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CERTAIN COEFFICIENT INEQUALITIES FOR SAKAGUCHI TYPE FUNCTIONS AND APPLICATIONS TO FRACTIONAL DERIVATIVES

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Abstract. In the present paper, sharp upper bounds of $|a_3 - \mu a_2^2|$ for the functions $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ belonging to a new subclass of Sakaguchi type functions are obtained. Also, application of our results for subclass of functions defined by convolution with a normalized analytic function are given. In particular, Fekete-Szegö inequalities for certain classes of functions defined through fractional derivatives are obtained.

1. Introduction

Let $\mathcal{A}$ be the class of analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

defined on $\Delta := \{z : z \in C \text{ and } |z| < 1\}$ and $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent functions.

For two functions $f, g \in \mathcal{A}$, we say that the function $f(z)$ is subordinate to $g(z)$ in Δ and write $f \prec g$ or $f(z) \prec g(z)$, if there exists an anlytic function $\omega(z)$ with $\omega(0) = 0$ and $|\omega(z)| < 1$, $(z \in \Delta)$, such that $f(z) = g(\omega(z))$, $(z \in \Delta)$. In particular, if the function g is univalent in Δ, the above subordination is equivalent to $f(0) = g(0)$ and $f(\Delta) \subset g(\Delta)$.

A function $f(z) \in \mathcal{A}$ is said to be in the class $M(\alpha, \lambda, t)$ if it satisfies

$$Re\left\{(1-\lambda)\left[\frac{(1-t)zf'(z)}{f(z)-f(zt)}\right] + \lambda\left[\frac{(1-t)(z^2f''(z)+zf'(z))}{zf'(z)-tzf'(zt)}\right]\right\} > \alpha, \tag{1.2}$$

$|t| \le 1$, $t \ne 1$, $0 \le \lambda \le 1$, for some $\alpha \in [0, 1)$ and for all $z \in \Delta$.

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For, $\lambda = 0$, the function $f(z) \in \mathcal{A}$ reduces to the Sakaguchi type class $S^*(\alpha, t)$ which satisfies

$$Re\left[\frac{(1-t)zf'(z)}{f(z)-f(zt)}\right] > \alpha,$$

$|t| \leq 1$, $t \neq 1$, for some $\alpha \in [0,1)$ and for all $z \in \Delta$ was introduced and studied by Owa et al. [6, 7].

For $\lambda = 0$, $\alpha = 0$ and $t = -1$, we get the class $S^*(0,-1)$ studied by Sakaguchi [8]. A function $f(z) \in S^*(\alpha,-1)$ is called Sakaguchi function of order α.

For, $\lambda = 1$, the function $f(z) \in \mathcal{A}$ reduces to the class $T(\alpha, t)$ which satisfies

$$Re\left[\frac{(1-t)(z^2 f''(z) + zf'(z))}{zf'(z) - tzf'(zt)}\right] > \alpha,$$

$|t| \leq 1$, $t \neq 1$, for some $\alpha \in [0,1)$ and for all $z \in \Delta$.

In this paper, we define the following class $M(\phi, \lambda, t)$ which is the generalization of the class $M(\alpha, \lambda, t)$.

Definition 1.1. *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$ be univalent starlike function with respect to 1 which maps the unit disk Δ onto a region in the right half plane which is symmetric with respect to the real axis, and let $B_1 > 0$. The function $f \in \mathcal{A}$ is in the class $M(\phi, \lambda, t)$ if*

$$(1-\lambda)\left[\frac{(1-t)zf'(z)}{f(z)-f(zt)}\right] + \lambda\left[\frac{(1-t)(z^2 f''(z) + zf'(z))}{zf'(z) - tzf'(zt)}\right] \prec \phi(z), \qquad (1.3)$$

$|t| \leq 1$, $t \neq 1$, $0 \leq \lambda \leq 1$.
For $\lambda = 0$ and $\lambda = 1$ in (1.3) we obtain the classes $S^*(\phi, t)$ and $T(\phi, t)$ respectively which were studied by Goyal et al. [1].

In the present paper, we obtain the Fekete-Szegö inequality for the functions in the subclass $M(\phi, \lambda, t)$. We also give application of our results to certain functions defined through convolution (or Hadamard product) and in particular, we consider the class $M^\gamma(\phi, \lambda, t)$ defined by fractional derivatives.

To prove our main results, we need the following lemmas:

Lemma 1.2. [3] *If $p_1(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is an analytic function with positive real part in Δ, then*

$$|c_2 - v c_1^2| \leq \begin{cases} -4v + 2 & \text{if } v \leq 0, \\ 2 & \text{if } 0 \leq v \leq 1, \\ 4v - 2 & \text{if } v \geq 1. \end{cases}$$

When $v < 0$ or $v > 1$, the equality holds if and only if $p_1(z)$ is $(1+z)/(1-z)$ or one of its rotations. If $0 < v < 1$, then equality holds if and only if $p_1(z)$ is $(1+z^2)/(1-z^2)$ or one of its rotations. If $v = 0$, the equality holds if and only if

$$p_1(z) = \left(\frac{1}{2} + \frac{1}{2}\gamma\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{1}{2}\gamma\right)\frac{1-z}{1+z} \quad (0 \leq \gamma \leq 1)$$

or one of its rotations. If $v = 1$, the equality holds if and only if $p_1(z)$ is the reciprocal of one of the functions such that the equality holds in the case of $v = 0$. Also the above upper bound is sharp and it can be improved as follows when $0 < v < 1$:

$$|c_2 - v c_1^2| + v|c_1|^2 \leq 2 \quad (0 < v \leq 1/2)$$

and

$$|c_2 - vc_1^2| + (1-v)|c_1|^2 \leq 2 \quad (1/2 < v \leq 1).$$

Lemma 1.3. [2] *If $p(z) = 1 + c_1z + c_2z^2 + \cdots$ is a function with positive real part, then for any complex number μ,*

$$|c_2 - \mu c_1^2| \leq 2\max\{1, |2\mu - 1|\}$$

and the result is sharp for the functions given by

$$p(z) = \frac{1+z^2}{1-z^2}, \quad p(z) = \frac{1+z}{1-z}.$$

2. Main Results

Our main result is the following:

Theorem 2.1. *If $f(z)$ given by (1.1) belongs to $M(\phi, \lambda, t)$, then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{1}{(1+2\lambda)(1-t)(2+t)}\left[B_2 + B_1^2\left(\frac{1+t}{1-t}\right)\frac{(1+3\lambda)}{(1+\lambda)^2} - \mu B_1^2\left(\frac{2+t}{1-t}\right)\frac{(1+2\lambda)}{(1+\lambda)^2}\right] & \text{if } \mu \leq \sigma_1, \\ \frac{B_1}{(1+2\lambda)(1-t)(2+t)} & \text{if } \sigma_1 \leq \mu \leq \sigma_2, \\ -\frac{1}{(1+2\lambda)(1-t)(2+t)}\left[B_2 + B_1^2\left(\frac{1+t}{1-t}\right)\frac{(1+3\lambda)}{(1+\lambda)^2} - \mu B_1^2\left(\frac{2+t}{1-t}\right)\frac{(1+2\lambda)}{(1+\lambda)^2}\right] & \text{if } \mu \geq \sigma_2 \end{cases}$$

where

$$\sigma_1 := \frac{(1-t)(1+\lambda)^2}{B_1(2+t)(1+2\lambda)}\left[-1 + \frac{B_2}{B_1} + B_1\left(\frac{1+t}{1-t}\right)\frac{(1+3\lambda)}{(1+\lambda)^2}\right]$$

and

$$\sigma_2 := \frac{(1-t)(1+\lambda)^2}{B_1(2+t)(1+2\lambda)}\left[1 + \frac{B_2}{B_1} + B_1\left(\frac{1+t}{1-t}\right)\frac{(1+3\lambda)}{(1+\lambda)^2}\right]$$

The result is sharp.

Proof. Let $f \in M(\phi, \lambda, t)$. Then there exists a Schwarz function $\omega(z) \in \mathcal{A}$ such that

$$(1-\lambda)\left[\frac{(1-t)zf'(z)}{f(z)-f(zt)}\right] + \lambda\left[\frac{(1-t)(z^2f''(z)+zf'(z))}{zf'(z)-tzf'(zt)}\right] = \phi(\omega(z)) \tag{2.1}$$

$(z \in \Delta; |t| \leq 1, t \neq 1)$.

If $p_1(z)$ is analytic and has positive real part in Δ and $p_1(0) = 1$, then

$$p_1(z) = \frac{1+\omega(z)}{1-\omega(z)} = 1 + c_1z + c_2z^2 + \cdots \quad (z \in \Delta). \tag{2.2}$$

From (2.2), we obtain

$$\omega(z) = \frac{c_1}{2}z + \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right)z^2 + \cdots \tag{2.3}$$

Let

$$\begin{aligned} p(z) &= (1-\lambda)\left[\frac{(1-t)zf'(z)}{f(z)-f(zt)}\right] + \lambda\left[\frac{(1-t)(z^2f''(z)+zf'(z))}{zf'(z)-tzf'(zt)}\right] \\ &= 1 + b_1z + b_2z^2 + \cdots \quad (z \in \Delta), \end{aligned} \tag{2.4}$$

which gives

$$b_1 = a_2(1-t)(1+\lambda) \tag{2.5}$$

and

$$b_2 = a_2^2(t^2-1)(1+3\lambda) + a_3(2-t-t^2)(1+2\lambda) \tag{2.6}$$

Since $\phi(z)$ is univalent and $p \prec \phi$, therefore using (2.3), we obtain

$$p(z) = \phi(\omega(z)) = 1 + \frac{B_1 c_1}{2} z + \left[\frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right) B_1 + \frac{1}{4} c_1^2 B_2\right] z^2 + \cdots \quad (z \in \Delta) \tag{2.7}$$

Now from (2.4), (2.5), (2.6) and (2.7), we have,

$$a_2(1-t)(1+\lambda) = \frac{B_1 c_1}{2}, \tag{2.8}$$

$$\begin{aligned} & a_2^2(t^2-1)(1+3\lambda) + a_3(2-t-t^2)(1+2\lambda) \\ & = \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right) B_1 + \frac{1}{4} c_1^2 B_2, \quad |t| \leq 1, t \neq 1 \end{aligned} \tag{2.9}$$

Therefore, we have

$$a_3 - \mu a_2^2 = \frac{B_1}{2(1+2\lambda)(1-t)(2+t)}[c_2 - v c_1^2] \tag{2.10}$$

where

$$v := \frac{1}{2}\left[1 - \frac{B_2}{B_1} - B_1\left(\frac{1+t}{1-t}\right)\frac{(1+3\lambda)}{(1+\lambda)^2} + \mu B_1\left(\frac{2+t}{1-t}\right)\frac{(1+2\lambda)}{(1+\lambda)^2}\right]$$

Our result now follows by an application of Lemma 1.2. To show that these bounds are sharp, we define the functions K_δ^ϕ $(\delta = 2, 3, \dots)$ by

$$\begin{aligned} & (1-\lambda)\left[\frac{(1-t)z(K_\delta^\phi(z))'}{K_\delta^\phi(z) - K_\delta^\phi(zt)}\right] + \lambda\left[\frac{(1-t)(z^2(K_\delta^\phi(z))'' + z(K_\delta^\phi(z))')}{z(K_\delta^\phi(z))' - tz(K_\delta^\phi(zt))'}\right] \\ & = \phi(z^{\delta-1}), \quad K_\delta^\phi(0) = 0 = (K_\delta^\phi(0))' - 1 \end{aligned}$$

and the function F_γ and G_γ $(0 \leq \gamma \leq 1)$ by

$$\begin{aligned} & (1-\lambda)\left[\frac{(1-t)z(F_\gamma(z))'}{F_\gamma(z) - F_\gamma(zt)}\right] + \lambda\left[\frac{(1-t)(z^2(F_\gamma(z))'' + z(F_\gamma(z))')}{z(F_\gamma(z))' - tz(F_\gamma(zt))'}\right] \\ & = \phi\left(\frac{z(z+\gamma)}{1+\gamma z}\right), \quad F_\gamma(0) = 0 = (F_\gamma(0))' - 1 \end{aligned}$$

and

$$\begin{aligned} & (1-\lambda)\left[\frac{(1-t)z(G_\gamma(z))'}{G_\gamma(z) - G_\gamma(zt)}\right] + \lambda\left[\frac{(1-t)(z^2(G_\gamma(z))'' + z(G_\gamma(z))')}{z(G_\gamma(z))' - tz(G_\gamma(zt))'}\right] \\ & = \phi\left(\frac{-z(z+\gamma)}{1+\gamma z}\right), \quad G_\gamma(0) = 0 = (G_\gamma(0))' - 1 \end{aligned}$$

Obviously, the functions K_δ^ϕ, F_γ, $G_\gamma \in M(\phi, \lambda, t)$. Also, we write $K^\phi := K_2^\phi$. If $\mu < \sigma_1$ or $\mu > \sigma_2$, then equality holds if and only if f is K^ϕ or one of its rotations. When $\sigma_1 < \mu < \sigma_2$, then equality holds if and only if f is K_3^ϕ or one of its rotations. If $\mu = \sigma_1$, then equality holds if and only if f is F_γ or one of its rotations. If $\mu = \sigma_2$, then equality holds if and only if f is G_γ or one of its rotations.

If $\sigma_1 \leq \mu \leq \sigma_2$, in view of Lemma 1.2, Theorem 2.1 can be improved.

Let $f(z)$ given by (1.1) belongs to $M(\phi,\lambda,t)$ and σ_3 be given by

$$\sigma_3 := \frac{(1-t)(1+\lambda)^2}{B_1(2+t)(1+2\lambda)}\left[\frac{B_2}{B_1}+B_1\left(\frac{1+t}{1-t}\right)\frac{(1+3\lambda)}{(1+\lambda)^2}\right]$$

If $\sigma_1<\mu\le\sigma_3$, then

$$\begin{aligned}&|a_3-\mu a_2^2|\\ &\quad+\frac{1}{B_1^2}\left[(B_1-B_2)\left(\frac{1-t}{2+t}\right)\frac{(1+\lambda)^2}{(1+2\lambda)}-B_1^2\left(\frac{1+t}{2+t}\right)\frac{(1+3\lambda)}{(1+2\lambda)}+\mu B_1^2\right]|a_2|^2\\ &\quad\le\frac{B_1}{(1+2\lambda)(1-t)(2+t)}.\end{aligned}$$

If $\sigma_3<\mu\le\sigma_2$, then

$$\begin{aligned}&|a_3-\mu a_2^2|\\ &\quad+\frac{1}{B_1^2}\left[(B_1+B_2)\left(\frac{1-t}{2+t}\right)\frac{(1+\lambda)^2}{(1+2\lambda)}+B_1^2\left(\frac{1+t}{2+t}\right)\frac{(1+3\lambda)}{(1+2\lambda)}-\mu B_1^2\right]|a_2|^2\\ &\quad\le\frac{B_1}{(1+2\lambda)(1-t)(2+t)}.\end{aligned}$$

□

Corollary 2.2. *For $\lambda=1$ in Theorem 2.1, $f(z)$ given by (1.1) belongs to $T(\phi,t)$, then*

$$|a_3-\mu a_2^2| \le \begin{cases}\frac{1}{3(1-t)(2+t)}\left[B_2+B_1^2\left(\frac{1+t}{1-t}\right)-\frac{3}{4}\mu B_1^2\left(\frac{2+t}{1-t}\right)\right] & \text{if } \mu\le\sigma_1,\\ \frac{B_1}{3(1-t)(2+t)} & \text{if } \sigma_1\le\mu\le\sigma_2,\\ -\frac{1}{3(1-t)(2+t)}\left[B_2+B_1^2\left(\frac{1+t}{1-t}\right)-\frac{3}{4}\mu B_1^2\left(\frac{2+t}{1-t}\right)\right] & \text{if } \mu\ge\sigma_2\end{cases}$$

where

$$\sigma_1 := \frac{4(1-t)}{3B_1(2+t)}\left[-1+\frac{B_2}{B_1}+B_1\left(\frac{1+t}{1-t}\right)\right]$$

and

$$\sigma_2 := \frac{4(1-t)}{3B_1(2+t)}\left[1+\frac{B_2}{B_1}+B_1\left(\frac{1+t}{1-t}\right)\right]$$

The result is sharp.
Also σ_3 is given by

$$\sigma_3 := \frac{4(1-t)}{3B_1(2+t)}\left[\frac{B_2}{B_1}+B_1\left(\frac{1+t}{1-t}\right)\right]$$

If $\sigma_1<\mu\le\sigma_3$, then

$$\begin{aligned}&|a_3-\mu a_2^2|\\ &\quad+\frac{1}{B_1^2}\left[(B_1-B_2)\frac{4}{3}\left(\frac{1-t}{2+t}\right)-B_1^2\frac{4}{3}\left(\frac{1+t}{2+t}\right)+\mu B_1^2\right]|a_2|^2\\ &\quad\le\frac{B_1}{3(1-t)(2+t)}.\end{aligned}$$

If $\sigma_3 < \mu \le \sigma_2$, *then*

$$\begin{aligned} &|a_3 - \mu a_2^2| \\ &\quad + \frac{1}{B_1^2}\left[(B_1+B_2)\frac{4}{3}\left(\frac{1-t}{2+t}\right) + B_1^2\frac{4}{3}\left(\frac{1+t}{2+t}\right) - \mu B_1^2\right]|a_2|^2 \\ &\quad \le \frac{B_1}{3(1-t)(2+t)}. \end{aligned}$$

Remark 2.3. *When* $\lambda = 0$ *in Theorem 2.1,* $f(z)$ *given by (1.1) belongs to* $S^*(\phi, t)$ *and the result coincides with a recent result of Goyal et al.* [1].

Theorem 2.4. *Let* $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$. *If* $f(z)$ *given by (1.1) belongs to* $M(\phi, \lambda, t)$, *then*

$$\begin{aligned} |a_3 - \mu a_2^2| &\le \frac{B_1}{(1+2\lambda)(1-t)(2+t)} \\ &\max\left\{1, \left|\frac{B_2}{B_1} + B_1\left(\frac{1+t}{1-t}\right)\frac{(1+3\lambda)}{(1+\lambda)^2} - \mu B_1\left(\frac{2+t}{1-t}\right)\frac{(1+2\lambda)}{(1+\lambda)^2}\right|\right\} \end{aligned} \tag{2.11}$$

This result is sharp.

Proof. By applying Lemma 1.3 in (2.10), we obtain the result (2.11).

The result (2.11) is sharp for the function defined by

$$(1-\lambda)\left[\frac{(1-t)zf'(z)}{f(z)-f(zt)}\right] + \lambda\left[\frac{(1-t)(z^2 f''(z) + zf'(z))}{zf'(z) - tzf'(zt)}\right] = \phi(z^2)$$

and

$$(1-\lambda)\left[\frac{(1-t)zf'(z)}{f(z)-f(zt)}\right] + \lambda\left[\frac{(1-t)(z^2 f''(z) + zf'(z))}{zf'(z) - tzf'(zt)}\right] = \phi(z)$$

□

Corollary 2.5. *Let* $\lambda = 0$ *in Theorem 2.4,* $f(z)$ *given by (1.1) belongs to* $S^*(\phi, t)$. *Let* $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$, *then*

$$|a_3 - \mu a_2^2| \le \frac{B_1}{(1-t)(2+t)}\max\left\{1, \left|\frac{B_2}{B_1} + B_1\left(\frac{1+t}{1-t}\right) - \mu B_1\left(\frac{2+t}{1-t}\right)\right|\right\}$$

This result is sharp.

Corollary 2.6. *Let* $\lambda = 1$ *in Theorem 2.4. Let* $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$ *and* $f(z)$ *given by (1.1) belongs to* $T(\phi, t)$, *then*

$$|a_3 - \mu a_2^2| \le \frac{B_1}{3(1-t)(2+t)}\max\left\{1, \left|\frac{B_2}{B_1} + B_1\left(\frac{1+t}{1-t}\right) - \mu B_1 \frac{3}{4}\left(\frac{2+t}{1-t}\right)\right|\right\}$$

This result is sharp.

3. Applications to Functions defined by Fractional Derivatives

For two analytic functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} g_n z^n$, their convolution (or Hadamard product) is defined to be the function $(f * g)z = z + \sum_{n=2}^{\infty} a_n g_n z^n$. For a fixed $g \in \mathcal{A}$, let $M^g(\phi, \lambda, t)$, $S^g(\phi, t)$, $T^g(\phi, t)$

be the classes of functions $f \in \mathcal{A}$ for which $f * g$ belongs to $M(\phi, \lambda, t)$, $S^*(\phi, t)$ and $T(\phi, t)$ respectively.

Definition 3.1. *Let $f(z)$ be analytic in a simply connected region of the z-plane containing the origin. The fractional derivative of f of order γ is defined by*

$$D_z^{\gamma} f(z) = \frac{1}{\Gamma(1-\gamma)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\gamma}} d\zeta \quad (0 \leq \gamma < 1) \tag{3.1}$$

where the multiplicity of $(z-\zeta)^{-\gamma}$ is removed by requiring that $log(z-\zeta)$ is real for $(z-\zeta) > 0$.

Using definition 3.1, Owa and Srivastava (see [4, 5]; see also [9, 10]) introduced a fractional derivative operator $\Omega^{\gamma} : \mathcal{A} \to \mathcal{A}$ defined by $(\Omega^{\gamma} f)(z) = \Gamma(2-\gamma) z^{\gamma} D_z^{\gamma} f(z)$, $(\gamma \neq 2, 3, 4, \dots)$. The classes $M^{\gamma}(\phi, \lambda, t)$, $S^{\gamma}(\phi, t)$, $T^{\gamma}(\phi, t)$ consists of functions $f \in \mathcal{A}$ for which $\Omega^{\gamma} f$ belongs to $M(\phi, \lambda, t)$, $S^*(\phi, t)$ and $T(\phi, t)$ respectively. The classes $M^{\gamma}(\phi, \lambda, t)$, $S^{\gamma}(\phi, t)$, $T^{\gamma}(\phi, t)$ is a special case of the classes $M^g(\phi, \lambda, t)$, $S^g(\phi, t)$ and $T^g(\phi, t)$ respectively, when

$$g(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\gamma)}{\Gamma(n+1-\gamma)} z^n \quad (z \in \Delta).$$

The classes $S^g(\phi, t)$ and $S^{\gamma}(\phi, t)$ were studied by Goyal et al. [1].

Now applying Theorem 2.1 for the function $(f * g)(z) = z + a_2 g_2 z^2 + a_3 g_3 z^3 + \cdots$ we get the following theorem after an obvious change of the parameter μ:

Theorem 3.2. *Let $g(z) = z + \sum_{n=2}^{\infty} g_n z^n$ $(g_n > 0)$. If $f(z)$ given by (1.1) belongs to $M^g(\phi, \lambda, t)$, then,*

$$|a_3 - \mu a_2^2|$$

$$\leq \begin{cases} \frac{1}{g_3(1+2\lambda)(1-t)(2+t)} \left[B_2 + B_1^2 \left(\frac{1+t}{1-t}\right) \frac{(1+3\lambda)}{(1+\lambda)^2} - \mu B_1^2 \frac{g_3}{g_2^2} \left(\frac{2+t}{1-t}\right) \frac{(1+2\lambda)}{(1+\lambda)^2} \right] & \text{if } \mu \leq \eta_1, \\ \frac{B_1}{g_3(1+2\lambda)(1-t)(2+t)} & \text{if } \eta_1 \leq \mu \leq \eta_2, \\ -\frac{1}{g_3(1+2\lambda)(1-t)(2+t)} \left[B_2 + B_1^2 \left(\frac{1+t}{1-t}\right) \frac{(1+3\lambda)}{(1+\lambda)^2} - \mu B_1^2 \frac{g_3}{g_2^2} \left(\frac{2+t}{1-t}\right) \frac{(1+2\lambda)}{(1+\lambda)^2} \right] & \text{if } \mu \geq \eta_2 \end{cases}$$

where

$$\eta_1 := \frac{g_2^2(1-t)(1+\lambda)^2}{g_3 B_1(2+t)(1+2\lambda)} \left[-1 + \frac{B_2}{B_1} + B_1 \left(\frac{1+t}{1-t}\right) \frac{(1+3\lambda)}{(1+\lambda)^2} \right]$$

and

$$\eta_2 := \frac{g_2^2(1-t)(1+\lambda)^2}{g_3 B_1(2+t)(1+2\lambda)} \left[1 + \frac{B_2}{B_1} + B_1 \left(\frac{1+t}{1-t}\right) \frac{(1+3\lambda)}{(1+\lambda)^2} \right]$$

The result is sharp.

Since $\Omega^{\gamma} f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\gamma)}{\Gamma(n+1-\gamma)} z^n$,

we have

$$g_2 := \frac{\Gamma(3)\Gamma(2-\gamma)}{\Gamma(3-\gamma)} = \frac{2}{2-\gamma} \tag{3.2}$$

and

$$g_3 := \frac{\Gamma(4)\Gamma(2-\gamma)}{\Gamma(4-\gamma)} = \frac{6}{(2-\gamma)(3-\gamma)} \tag{3.3}$$

For g_2, g_3 given by (3.2) and (3.3) respectively, Theorem 3.2 reduces to the following:

Theorem 3.3. *Let $\gamma < 2$. If $f(z)$ given by (1.1) belongs to $M^\gamma(\phi, \lambda, t)$, then*

$$|a_3 - \mu a_2^2|$$
$$\leq \begin{cases} \frac{(2-\gamma)(3-\gamma)}{6(1+2\lambda)(1-t)(2+t)} \left[B_2 + B_1^2 \left(\frac{1+t}{1-t}\right) \frac{(1+3\lambda)}{(1+\lambda)^2} - \frac{3}{2}\mu B_1^2 \frac{(2-\gamma)}{(3-\gamma)} \left(\frac{2+t}{1-t}\right) \frac{(1+2\lambda)}{(1+\lambda)^2}\right] & \text{if } \mu \leq \eta_1^*, \\ \frac{B_1(2-\gamma)(3-\gamma)}{6(1+2\lambda)(1-t)(2+t)} & \text{if } \eta_1^* \leq \mu \leq \eta_2^*, \\ -\frac{(2-\gamma)(3-\gamma)}{6(1+2\lambda)(1-t)(2+t)} \left[B_2 + B_1^2 \left(\frac{1+t}{1-t}\right) \frac{(1+3\lambda)}{(1+\lambda)^2} - \frac{3}{2}\mu B_1^2 \frac{(2-\gamma)}{(3-\gamma)} \left(\frac{2+t}{1-t}\right) \frac{(1+2\lambda)}{(1+\lambda)^2}\right] & \text{if } \mu \geq \eta_2^*, \end{cases}$$

where

$$\eta_1^* := \frac{2(3-\gamma)(1-t)(1+\lambda)^2}{3(2-\gamma)B_1(2+t)(1+2\lambda)} \left[-1 + \frac{B_2}{B_1} + B_1 \left(\frac{1+t}{1-t}\right) \frac{(1+3\lambda)}{(1+\lambda)^2}\right]$$

and

$$\eta_2^* := \frac{2(3-\gamma)(1-t)(1+\lambda)^2}{3(2-\gamma)B_1(2+t)(1+2\lambda)} \left[1 + \frac{B_2}{B_1} + B_1 \left(\frac{1+t}{1-t}\right) \frac{(1+3\lambda)}{(1+\lambda)^2}\right]$$

The result is sharp.

Corollary 3.4. *For $\lambda = 1$ in Theorem 3.2, $f(z)$ given by (1.1) belongs to $T^g(\phi, t)$, then*

$$|a_3 - \mu a_2^2|$$
$$\leq \begin{cases} \frac{1}{3g_3(1-t)(2+t)} \left[B_2 + B_1^2 \left(\frac{1+t}{1-t}\right) - \frac{3}{4}\mu B_1^2 \frac{g_3}{g_2^2} \left(\frac{2+t}{1-t}\right)\right] & \text{if } \mu \leq \eta_1, \\ \frac{B_1}{3g_3(1-t)(2+t)} & \text{if } \eta_1 \leq \mu \leq \eta_2, \\ -\frac{1}{3g_3(1-t)(2+t)} \left[B_2 + B_1^2 \left(\frac{1+t}{1-t}\right) - \frac{3}{4}\mu B_1^2 \frac{g_3}{g_2^2} \left(\frac{2+t}{1-t}\right)\right] & \text{if } \mu \geq \eta_2, \end{cases}$$

where

$$\eta_1 := \frac{4g_2^2(1-t)}{3g_3B_1(2+t)} \left[-1 + \frac{B_2}{B_1} + B_1 \left(\frac{1+t}{1-t}\right)\right]$$

and

$$\eta_2 := \frac{4g_2^2(1-t)}{3g_3B_1(2+t)} \left[1 + \frac{B_2}{B_1} + B_1 \left(\frac{1+t}{1-t}\right)\right]$$

The result is sharp.

Remark 3.5. *When $\lambda = 0$ in Theorem 3.2, $f(z)$ given by (1.1) belongs to $S^g(\phi, t)$ and the result coincides with a result of Goyal et al.* [1].

Corollary 3.6. *For $\lambda = 1$ in Theorem 3.3, $f(z)$ given by (1.1) belongs to $T^{\gamma}(\phi, t)$, then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{(2-\gamma)(3-\gamma)}{18(1-t)(2+t)}\left[B_2 + B_1^2\left(\frac{1+t}{1-t}\right) - \frac{9}{8}\mu B_1^2\frac{(2-\gamma)}{(3-\gamma)}\left(\frac{2+t}{1-t}\right)\right] & \text{if } \mu \leq \eta_1^*, \\ \frac{B_1(2-\gamma)(3-\gamma)}{18(1-t)(2+t)} & \text{if } \eta_1^* \leq \mu \leq \eta_2^*, \\ -\frac{(2-\gamma)(3-\gamma)}{18(1-t)(2+t)}\left[B_2 + B_1^2\left(\frac{1+t}{1-t}\right) - \frac{9}{8}\mu B_1^2\frac{(2-\gamma)}{(3-\gamma)}\left(\frac{2+t}{1-t}\right)\right] & \text{if } \mu \geq \eta_2^*, \end{cases}$$

where

$$\eta_1^* := \frac{8}{9}\left(\frac{3-\gamma}{2-\gamma}\right)\frac{(1-t)}{B_1(2+t)}\left[-1 + \frac{B_2}{B_1} + B_1\left(\frac{1+t}{1-t}\right)\right]$$

and

$$\eta_2^* := \frac{8}{9}\left(\frac{3-\gamma}{2-\gamma}\right)\frac{(1-t)}{B_1(2+t)}\left[1 + \frac{B_2}{B_1} + B_1\left(\frac{1+t}{1-t}\right)\right]$$

The result is sharp.

Remark 3.7. *For $\lambda = 0$ in Theorem 3.3, $f(z)$ given by (1.1) belongs to $S^{\gamma}(\phi, t)$ and the result coincides with a result of Goyal et al.* [1].

Theorem 3.8. *Let $g(z) = z + \sum_{n=2}^{\infty} g_n z^n$ $(g_n > 0)$ and let the function $\phi(z)$ be given by $\phi(z) = 1 + \sum_{n=1}^{\infty} B_n z^n$. If $f(z)$ given by (1.1) belongs to $M^g(\phi, \lambda, t)$, then*

$$|a_3 - \mu a_2^2| \leq \frac{B_1}{g_3(1+2\lambda)(1-t)(2+t)} \max\left\{1, \left|\frac{B_2}{B_1} + B_1\left(\frac{1+t}{1-t}\right)\frac{(1+3\lambda)}{(1+\lambda)^2} - \mu B_1\frac{g_3}{g_2^2}\left(\frac{2+t}{1-t}\right)\frac{(1+2\lambda)}{(1+\lambda)^2}\right|\right\}$$

The proof of Theorem 3.8 is similar to Theorem 2.4, so the details are omitted.

Corollary 3.9. *Let $\lambda = 0$ in Theorem 3.8, $f(z)$ given by (1.1) belongs to $S^g(\phi, t)$, then*

$$|a_3 - \mu a_2^2| \leq \frac{B_1}{g_3(1-t)(2+t)} \max\left\{1, \left|\frac{B_2}{B_1} + B_1\left(\frac{1+t}{1-t}\right) - \mu B_1\frac{g_3}{g_2^2}\left(\frac{2+t}{1-t}\right)\right|\right\}$$

Corollary 3.10. *Let $\lambda = 1$ in Theorem 3.8, $f(z)$ given by (1.1) belongs to $T^g(\phi, t)$, then*

$$|a_3 - \mu a_2^2| \leq \frac{B_1}{3g_3(1-t)(2+t)} \max\left\{1, \left|\frac{B_2}{B_1} + B_1\left(\frac{1+t}{1-t}\right) - \frac{3}{4}\mu B_1\frac{g_3}{g_2^2}\left(\frac{2+t}{1-t}\right)\right|\right\}$$

For g_2, g_3 given by (3.2) and (3.3) respectively, Theorem 3.8 reduces to the following:

Theorem 3.11. *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$. If $f(z)$ given by (1.1) belongs to $M^{\gamma}(\phi, \lambda, t)$, then*

$$|a_3 - \mu a_2^2| \leq \frac{B_1(2-\gamma)(3-\gamma)}{6(1+2\lambda)(1-t)(2+t)}$$
$$\max\left\{1, \left|\frac{B_2}{B_1} + B_1\left(\frac{1+t}{1-t}\right)\frac{(1+3\lambda)}{(1+\lambda)^2} - \frac{3}{2}\mu B_1\left(\frac{2-\gamma}{3-\gamma}\right)\left(\frac{2+t}{1-t}\right)\frac{(1+2\lambda)}{(1+\lambda)^2}\right|\right\}$$

The result is sharp.

Corollary 3.12. *Let $\lambda = 0$ in Theorem 3.11, $f(z)$ given by (1.1) belongs to $S^{\gamma}(\phi, t)$, then*

$$|a_3 - \mu a_2^2| \leq \frac{B_1(2-\gamma)(3-\gamma)}{6(1-t)(2+t)}$$
$$\max\left\{1, \left|\frac{B_2}{B_1} + B_1\left(\frac{1+t}{1-t}\right) - \frac{3}{2}\mu B_1\left(\frac{2-\gamma}{3-\gamma}\right)\left(\frac{2+t}{1-t}\right)\right|\right\}.$$

Corollary 3.13. *Let $\lambda = 1$ in Theorem 3.11, $f(z)$ given by (1.1) belongs to $T^{\gamma}(\phi, t)$, then*

$$|a_3 - \mu a_2^2| \leq \frac{B_1(2-\gamma)(3-\gamma)}{18(1-t)(2+t)}$$
$$\max\left\{1, \left|\frac{B_2}{B_1} + B_1\left(\frac{1+t}{1-t}\right) - \frac{9}{8}\mu B_1\left(\frac{2-\gamma}{3-\gamma}\right)\left(\frac{2+t}{1-t}\right)\right|\right\}.$$

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