\delta^2 = 1$ and for any vector field X, Y on M.The structure defined above is called a δ- Lorentzian almost contact metric structure. Manifold M together with the structure $(\phi, \xi, \eta, g, \delta)$ is also called a δ Lorentzian kenmotsu manifold if

$$(\nabla\phi)(Y) = g(\phi(X), Y)\xi + \delta\eta(Y)\phi(X)$$

more generally, S. S. Pujar introduce the definition.

Definition 2.2. A δ- Lorentzian almost contact metric manifold M $(\phi, \xi, \eta, g, \delta)$ is called a Lorentzian β-kenmotsu manifold if

$$(\nabla\phi)(Y) = \beta\{g(\phi(X), Y)\xi + \delta\eta(Y)\phi(X)\} \tag{2.2}$$

where ∇ is the Levi-Civita connection with respect to g. β is a smooth function on M and X,Y are vector fields on M and δ is such that $\delta^2 = 1$ or $\delta = \pm 1$. If $\delta = 1$, then δ- Lorentzian β- kenmotsu manifold is usual Lorentzian β- kenmotsu manifold and is called the time like manifold. In this case ξ is called a time like vector field. From (2.2) it follows that

$$\nabla_X \xi = \delta\beta\{X + \eta(X)\xi\} \tag{2.3}$$

$$(\nabla_X \eta)Y = \beta\{g(X, Y) + \delta\eta(X)\eta(Y)\} \tag{2.4}$$

$$R(X, Y)\xi = \beta^2\{\eta(Y)X - \eta(X)Y\} + \delta\{(X\beta)\phi^2 Y - (Y\beta)\phi^2 X\} \tag{2.5}$$

$$R(\xi, Y)\xi = \{\beta^2 + \delta(\xi\beta)\}\phi^2 Y, R(\xi, \xi)\xi = 0 \tag{2.6}$$

$$R(\xi, Y)X = \beta^2[\delta g(X,Y)\xi - \eta(X)Y] + \delta[(X\beta)\phi^2 Y - g(\phi X, \phi Y)(grad\beta)] \quad (2.7)$$

$$S(Y,\xi) = 2n\beta^2\eta(Y) - (2n-1)\delta(Y\beta) + \delta\eta(Y)(\xi\beta) \quad (2.8)$$

$$S(\xi,\xi) = -2n[\beta^2 + \delta(\xi\beta)] \quad (2.9)$$

$$QY = 2n\beta^2 Y, \text{where } \beta \text{ is constant.} \quad (2.10)$$

where R is the curvature tensor of type $(1,3)$ of the manifold and Q is the symmetric endomorphism of the tangent space at each point of the manifold corresponding to the Ricci tensor S, that $g(QX,Y) = S(X,Y)$ for any vector fields X, Y on M. The ξ-sectional curvature $K(\xi, X) = g(R(\xi, X)\xi, X)$ for a unit vector filed X orthogonal to ξ plays an important role in the study of an almost contact metric manifold. In our paper we consider a δ-Lorentzian β-Kenmotsu manifold of non-vanishing ξ-sectional curvature.

In the next section, we prove the sum of the associated 1- forms Weakly Symmetric δ-Lorentzian β-Kenmotsu manifold of non-vanishing ξ- sectional curvature is nonzero everywhere.

3. Weakly Symmetric δ-Lorentzian β-Kenmotsu manifolds

Definition 3.1. A δ-Lorentzian β-Kenmotsu manifold $M^{2n+1}, g)$ $(n > 1)$ is said to be weakly symmetric if its Riemannian curvature tensor $\bar{R}$ of a type $(0,4)$ satisfies (1.2). Let $e_i : i = 1, 2, ..., (2n+1)$ be an orthonormal basis of the tangent space $T_p(M)$ at any point P of the manifold. After, setting $Y = V = e_i$ in equation (1.2) and taking summation over i, $1 \leq i \leq 2n+1$, we get

$$(\nabla_X S)(Z,U) = A(X)S(Z,U) + B(Z)S(X,U) + D(U)S(X,Z) \\ + B(R(X,Z)U) + D(R(X,U)Z) \quad (3.1)$$

Now, putting $X = Z = U = \xi$, in equation (3.1) and using (2.5) and (2.9), we get

$$A(\xi) + B(\xi) + D(\xi) = \frac{2\beta(\xi\beta) + \delta\xi(\xi\beta)}{\beta^2 + \delta(\xi\beta)} \quad (3.2)$$

provided that $\beta^2 + \delta(\xi\beta) \neq 0$. The ξ- sectional curvature $K(\xi, X)$ of a δ- Lorentzian β- Kenmotsu manifold for a unit vector field X orthogonal to ξ is given by $K(\xi, X) = g(R(\xi, X)\xi, X)$.Hence equation (2.6) yields $K(\xi, X) = \beta^2 + \delta(\xi\beta)$. If $\beta^2 + \delta(\xi\beta) = 0$, then the manifold is of vanishing ξ- sectional curvature. Hence we can state the following.

Theorem 3.2. *In a weakly symmetric δ- Lorentzian β- kenmotsu manifold $(M^{(2n+1)}, g)$ $(n > 1)$ of non-vanishing ξ- sectional curvature, relation* (3.2) *holds.*

Next, substituting X and Z by ξ in equation (3.1) and then using (2.9) we obtain

$$(\nabla_\xi S)(\xi, U) = [A(\xi)+B(\xi)]S(\xi,U)+[\beta^2+\delta(\xi\beta)][(-2n+1)D(U)+\eta(U)D(\xi)] \quad (3.3)$$

Again, we have

$$\begin{aligned}(\nabla_\xi S)(\xi, U) &= \nabla_\xi S(\xi, U) - S(\nabla_\xi \xi, U) - S(\xi, \nabla_\xi U)\\ &= \nabla_\xi S(\xi, U) - S(\xi, \nabla_\xi U) \text{ (using equation (2.8))}\\ &= [4n(\beta(\xi\beta))]\eta(U) - (2n-1)\delta U(\xi\beta) + \delta\eta(U)\xi(\xi\beta) \qquad (3.4)\end{aligned}$$

From equations (3.2), (3.3) and (3.4), we get

$$\begin{aligned}D(U) &= \frac{[4n\beta(\xi\beta) + \delta\xi(\xi\beta)]\eta(U)}{(-2n+1)(\beta^2 + \delta(\xi\beta))}\\ &- \frac{(2n-1)\delta U(\xi\beta)}{(-2n+1)((\beta^2 + \delta(\xi\beta))}\\ &+ D(\xi)\left[\frac{(2n-1)[\beta^2\eta(U) - \delta(U\beta)]}{(-2n+1)(\beta^2 + \delta(\xi\beta))}\right]\\ &- \left[\frac{2\beta(\xi\beta) + \delta\xi(\xi\beta)}{(-2n+1)(\beta^2 + \delta(\xi\beta))^2}\right][2n\beta^2\eta(U) - (2n-1)\delta(U\beta) + \delta\eta(U)(\xi\beta)]\end{aligned} \qquad (3.5)$$

for any vector field U, provided that $\beta^2 + \delta(\xi\beta) \neq 0$. Next, setting $X = U = \xi$ in equation (3.1) and proceeding in a similar manner as above we get

$$\begin{aligned}B(Z) &= \frac{[4n\beta(\xi\beta) + \delta\xi(\xi\beta)]\eta(Z)}{(-2n+1)(\beta^2 + \delta(\xi\beta))}\\ &- \frac{(2n-1)\delta Z(\xi\beta)}{(-2n+1)((\beta^2 + \delta(\xi\beta))}\\ &+ D(\xi)\left[\frac{(2n-1)[\beta^2\eta(Z) - \delta(Z\beta)]}{(-2n+1)(\beta^2 + \delta(\xi\beta))}\right]\\ &- \left[\frac{2\beta(\xi\beta) + \delta\xi(\xi\beta)}{(-2n+1)(\beta^2 + \delta(\xi\beta))^2}\right][2n\beta^2\eta(Z) - (2n-1)\delta(Z\beta) + \delta\eta(Z)(\xi\beta)]\end{aligned} \qquad (3.6)$$

for any vector field Z, provided that $\beta^2 + \delta(\xi\beta) \neq 0$. This leads to the following:

Theorem 3.3. *In a weakly symmetric δ- Lorentzian β- kenmotsu manifold $(M^{(2n+1)}, g)$ $(n > 1)$ of non-vanishing ξ- sectional curvature, the associated 1-forms D and B are given by relation* (3.5) *and* (3.6)*, respectively.*

Again, setting $Z = U = \xi$ in equation (3.1) we get

$$\begin{aligned}(\nabla_X S)(\xi, \xi) &= A(X)S(\xi, \xi) + [B(\xi) + D(\xi)]S(X, \xi)\\ &+ B(R(X, \xi)\xi) + D(R(X, \xi)\xi)\\ &= -2n(\beta^2 + \delta(\xi\beta))A(X) + [B(\xi) + D(\xi)]S(X, \xi)\\ &- (\beta^2 + \delta(\xi\beta))[\eta(X)B(\xi) + D(\xi) + B(X) + D(X)] \qquad (3.7)\end{aligned}$$

Now we have

$$(\nabla_X S)(\xi, \xi) = \nabla_X S(\xi, \xi) - 2S(\nabla_X \xi, \xi),$$

which yields by using equations (2.3) and (2.8),that

$$(\nabla_X S)(\xi, \xi) = -2\beta(X\beta) - 2n\delta X(\xi\beta). \qquad (3.8)$$

In view of equations (3.5), (3.6), (3.7) and (3.8) yields

$$\begin{aligned} A(X)+B(X)+D(X) &= \frac{2n\delta X(\xi\beta)}{\beta^2+\delta(\xi\beta)} \\ &- \frac{[4n\beta(\xi\beta)+\delta\xi(\xi\beta)]\eta(X)}{2n(\beta^2+\delta(\xi\beta))} \\ &+ \frac{(2n-1)\delta X(\xi\beta)+\beta(X\beta)}{2n(\beta^2+\delta(\xi\beta))} \\ &+ \left[\frac{2\beta(\xi\beta)+\delta\xi(\xi\beta)}{2n(\beta^2+\delta(\xi\beta))^2}\right][2n\beta^2\eta(X)-(2n-1)\delta(X\beta)+\delta\eta(X)(\xi\beta)] \end{aligned} \tag{3.9}$$

for any vector field X, provided that $\beta^2+\delta(\xi\beta)\neq 0$. This leads to the following:

Theorem 3.4. *In a weakly symmetric δ- Lorentzian β- kenmotsu manifold $(M^{(2n+1)},g)$ $(n>1)$ of non-vanishing ξ- sectional curvature, the sum of the associated 1-forms is given by relation (3.9).*

In particular, if $\phi(grad\alpha)=grad\beta$ then $(\xi\beta)=0$ and hence relation (3.9) to the following form

$$A(X)+B(X)+D(X)=\frac{\beta(X\beta)}{n\beta^2} \tag{3.10}$$

for any vector field X, provided that $\beta^2\neq 0$.

Corollary 3.5. *If a weakly symmetric $\beta\neq 0$, δ- Lorentzian β- kenmotsu manifold $(M^{(2n+1)},g)$ $(n>1)$ satisfies the condition $\phi(grad\alpha)=grad\beta$, then the sum of the associated 1-forms is given by relation (3.10).*

If $\beta=1$ then equation (3.9) yields

$$\begin{aligned} A(X)+B(X)+D(X) &= \frac{2n\delta X(\xi)}{1+\delta(\xi)} \\ &- \frac{[4n(\xi)+\delta\xi(\xi)]\eta(X)}{2n(1+\delta(\xi))} \\ &+ \frac{(2n-1)\delta X(\xi)+X}{2n(1+\delta(\xi)} \\ &+ \left[\frac{2(\xi)+\delta\xi(\xi)}{2n(1+\delta(\xi))^2}\right][2n\eta(X)-(2n-1)\delta(X)+\delta\eta(X)(\xi)] \end{aligned} \tag{3.11}$$

Corollary 3.6. *There is no weakly symmetric δ- Lorentzian β- kenmotsu manifold $(M^{(2n+1)},g)$ $(n>1)$,unless the sum of the associated 1-forms is given by relation (3.11).*

If $\beta=0$, then (3.9) yields

$$A(X)+B(X)+D(X)=0 \tag{3.12}$$

for all X.This leads to the following:

Corollary 3.7. *There is no weakly symmetric cosympletic δ- Lorentzian β- kenmotsu manifold $(M^{(2n+1)},g)$ $(n>1)$,unless the sum of the associated 1-forms is everywhere zero.*

In the next section, we prove the sum of the associated 1- forms Weakly Ricci Symmetric δ-Lorentzian β-Kenmotsu manifold of non-vanishing ξ- sectional curvature is nonzero everywhere.

4. Weakly Ricci Symmetric δ-Lorentzian β-Kenmotsu manifolds

Definition 4.1. A δ-Lorentzian β-Kenmotsu manifold (M^{2n+1}, g) $(n > 1)$ is said to be weakly Ricci symmetric if its Ricci tensor of type (0, 2) is not identically zero and satisfies relation (1.3).

Theorem 4.2. *In a weakly Ricci symmetric δ- Lorentzian β- kenmotsu manifold $(M^{(2n+1)}, g)$ $(n > 1)$ of non-vanishing ξ- sectional curvature, thefollowing relations hold:*

$$A(\xi) + B(\xi) + C(\xi) = \frac{2\beta(\xi\beta) + \delta\xi(\xi\beta)}{\beta^2 + \delta(\xi\beta)} \tag{4.1}$$

$$\begin{aligned}[r - 2n\beta^2 - \delta(\xi\beta)][A(\xi) + B(\xi)] &= \frac{r(3\beta(\xi\beta) + \delta\xi(\xi\beta) + \delta\beta^3)}{\beta^2 + \delta\delta()\xi\beta} \\ &\quad - (6n + (2n+1)\delta - 1)\beta(\xi\beta) - \delta\xi(\xi\beta) - 2n(2n+1)\beta^3 \\ &\quad + (2n-1)\delta[div(grad.\beta) - (\rho_1\beta) - (\rho_2\beta)] \end{aligned} \tag{4.2}$$

where r is the scaler curvature of the manifold, div denotes the divergence,ρ_1, ρ_2 being the associated vector fields corresponding to the 1-form A and B, respectively.

Proof. From equation (1.3) it follows that

$$(\nabla_X S)(Y, \xi) = A(X)S(Y, \xi) + B(Y)S(X, \xi) + C(\xi)S(Y, X) \tag{4.3}$$

In view of (2.8) we obtain from (4.3)

$$\begin{aligned} &A(X)[2n\beta^2\eta(Y) - (2n-1)\delta(Y\beta) + \delta\eta(Y)(\xi\beta)] \\ &\quad + B(Y)[2n\beta^2\eta(X) - (2n-1)\delta(X\beta) + \delta\eta(X)(\xi\beta)] + C(\xi)S(Y, X) \\ &\quad = 4n\beta(X\beta)\eta(Y) - (2n-1)X(Y\beta)\delta + \delta X(\xi\beta)\eta(Y) + [2n\beta^3 + \delta\beta(\xi\beta)]g(X, Y) \\ &\qquad + (2n-1)[(\nabla_X Y\beta)\delta + \beta(Y\beta)\eta(X)] - \delta\beta S(Y, X) \end{aligned} \tag{4.4}$$

where (2.9) has been used. Setting $X = Y = \xi$ in (4.4) and then using (2.9) we obtain relation (4.1). Let $e_i, i = 1, 2..., (2n+1)$ be an orthonormal basis of the tangent space $T_P M$ at any point of the manifold. then setting $X = Y = e_i$ in (4.4) and taking summation over i, $1 \leq i \leq 2n+1$ and then using (2.8) we obtain

$$\begin{aligned} &[A(\xi) + B(\xi)](2n\beta^2 + \delta(\xi\beta)) - (2n-1)\delta[(\rho_1\beta) + (\rho_2\beta)] + rC(\xi) \\ &\quad = (6n + (2n+1)\delta - 1)\beta(\xi\beta) + \delta\xi(\xi\beta) + 2n(2n+1)\beta^3 \\ &\qquad - (2n-1)\mathrm{div}(\mathrm{grad}\beta)\delta - \delta\beta r \end{aligned} \tag{4.5}$$

where $r = \sum_{i=1}^{2n+1} S(e_i, e_i)$ eliminating $C(\xi)$ from (4.1) and (4.5) we obtain (4.2). This proves the theorem. □

5. Example of δ-Lorentzian β-Kenmotsu manifolds

We consider the 3-dim. manifold $M = (x, y, z) \in R^3 : Z \neq 0$, where (x, y, z) are the standard coordinates in R^3. Let e_1, e_2, e_3 be a linearly independent global frame on M given by

$$e_1 = e^{-z}\frac{\partial}{\partial y},\ e_2 = e^{-z}(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}),\ e_3 = \beta\frac{\partial}{\partial z}$$

Let g be the an indefinite metric defined by

$$g(e_1, e_1) = g(e_2, e_2) = 1,\ g(e_3, e_3) = -\delta$$
$$g(e_1, e_2) = g(e_2, e_3) = g(e_1, e_3) = 0$$

and the δ- Lorentzian metric g is thus given by

$$\begin{aligned} g &= g_{11}(dx)^2 + g_{22}(dy)^2 + g_{33}(dz)^2 + 2g_{12}dx \wedge dy \\ &= 2e^{2z}(dx)^2 + e^{2z}(dy)^2 - \frac{\delta}{\beta^2}(dz)^2 - 2e^{2z}dx \wedge dy \end{aligned}$$

$$(g_{ij}) = \begin{pmatrix} 2e^{2z} & -2e^{2z} & 0 \\ -e^{2z} & e^{2z} & 0 \\ 0 & 0 & \frac{\delta}{\beta^2} \end{pmatrix}$$

where $\delta = \pm 1$. If $\delta = -1$, then δ-Lorentzian metric g becomes a Riemannian positive definite metric on M so that in this case the characteristic vector field ξ becomes aspace like and if $\delta = 1$, Then it becomes a light like. Let η be the 1-form defined by

$$\eta(X) = \delta g(X, \xi)$$

for any vector field X on M^3. Let ϕ be the tensor field of type (1, 1) defined by

$$\phi(e_1) = -e_1,\ \phi(e_2) = -e_2,\ \phi(e_3) = 0$$

using the linearity property of g and ϕ, one can deduce

$$\phi^2 X = X + \eta(X)\xi,\ \eta(X) = -1,\ g(\xi, \xi) = -\delta$$
$$g(\phi X, \phi Y) = g(X, Y) + \delta\eta(X)\eta(Y).$$

Also, $\eta(e_1) = 0,\ \eta(e_2) = 0,\ \eta(e_3) = -1$ for any vector field X and Y on M. Let ∇ be the Levi-Civita connection with respect to g. Then we have

$$[e_1, e_2] = 0,\ [e_1, e_3] = \delta\beta e_1,\ [e_2, e_3] = \delta\beta e_2$$

Using Koszule's formula for Levi-Civita connection ∇ with respect to g, that is

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &\quad - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]) \end{aligned}$$

one can easily calculate

$$\nabla_{e_1} e_3 = \delta\beta e_1,\ \nabla_{e_3} e_3 = 0,\ \nabla_{e_2} e_3 = \delta\beta e_2$$
$$\nabla_{e_2} e_2 = -\delta\beta e_3,\ \nabla_{e_1} e_2 = 0,\ \nabla_{e_2} e_1 = 0$$
$$\nabla_{e_1} e_1 = \delta\beta e_3,\ \nabla_{e_3} e_2 = 0,\ \nabla_{e_3} e_1 = 0$$

with these information the structure $(\phi, \xi, \eta, g, \delta)$ satisfies (2.2) and (2.3). Hence $M^3(\phi, \xi, \eta, g, \delta)$ defines a δ-Lorentzian β-Kenmotsu manifold.

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