

THE ONE-DIMENSIONAL HEAT EQUATION AS A
FIRST-ORDER SYSTEM :
FORMAL SOLUTIONS BY MEANS OF THE LAPLACE
TRANSFORM

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ABSTRACT. In this paper an extended heat equation problem as a linear first-order system of partial differential equations is considered. The classical problems in a strip S are assigned to our problems. Formal solutions are given by one-dimensional Laplace transform.

1. THE PROBLEM

We consider a first-order system of partial differential equations for two real valued functions $u(x, t), v(x, t)$ and given functions $f_i(x, t)$:

$$\begin{aligned} u_x &= v + f_1(x, t), & u, v &\in \mathbf{C}^1(\mathbb{R}_+ \times \mathbb{R}_+), \\ u_t - v_x &= f_2(x, t), & f_i &\in \mathbf{C}(\mathbb{R}_+ \times \mathbb{R}_+), i = 1, 2. \end{aligned} \quad (1)$$

In the case $f_1(x, t) \not\equiv 0$ this system is a more general problem as the conventional one-dimensional heat equation [1, 12] with the inhomogeneous term $f_2(x, t)$ and the solution $u(x, t)$. We will see this below, when we look at the formal solutions. We are looking for solutions of (1) in the strip $S = (0, l) \times \mathbb{R}_+$ of the first quadrant of the (x, t) -plane, $l \in \mathbb{R}_+$.

Just like in the classical case of the heat equation [6, 8] we consider in S the four

Problems \mathbf{P}^{ij} :

Give a solution $[u, v]^T$ of (1) in the domain $0 < x < l, 0 < t < \infty$, with :
 u satisfies $\lim_{t \rightarrow +0} u(x, t) = f(x), 0 < x < l, f(x) \in \mathbf{C}(0, l)$, and
for a fixed $\mathbf{P}^{ij}, i, j \in \{1, 2\}$, the solution satisfies

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$$\begin{aligned}
 \mathbf{P}^{11} : \quad & \lim_{x \rightarrow +0} u(x, t) = u_0(t), & \lim_{x \rightarrow l-0} u(x, t) = u_l(t), \\
 \mathbf{P}^{22} : \quad & \lim_{x \rightarrow +0} v(x, t) = v_0(t), & \lim_{x \rightarrow l-0} v(x, t) = v_l(t), \\
 \mathbf{P}^{12} : \quad & \lim_{x \rightarrow +0} u(x, t) = u_0(t), & \lim_{x \rightarrow l-0} v(x, t) = v_l(t), \\
 \mathbf{P}^{21} : \quad & \lim_{x \rightarrow +0} v(x, t) = v_0(t), & \lim_{x \rightarrow l-0} u(x, t) = u_l(t),
 \end{aligned} \tag{2}$$

where f and the functions $u_0, u_l, v_0, v_l \in \mathbf{C}(\mathbb{R}_+)$ are given.

2. THE PROBLEM IN THE IMAGE RANGE, THE SOLUTION IN THE IMAGE RANGE

We now assume that the Laplace transform exists for all quantities which we are using. We transform the problems (1), (2) from the t -domain (originals) into the s -range (images) by Laplace transform:

Let

$$L[u(x, t); t](s) = \int_0^\infty e^{-st} u(x, t) dt = w_1(x, s), \text{ shortly } u(x, t) \circ \rightarrow \bullet w_1(x, s).$$

The transformation of the other quantities from (1), (2) into the s -range is described in the following manner :

$$\begin{aligned}
 v(x, t) \circ \rightarrow \bullet w_2(x, s), & & v_x(x, t) \circ \rightarrow \bullet w_{2,x}(x, s), \\
 u_t(x, t) \circ \rightarrow \bullet s \cdot w_1(x, s) - f(x), & & u_x(x, t) \circ \rightarrow \bullet w_{1,x}(x, s), \\
 v_0(t) \circ \rightarrow \bullet w_2^0(s), & & v_l(t) \circ \rightarrow \bullet w_2^l(s), \\
 u_0(t) \circ \rightarrow \bullet w_1^0(s), & & u_l(t) \circ \rightarrow \bullet w_1^l(s), \\
 f_i(x, t) \circ \rightarrow \bullet \varphi_i(x, s), \quad i = 1, 2.
 \end{aligned} \tag{3}$$

In the s -range we get the following system of ordinary differential equations

$$\begin{bmatrix} w_{1,x} \\ w_{2,x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ s & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + \begin{bmatrix} \varphi_1(x, s) \\ -f(x) - \varphi_2(x, s) \end{bmatrix} \stackrel{Df}{=} \begin{bmatrix} 0 & 1 \\ s & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + \begin{bmatrix} r_1(x, s) \\ r_2(x, s) \end{bmatrix}. \tag{4}$$

Considering (2) we are now looking for solutions of the four boundary value problems \mathbf{P}^{ij} (4), (4a) in the s -range:

$$\begin{aligned}
\Pi^{11} &: w_1(0, s) = w_1^0(s), \quad w_1(l, s) = w_1^l(s), \\
\Pi^{22} &: w_2(0, s) = w_2^0(s), \quad w_2(l, s) = w_2^l(s), \\
\Pi^{12} &: w_1(0, s) = w_1^0(s), \quad w_2(l, s) = w_2^l(s), \\
\Pi^{21} &: w_2(0, s) = w_2^0(s), \quad w_1(l, s) = w_1^l(s).
\end{aligned} \tag{4a}$$

The functions $w_i^0(s), w_i^l(s), i = 1, 2$, are well-known from (3).

We can write problem Π^{11} using the "boundary matrices" B_0^{11}, B_l^{11} and the right-hand side β^{11} :

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w_1(0, s) \\ w_2(0, s) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} w_1(l, s) \\ w_2(l, s) \end{bmatrix} \stackrel{Df}{=} B_0^{11}w(0, s) + B_l^{11}w(l, s) = \begin{bmatrix} w_1^0(s) \\ w_1^l(s) \end{bmatrix} \stackrel{Df}{=} \beta^{11}.$$

Similarly we rewrite the remaining problems Π^{ij} from (4a) which have the corresponding boundary matrices B_0^{ij}, B_l^{ij} and the right-hand sides β^{ij} .

A particular solution of the initial value problem for a first-order system of linear inhomogeneous ordinary differential equations with constant coefficients can be found by the method referred to as "Variation of the Parameters".

For boundary value problems of the form (4),(4a) (linear inhomogeneous ordinary first-order systems with constant coefficients and totally separated boundary conditions) the method works too [4]. This is used in the following.

To construct the columns of the fundamental matrix \mathcal{W} which contains the principal solutions of the homogeneous system from (4) we make use of the eigenvalues $\lambda_1 = \sqrt{s}, \lambda_2 = -\sqrt{s}$ of the coefficient matrix of the homogeneous system from (4) and of the corresponding eigenvectors

$$e_1 = [1, \sqrt{s}]^T, \quad e_2 = [-1, \sqrt{s}]^T :$$

$$\mathcal{W}(x, s) = \begin{bmatrix} e^{\sqrt{s}x} & -e^{-\sqrt{s}x} \\ \sqrt{s}e^{\sqrt{s}x} & \sqrt{s}e^{-\sqrt{s}x} \end{bmatrix}. \tag{5}$$

As an exemplary case, in the sequel we give focus on the problem Π^{11} from (4), (4a), where $r(x, s) = [r_1(x, s), r_2(x, s)]^T$:

Using (5) and the boundary matrices B_0^{11}, B_l^{11} defined above, we define

$$M_{11} \equiv B_0^{11}\mathcal{W}(0, s) + B_l^{11}\mathcal{W}(l, s) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \mathcal{W}(0, s) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \mathcal{W}(l, s). \tag{6}$$

Thus, we obtain the solution of the inhomogeneous problem Π^{11} from (4), (4a):

$$w(x, s) = \mathcal{W}(x, s) M_{11}^{-1} \beta^{11} + \int_0^l \mathcal{G}^{11}(x, \xi, s) r(\xi, s) d\xi. \quad (7)$$

Since the function \mathcal{G}^{11} describes the influence of the left-hand and the right-hand boundary of S on the solution of the problem $\mathbf{\Pi}^{11}$ at the position x , $x \in (0, l)$, it is named influence function or Greens function.

With (7) we have

$$\mathcal{G}^{11}(x, \xi, s) = \begin{cases} \mathcal{W}(x, s) M_{11}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \mathcal{W}(0, s) \mathcal{W}^{-1}(\xi, s), & \xi \leq x, \\ -\mathcal{W}(x, s) M_{11}^{-1} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \mathcal{W}(l, s) \mathcal{W}^{-1}(\xi, s), & x < \xi. \end{cases} \quad (8)$$

Let us abbreviate

$$\mathcal{S}(x) \stackrel{Df}{=} \sinh(\sqrt{s}x), \quad \mathcal{C}(x) \stackrel{Df}{=} \cosh(\sqrt{s}x).$$

Then, Green's function can be formulated as:

$$\mathcal{G}_{\xi \leq x}^{11} = \frac{1}{2 \cdot \mathcal{S}(l)} \begin{bmatrix} -\mathcal{S}(x-\xi-l) - \mathcal{S}(x+\xi-l) & \frac{1}{\sqrt{s}} \{-\mathcal{C}(x-\xi-l) + \mathcal{C}(x+\xi-l)\} \\ -\sqrt{s} \{\mathcal{C}(x-\xi-l) + \mathcal{C}(x+\xi-l)\} & -\mathcal{S}(x-\xi-l) + \mathcal{S}(x+\xi-l) \end{bmatrix},$$

$$\mathcal{G}_{x < \xi}^{11} = \frac{1}{2 \cdot \mathcal{S}(l)} \begin{bmatrix} -\mathcal{S}(x-\xi+l) - \mathcal{S}(x+\xi-l) & \frac{1}{\sqrt{s}} \{-\mathcal{C}(x-\xi+l) + \mathcal{C}(x+\xi-l)\} \\ -\sqrt{s} \{\mathcal{C}(x-\xi+l) + \mathcal{C}(x+\xi-l)\} & -\mathcal{S}(x-\xi+l) + \mathcal{S}(x+\xi-l) \end{bmatrix}.$$

So we have in accordance with (7) the complete solution of $\mathbf{\Pi}^{11}$:

$$\begin{aligned}
w_1^{11}(x, s) &= \frac{\sinh \sqrt{s}(l-x)}{\sinh \sqrt{sl}} w_1^0(s) + \frac{\sinh \sqrt{s}x}{\sinh \sqrt{sl}} w_1^l(s) \\
&\quad - \frac{1}{2} \int_0^x \frac{\sinh \sqrt{s}(x-\xi-l)}{\sinh \sqrt{sl}} r_1(\xi, s) d\xi - \frac{1}{2} \int_x^l \frac{\sinh \sqrt{s}(x-\xi+l)}{\sinh \sqrt{sl}} r_1(\xi, s) d\xi \\
&\quad - \frac{1}{2} \int_0^x \frac{\cosh \sqrt{s}(x-\xi-l)}{\sqrt{s} \sinh \sqrt{sl}} r_2(\xi, s) d\xi - \frac{1}{2} \int_x^l \frac{\cosh \sqrt{s}(x-\xi+l)}{\sqrt{s} \sinh \sqrt{sl}} r_2(\xi, s) d\xi \\
&\quad - \frac{1}{2} \int_0^l \frac{\sinh \sqrt{s}(x+\xi-l)}{\sinh \sqrt{sl}} r_1(\xi, s) d\xi + \frac{1}{2} \int_0^l \frac{\cosh \sqrt{s}(x+\xi-l)}{\sqrt{s} \sinh \sqrt{sl}} r_2(\xi, s) d\xi,
\end{aligned} \tag{9}$$

$$\begin{aligned}
w_2^{11}(x, s) &= -\frac{\sqrt{s} \cosh \sqrt{s}(l-x)}{\sinh \sqrt{sl}} w_1^0(s) + \frac{\sqrt{s} \cosh \sqrt{s}x}{\sinh \sqrt{sl}} w_1^l(s) \\
&\quad - \frac{1}{2} \int_0^x \frac{\sqrt{s} \cosh \sqrt{s}(x-\xi-l)}{\sinh \sqrt{sl}} r_1(\xi, s) d\xi - \frac{1}{2} \int_x^l \frac{\sqrt{s} \cosh \sqrt{s}(x-\xi+l)}{\sinh \sqrt{sl}} r_1(\xi, s) d\xi \\
&\quad - \frac{1}{2} \int_0^x \frac{\sinh \sqrt{s}(x-\xi-l)}{\sinh \sqrt{sl}} r_2(\xi, s) d\xi - \frac{1}{2} \int_x^l \frac{\sinh \sqrt{s}(x-\xi+l)}{\sinh \sqrt{sl}} r_2(\xi, s) d\xi \\
&\quad - \frac{1}{2} \int_0^l \frac{\sqrt{s} \cosh \sqrt{s}(x+\xi-l)}{\sinh \sqrt{sl}} r_1(\xi, s) d\xi + \frac{1}{2} \int_0^l \frac{\sinh \sqrt{s}(x+\xi-l)}{\sinh \sqrt{sl}} r_2(\xi, s) d\xi.
\end{aligned} \tag{9a}$$

Inserting the expressions (9), (9a) into the system (4) and looking at the boundary conditions of $\mathbf{\Pi}^{11}$, we see that $[w_1^{11}, w_2^{11}]^T$ is indeed the solution of the inhomogeneous problem $\mathbf{\Pi}^{11}$.

In a similar manner we discuss the remaining problems $\mathbf{\Pi}^{22}$, $\mathbf{\Pi}^{12}$, $\mathbf{\Pi}^{21}$ from (4), (4a); we introduce the corresponding quantities

$$(B_0^{22}, B_l^{22}, \beta^{22}, M_{22}), \quad (B_0^{12}, B_l^{12}, \beta^{12}, M_{12}), \quad (B_0^{21}, B_l^{21}, \beta^{21}, M_{21})$$

and obtain in analogy with (6), (7), (8), (9), (9a) the solutions

$$\begin{bmatrix} w_1^{22}(x, s) \\ w_2^{22}(x, s) \end{bmatrix}, \quad \begin{bmatrix} w_1^{12}(x, s) \\ w_2^{12}(x, s) \end{bmatrix}, \quad \begin{bmatrix} w_1^{21}(x, s) \\ w_2^{21}(x, s) \end{bmatrix}. \tag{10}$$

Here, we don't formulate the complete solutions (10) of the problems $\mathbf{\Pi}^{22}$, $\mathbf{\Pi}^{12}$, $\mathbf{\Pi}^{21}$ in analogy with (9), (9a). We give only the solutions of the corresponding homogeneous problems ($r_1(x, s) = r_2(x, s) \equiv 0$ in (4)). So we can demonstrate at least, which combinations of hyperbolic functions play an important part in the individual

problems $\mathbf{\Pi}^{ij}$, because only those quotients of hyperbolic functions, which appear in the solution of the homogeneous problem also play a role in the solution of the inhomogeneous problem. From these quotients we need the originals of the Laplace transform to get the formal solution of the problems $\mathbf{\Pi}^{ij}$:

$$w_{1,hom}^{22}(x, s) = \frac{1}{\sinh \sqrt{s}l} \left\{ -\frac{1}{\sqrt{s}} \cosh \sqrt{s}(l-x) \cdot w_2^0(s) + \frac{1}{\sqrt{s}} \cosh \sqrt{s}x \cdot w_2^l(s) \right\},$$

$$w_{2,hom}^{22}(x, s) = \frac{1}{\sinh \sqrt{s}l} \left\{ \sinh \sqrt{s}(l-x) \cdot w_2^0(s) + \sinh \sqrt{s}x \cdot w_2^l(s) \right\},$$
(11)

$$w_{1,hom}^{12}(x, s) = \frac{1}{\cosh \sqrt{s}l} \left\{ \cosh \sqrt{s}(l-x) \cdot w_1^0(s) + \frac{1}{\sqrt{s}} \sinh \sqrt{s}x \cdot w_2^l(s) \right\},$$

$$w_{2,hom}^{12}(x, s) = \frac{1}{\cosh \sqrt{s}l} \left\{ -\sqrt{s} \sinh \sqrt{s}(l-x) \cdot w_1^0(s) + \cosh \sqrt{s}x \cdot w_2^l(s) \right\},$$
(11a)

$$w_{1,hom}^{21}(x, s) = \frac{1}{\cosh \sqrt{s}l} \left\{ -\frac{1}{\sqrt{s}} \sinh \sqrt{s}(l-x) \cdot w_2^0(s) + \cosh \sqrt{s}x \cdot w_1^l(s) \right\},$$

$$w_{2,hom}^{21}(x, s) = \frac{1}{\cosh \sqrt{s}l} \left\{ \cosh \sqrt{s}(l-x) \cdot w_2^0(s) + \sqrt{s} \sinh \sqrt{s}x \cdot w_1^l(s) \right\}.$$

3. FORMAL SOLUTIONS OF THE PROBLEMS \mathbf{P}^{ij} : THE SOLUTIONS OF THE PROBLEMS $\mathbf{\Pi}^{ij}$ WILL BE TRANSFORMED BACK

We know that under the Laplace transform the originals of particular quotients of hyperbolic functions are mainly Jacobian theta functions. We make use of theta functions as real valued functions of real arguments and we write them as follows [7] :

$$\Theta_1(x, t) = \frac{1}{\sqrt{\pi t}} \sum_{n=-\infty}^{\infty} (-1)^n e^{-\frac{(x+n-\frac{1}{2})^2}{t}} = 2 \sum_{n=0}^{\infty} (-1)^n e^{-\pi^2 t (n+\frac{1}{2})^2} \cdot \sin[(2n+1)\pi x],$$

$$\Theta_2(x, t) = \frac{1}{\sqrt{\pi t}} \sum_{n=-\infty}^{\infty} (-1)^n e^{-\frac{(x+n)^2}{t}} = 2 \sum_{n=0}^{\infty} e^{-\pi^2 t (n+\frac{1}{2})^2} \cdot \cos[(2n+1)\pi x],$$
(12)

$$\Theta_3(x, t) = \frac{1}{\sqrt{\pi t}} \sum_{n=-\infty}^{\infty} e^{-\frac{(x+n)^2}{t}} = \sum_{n=0}^{\infty} \varepsilon_n e^{-\pi^2 t n^2} \cdot \cos[2\pi n x],$$

$$\Theta_4(x, t) = \frac{1}{\sqrt{\pi t}} \sum_{n=-\infty}^{\infty} e^{-\frac{(x+n+\frac{1}{2})^2}{t}} = \sum_{n=0}^{\infty} (-1)^n \varepsilon_n e^{-\pi^2 t n^2} \cdot \cos[2\pi n x],$$

$\varepsilon_0 = 1, \varepsilon_n = 2, n > 0.$

Such theta functions are entire, transcendent functions of their arguments in the upper half-plane $H = \{(x, t) | -\infty < x < \infty, t > 0\}$. See the charts of the theta functions in [10] and put ibidem $q = e^{-\pi^2 t}$.

For $t \rightarrow +0$ we have a characteristic behavior on the boundary of H . We observe, see [3], that the terms, appearing in (12) in $\Theta_3(x, t)$,

$$k_t(x, n) \stackrel{Df}{=} \frac{1}{\sqrt{\pi t}} e^{-\frac{(x+n)^2}{t}}, \quad n \in \mathbb{Z},$$

establish w.r.t. the variable t a "sequence of type δ ", i.e. they converge for $t \rightarrow +0$ to the δ -distribution $\delta(x+n)$ in the sense of convergence, which is valid in $\mathcal{D}'(\mathbb{R})$ (\mathcal{D}' space of distributions). The sum $\delta(x+n)$ converges on its part in \mathcal{D}' [11], thus

$$\lim_{t \rightarrow +0} \Theta_3(x, t) = \delta_{\Theta_3}(x) = \sum_{n=-\infty}^{\infty} \delta(x+n)$$

is a distribution (Dirac comb), which assigns each $\varphi \in \mathcal{D}$ (\mathcal{D} space of test functions) the (per definition) finite sum $\sum_{k=-\infty}^{\infty} \varphi(k)$.

Similar statements hold for the remaining Θ -functions in (12).

The theta functions $\Theta_i(\frac{x}{2l}, \frac{t}{l^2})$, $i = 1, 2, 3, 4$, $(x, t) \in H$, satisfy the differential equation

$$\Theta_t(\frac{x}{2l}, \frac{t}{l^2}) = \Theta_{xx}(\frac{x}{2l}, \frac{t}{l^2}).$$

This can be demonstrated with the help of (12).

Now we have to transform back quotients of hyperbolic functions which are relevant for our problem (4),(4a). For this purpose we give two tables [5, 7, 9]. We suppose $|\nu| < l, \nu \in \mathbb{R}$.

The tables *Tab.1, Tab.2* present the originals of hyperbolic functions in which we are interested: the headers for a general argument ν , the subsequent five lines for the arguments named concretly in the first column.

ν	$\frac{\sinh \sqrt{s\nu}}{\sinh \sqrt{sl}} \bullet \rightarrow \circ \frac{1}{l} \frac{\partial}{\partial \nu} \Theta_4(\frac{\nu}{2l}, \frac{t}{l^2})$	$\frac{\cosh \sqrt{s\nu}}{\sqrt{s} \sinh \sqrt{sl}} \bullet \rightarrow \circ \frac{1}{l} \Theta_4(\frac{\nu}{2l}, \frac{t}{l^2})$
x	$\frac{1}{l} \Theta_{4,x}(\frac{x}{2l}, \frac{t}{l^2})$	$\frac{1}{l} \Theta_4(\frac{x}{2l}, \frac{t}{l^2})$
$l-x$	$-\frac{1}{l} \Theta_{3,x}(\frac{x}{2l}, \frac{t}{l^2})$	$\frac{1}{l} \Theta_3(\frac{x}{2l}, \frac{t}{l^2})$
$x-\xi-l$	$\frac{1}{l} \Theta_{3,x}(\frac{x-\xi}{2l}, \frac{t}{l^2})$	$\frac{1}{l} \Theta_3(\frac{x-\xi}{2l}, \frac{t}{l^2})$
$x-\xi+l$	$\frac{1}{l} \Theta_{3,x}(\frac{x-\xi}{2l}, \frac{t}{l^2})$	$\frac{1}{l} \Theta_3(\frac{x-\xi}{2l}, \frac{t}{l^2})$
$x+\xi-l$	$\frac{1}{l} \Theta_{3,x}(\frac{x+\xi}{2l}, \frac{t}{l^2})$	$\frac{1}{l} \Theta_3(\frac{x+\xi}{2l}, \frac{t}{l^2})$

Tab.1

ν	$\frac{\cosh \sqrt{s\nu}}{\cosh \sqrt{sl}} \bullet \rightarrow \circ \frac{1}{l} \frac{\partial}{\partial \nu} \Theta_1 \left(\frac{\nu}{2l}, \frac{t}{l^2} \right)$	$\frac{\sinh \sqrt{s\nu}}{\sqrt{s} \cosh \sqrt{sl}} \bullet \rightarrow \circ \frac{1}{l} \Theta_1 \left(\frac{\nu}{2l}, \frac{t}{l^2} \right)$
x	$\frac{1}{l} \Theta_{1,x} \left(\frac{x}{2l}, \frac{t}{l^2} \right)$	$\frac{1}{l} \Theta_1 \left(\frac{x}{2l}, \frac{t}{l^2} \right)$
$l-x$	$-\frac{1}{l} \Theta_{2,x} \left(\frac{x}{2l}, \frac{t}{l^2} \right)$	$\frac{1}{l} \Theta_2 \left(\frac{x}{2l}, \frac{t}{l^2} \right)$
$x-\xi-l$	$-\frac{1}{l} \Theta_{2,x} \left(\frac{x-\xi}{2l}, \frac{t}{l^2} \right)$	$-\frac{1}{l} \Theta_2 \left(\frac{x-\xi}{2l}, \frac{t}{l^2} \right)$
$x-\xi+l$	$\frac{1}{l} \Theta_{2,x} \left(\frac{x-\xi}{2l}, \frac{t}{l^2} \right)$	$\frac{1}{l} \Theta_2 \left(\frac{x-\xi}{2l}, \frac{t}{l^2} \right)$
$x+\xi-l$	$-\frac{1}{l} \Theta_{2,x} \left(\frac{x+\xi}{2l}, \frac{t}{l^2} \right)$	$-\frac{1}{l} \Theta_2 \left(\frac{x+\xi}{2l}, \frac{t}{l^2} \right)$

Tab.2

The terms of the solutions in (9), (9a), (11), (11a) contain products of functions depended on the variable s . Therefore we have convolutions in the t -domain.

Furthermore we have not considered hitherto in *Tab.1*, *Tab.2* terms of such kind:

$$s \cdot \frac{\cosh \sqrt{s\nu}}{\sqrt{s} \sinh \sqrt{sl}} \cdot h(s) \quad , \quad s \cdot \frac{\sinh \sqrt{s\nu}}{\sqrt{s} \cosh \sqrt{sl}} \cdot h(s).$$

We treat such terms by the product rule pertaining to the s -range [7]

$$\begin{aligned} s \cdot g(s) &= s \cdot g_1(s) \cdot g_2(s) \bullet \rightarrow \circ \frac{d}{dt} \left\{ \int_0^t f_1(t-\tau) f_2(\tau) d\tau \right\} , \\ &= \int_0^t f_{1,t}(t-\tau) f_2(\tau) d\tau + f_1(0) f_2(t), \end{aligned} \quad (13)$$

$$g_i(s) \bullet \rightarrow \circ f_i(t), \quad i = 1, 2.$$

With (13) and *Tab.1* we have

$$s \cdot \frac{\cosh \sqrt{s\nu}}{\sqrt{s} \sinh \sqrt{sl}} \cdot h(s) \bullet \rightarrow \circ \frac{1}{l} \int_0^t \Theta_{4,t} \left(\frac{\nu}{2l}, \frac{t-\tau}{l^2} \right) H(\tau) d\tau + \frac{1}{l} \Theta_4 \left(\frac{\nu}{2l}, 0 \right) H(t), \quad (14)$$

$$h(s) \bullet \rightarrow \circ H(t),$$

and especially in the t -domain for

$$\begin{aligned} \nu = x, h(s) = w_1^l(s) & : \frac{1}{l} \int_0^t \Theta_{4,t} \left(\frac{x}{2l}, \frac{t-\tau}{l^2} \right) u_l(\tau) d\tau + \frac{1}{l} \Theta_4 \left(\frac{x}{2l}, 0 \right) u_l(t), \\ \nu = l-x, h(s) = w_1^0(s) & : \frac{1}{l} \int_0^t \Theta_{3,t} \left(\frac{x}{2l}, \frac{t-\tau}{l^2} \right) u_0(\tau) d\tau + \frac{1}{l} \Theta_3 \left(\frac{x}{2l}, 0 \right) u_0(t). \end{aligned}$$

In the same way we get the originals of the remaining ν -values (see *Tab.1*) with $h(s) = \varphi_1(\xi, s)$ in (14).

In exactly the same manner we obtain with (13) and *Tab.2*

$$\begin{aligned} s \cdot \frac{\sinh \sqrt{s\nu}}{\sqrt{s} \cosh \sqrt{sl}} \cdot h(s) \bullet \rightarrow \circ & \frac{1}{l} \int_0^t \Theta_{1,t} \left(\frac{\nu}{2l}, \frac{t-\tau}{l^2} \right) H(\tau) d\tau + \frac{1}{l} \Theta_1 \left(\frac{\nu}{2l}, 0 \right) H(t), \quad (15) \\ h(s) \bullet \rightarrow \circ & H(t), \end{aligned}$$

and especially in the t -domain for

$$\begin{aligned} \nu = x, h(s) = w_1^l(s) & : \frac{1}{l} \int_0^t \Theta_{1,t} \left(\frac{x}{2l}, \frac{t-\tau}{l^2} \right) u_l(\tau) d\tau + \frac{1}{l} \Theta_1 \left(\frac{x}{2l}, 0 \right) u_l(t), \\ \nu = l-x, h(s) = w_1^0(s) & : \frac{1}{l} \int_0^t \Theta_{2,t} \left(\frac{x}{2l}, \frac{t-\tau}{l^2} \right) u_0(\tau) d\tau + \frac{1}{l} \Theta_2 \left(\frac{x}{2l}, 0 \right) u_0(t). \end{aligned}$$

In the same way we get the originals of the remaining ν -values (see *Tab.2*) with $h(s) = \varphi_1(\xi, s)$ in (15).

Now that the transformation into the t -domain has been accomplished for the solutions (11), (11a) of the homogeneous problems and for the solutions of the corresponding inhomogeneous problems, we translate the solutions (9), (9a) as well as the not explicitly listed solutions (10) term by term into the t -domain and obtain formally explicit solutions for the problems \mathbf{P}^{ij} given in (1), (2).

Setting

$$\begin{aligned} \Theta_i^\oplus(x, \xi, t, \tau) &= \Theta_i \left(\frac{x-\xi}{2l}, \frac{t-\tau}{l^2} \right) + \Theta_i \left(\frac{x+\xi}{2l}, \frac{t-\tau}{l^2} \right), \\ \Theta_i^\ominus(x, \xi, t, \tau) &= \Theta_i \left(\frac{x-\xi}{2l}, \frac{t-\tau}{l^2} \right) - \Theta_i \left(\frac{x+\xi}{2l}, \frac{t-\tau}{l^2} \right), \\ & i \in \{2, 3\}, \end{aligned}$$

we can write the formal solutions for our problems in the strip S as:

$$\begin{aligned}
u^{11}(x, t) &= -\frac{1}{l} \int_0^t \Theta_{3,x} \left(\frac{x}{2l}, \frac{t-\tau}{l^2} \right) u_0(\tau) d\tau + \frac{1}{l} \int_0^t \Theta_{4,x} \left(\frac{x}{2l}, \frac{t-\tau}{l^2} \right) u_l(\tau) d\tau \\
&\quad - \frac{1}{2l} \int_0^l \int_0^t \Theta_{3,x}^{\oplus}(x, \xi, t, \tau) f_1(\xi, \tau) d\tau d\xi + \frac{1}{2l} \int_0^l \int_0^t \Theta_{3,x}^{\ominus}(x, \xi, t, \tau) f_2(\xi, \tau) d\tau d\xi \\
&\quad + \frac{1}{2l} \int_0^l \Theta_{3,x}^{\ominus}(x, \xi, t, 0) f(\xi) d\xi,
\end{aligned} \tag{P^{11}}$$

$$\begin{aligned}
v^{11}(x, t) &= -\frac{1}{l} \int_0^t \Theta_{3,t} \left(\frac{x}{2l}, \frac{t-\tau}{l^2} \right) u_0(\tau) d\tau + \frac{1}{l} \int_0^t \Theta_{4,t} \left(\frac{x}{2l}, \frac{t-\tau}{l^2} \right) u_l(\tau) d\tau \\
&\quad - \frac{1}{2l} \int_0^l \int_0^t \Theta_{3,t}^{\oplus}(x, \xi, t, \tau) f_1(\xi, \tau) d\tau d\xi - \frac{1}{2l} \int_0^l \Theta_{3,t}^{\oplus}(x, \xi, \tau, \tau) f_1(\xi, t) d\xi \\
&\quad + \frac{1}{2l} \int_0^l \int_0^t \Theta_{3,x}^{\ominus}(x, \xi, t, \tau) f_2(\xi, \tau) d\tau d\xi + \frac{1}{2l} \int_0^l \Theta_{3,x}^{\ominus}(x, \xi, t, 0) f(\xi) d\xi \\
&\quad - \frac{1}{l} \Theta_{3,x} \left(\frac{x}{2l}, 0 \right) u_0(t) + \frac{1}{l} \Theta_{4,x} \left(\frac{x}{2l}, 0 \right) u_l(t),
\end{aligned}$$

$$\begin{aligned}
u^{22}(x, t) &= -\frac{1}{l} \int_0^t \Theta_3 \left(\frac{x}{2l}, \frac{t-\tau}{l^2} \right) v_0(\tau) d\tau + \frac{1}{l} \int_0^t \Theta_4 \left(\frac{x}{2l}, \frac{t-\tau}{l^2} \right) v_l(\tau) d\tau \\
&\quad - \frac{1}{2l} \int_0^l \int_0^t \Theta_{3,x}^{\ominus}(x, \xi, t, \tau) f_1(\xi, \tau) d\tau d\xi + \frac{1}{2l} \int_0^l \int_0^t \Theta_{3,x}^{\oplus}(x, \xi, t, \tau) f_2(\xi, \tau) d\tau d\xi \\
&\quad + \frac{1}{2l} \int_0^l \Theta_{3,x}^{\oplus}(x, \xi, t, 0) f(\xi) d\xi,
\end{aligned} \tag{P^{22}}$$

$$\begin{aligned}
v^{22}(x, t) &= -\frac{1}{l} \int_0^t \Theta_{3,x} \left(\frac{x}{2l}, \frac{t-\tau}{l^2} \right) v_0(\tau) d\tau + \frac{1}{l} \int_0^t \Theta_{4,x} \left(\frac{x}{2l}, \frac{t-\tau}{l^2} \right) v_l(\tau) d\tau \\
&\quad - \frac{1}{2l} \int_0^l \int_0^t \Theta_{3,t}^{\ominus}(x, \xi, t, \tau) f_1(\xi, \tau) d\tau d\xi - \frac{1}{2l} \int_0^l \Theta_{3,t}^{\ominus}(x, \xi, \tau, \tau) f_1(\xi, t) d\xi \\
&\quad + \frac{1}{2l} \int_0^l \int_0^t \Theta_{3,x}^{\oplus}(x, \xi, t, \tau) f_2(\xi, \tau) d\tau d\xi + \frac{1}{2l} \int_0^l \Theta_{3,x}^{\oplus}(x, \xi, t, 0) f(\xi) d\xi,
\end{aligned}$$

$$\begin{aligned}
u^{12}(x, t) &= -\frac{1}{l} \int_0^t \Theta_{2,x} \left(\frac{x}{2l}, \frac{t-\tau}{l^2} \right) u_0(\tau) d\tau + \frac{1}{l} \int_0^t \Theta_1 \left(\frac{x}{2l}, \frac{t-\tau}{l^2} \right) v_l(\tau) d\tau \\
&\quad - \frac{1}{2l} \int_0^l \int_0^t \Theta_{2,x}^{\oplus}(x, \xi, t, \tau) f_1(\xi, \tau) d\tau d\xi + \frac{1}{2l} \int_0^l \int_0^t \Theta_{2,x}^{\ominus}(x, \xi, t, \tau) f_2(\xi, \tau) d\tau d\xi \\
&\quad + \frac{1}{2l} \int_0^l \Theta_{2,x}^{\ominus}(x, \xi, t, 0) f(\xi) d\xi,
\end{aligned} \tag{P^{12}}$$

$$\begin{aligned}
v^{12}(x, t) &= -\frac{1}{l} \int_0^t \Theta_{2,t} \left(\frac{x}{2l}, \frac{t-\tau}{l^2} \right) u_0(\tau) d\tau + \frac{1}{l} \int_0^t \Theta_{1,x} \left(\frac{x}{2l}, \frac{t-\tau}{l^2} \right) v_l(\tau) d\tau \\
&\quad - \frac{1}{2l} \int_0^l \int_0^t \Theta_{2,t}^{\oplus}(x, \xi, t, \tau) f_1(\xi, \tau) d\tau d\xi - \frac{1}{2l} \int_0^l \Theta_{2,t}^{\oplus}(x, \xi, \tau, \tau) f_1(\xi, t) d\xi \\
&\quad + \frac{1}{2l} \int_0^l \int_0^t \Theta_{2,x}^{\ominus}(x, \xi, t, \tau) f_2(\xi, \tau) d\tau d\xi + \frac{1}{2l} \int_0^l \Theta_{2,x}^{\ominus}(x, \xi, t, 0) f(\xi) d\xi \\
&\quad - \frac{1}{l} \Theta_2 \left(\frac{x}{2l}, 0 \right) u_0(t),
\end{aligned}$$

$$\begin{aligned}
u^{21}(x, t) &= -\frac{1}{l} \int_0^t \Theta_2 \left(\frac{x}{2l}, \frac{t-\tau}{l^2} \right) v_0(\tau) d\tau + \frac{1}{l} \int_0^t \Theta_{1,x} \left(\frac{x}{2l}, \frac{t-\tau}{l^2} \right) u_l(\tau) d\tau \\
&\quad - \frac{1}{2l} \int_0^l \int_0^t \Theta_{2,x}^{\ominus}(x, \xi, t, \tau) f_1(\xi, \tau) d\tau d\xi + \frac{1}{2l} \int_0^l \int_0^t \Theta_{2,x}^{\oplus}(x, \xi, t, \tau) f_2(\xi, \tau) d\tau d\xi \\
&\quad + \frac{1}{2l} \int_0^l \Theta_{2,x}^{\oplus}(x, \xi, t, 0) f(\xi) d\xi,
\end{aligned} \tag{P^{21}}$$

$$\begin{aligned}
v^{21}(x, t) &= -\frac{1}{l} \int_0^t \Theta_{2,t} \left(\frac{x}{2l}, \frac{t-\tau}{l^2} \right) u_0(\tau) d\tau + \frac{1}{l} \int_0^t \Theta_{1,x} \left(\frac{x}{2l}, \frac{t-\tau}{l^2} \right) v_l(\tau) d\tau \\
&\quad - \frac{1}{2l} \int_0^l \int_0^t \Theta_{2,t}^{\oplus}(x, \xi, t, \tau) f_1(\xi, \tau) d\tau d\xi - \frac{1}{2l} \int_0^l \Theta_{2,t}^{\oplus}(x, \xi, \tau, \tau) f_1(\xi, t) d\xi \\
&\quad + \frac{1}{2l} \int_0^l \int_0^t \Theta_{2,x}^{\ominus}(x, \xi, t, \tau) f_2(\xi, \tau) d\tau d\xi + \frac{1}{2l} \int_0^l \Theta_{2,x}^{\ominus}(x, \xi, t, 0) f(\xi) d\xi \\
&\quad - \frac{1}{l} \Theta_2 \left(\frac{x}{2l}, 0 \right) u_0(t).
\end{aligned}$$

Each permitted input f, u_0, v_0, u_l, v_l , on the three sides of the strip S according to (1),(2) leads to a formal solution $(u(x, t), v(x, t))$, $0 < t \leq t_e$, $x \in (0, l)$, i.e.

each permitted initial state has a solution for the interior of the strip and for the closure of the strip at $t = t_e, x \in (0, l)$.

The problem (1),(2) therefore describes evolutionary problems. The point is that the problems \mathbf{P}^{ij} are initial value problems in the strip S .

It needs to be explained in full detail that the formal solutions $(\mathbf{P}^{11}), (\mathbf{P}^{22}), (\mathbf{P}^{12}), (\mathbf{P}^{21})$ solve the problem (1), (2), i.e. that they satisfy the assumption of a theorem concerning existence and unity. We recall that [1, 12] contain proofs of theorems concerning existence and unity of the solution of the homogeneous and the inhomogeneous heat equation of second order in the strip S . In order to prove such a theorem for the problem (1), (2), more work is needed.

If we reduce (1), (2) to the classical heat equation of second order and we adopt the assumptions of the theorems in [1, 12] and we look at the remaining parts of the solution $u^{11}(x, t)$ of $(\mathbf{P}^{11}), u^{22}(x, t)$ of (\mathbf{P}^{22}) , we obtain solutions which are identical to those in [1, 12].

REFERENCES

- [1] J.R. Cannon, *The One-Dimensional Heat Equation*, Addison-Wesley Publishing Company, Menlo Park, 1984, pp. 62-63, pp. 327-340, (Th.6.3.1, Th.6.4.1, Th.19.3.4, Th. 19.3.5) .
- [2] G. Doetsch, *Einführung in Theorie und Anwendung der Laplace-Transformation*, Birkhäuser Verlag, Basel und Stuttgart, 2. Auflage 1970, pp. 297-308 .
- [3] I.M. Gelfand, G.E. Schilov, *Verallgemeinerte Funktionen (Distributionen)*, Band 1, Deutscher Verlag der Wissenschaften, Berlin, 1960, pp. 45-47.
- [4] M. Hermann, *Numerik gewöhnlicher Differentialgleichungen, Anfangs- und Randwertprobleme*, Oldenbourg Verlag, München, Wien, 2004, pp. 195-209 .
- [5] Jahnke-Emde-Lösch, *Tafeln höherer Funktionen*, 6. Auflage, Revised by F. Lösch, Teubner, Stuttgart, 1960, pp. 82-85 .
- [6] J. Kevorkian, *Partial Differential Equations : Analytical Solution Techniques*, Springer, New York, Berlin, Heidelberg, 2000, p. 39 .
- [7] F. Oberhettinger, L. Badii, *Tables of Laplace Transforms*, Springer, Berlin-Heidelberg-New York, 1973, p. 421,422,295,208,209 .
- [8] A.D Polyandin, V.F. Zaitsev, *Handbuch der linearen Differentialgleichungen*, Spektrum Akademischer Verlag, Heidelberg, Berlin, Oxford, 1996, pp. 345-358 .
- [9] A.P. Prudnikov, Yu.A. Brychkov, O.I. Marichev, *Integrals and Series, Vol. 5, Inverse Laplace Transforms*, Gordon and Breach Science Publishers, New York, Reading, 1992, pp. 69-71 .
- [10] W.J. Thompson, *Atlas for Computing Mathematical Functions*, Wiley and Sons INC. , New York, 1997, pp. 525-528 .
- [11] W. Walter, *Einführung in die Theorie der Distributionen*, 3. Auflage, BI Wissenschaftsverlag, Mannheim, 1994, p. 47,57 .
- [12] D.V. Widder, *The Heat Equation*, Academic Press, NewYork-San Francisco, 1975, pp. 86 - 106 (especially Theorems 9, 10) .

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