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COUPLED COINCIDENCE POINT THEOREMS FOR COMPATIBLE MAPPINGS IN PARTIALLY ORDERED METRIC SPACES

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Abstract. In this paper, we present some coupled coincidence results for mixed g- monotone mappings in partially ordered complete metric spaces which are generalization of many recent results. Moreover, an example is given to illustrate our main result.

1. Introduction

The Banach contraction principle is one of the pivotal results of metric fixed point theory. It has many applications in a number of branches of mathematics. Generalizations of the above principle have been active area of research. Moreover, the existence of a fixed point for contractive mappings in partially ordered metric spaces has attracted many mathematicians (cf, [1] - [8]) and the references therein. In [3], Bhaskar and Lakshmikantham introduced the notion of a mixed monotone mapping and proved some coupled fixed point theorems for a mixed monotone mapping. Afterwards, Lakshmikantham and Ciric [7]introduced the concept of mixed g- monotone mappings and proved coupled coincidence results for two mappings F and g where F has the mixed g- monotone property and the functions F and g commute. It is well-known that the concept of commuting maps has been weakened in various ways. One such notion which is weaker than commuting is the concept of compatibility introduced by Jungck [5]. In [4], Choudhury and Kundu defined the concept of compatibility of F and g. The purpose of this paper is to present some coupled coincidence point theorems for mixed g- monotone mappings in the context of a complete metric space endowed with a partial order. We also present an applicable example.

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2. **Preliminaries**

Let us recall the Definition of the monotonic function $f : X \to X$ in the partially order set $(X, \preceq)$. We say that f is non-decreasing if for $x, y \in X$, $x \preceq y$, we have $fx \preceq fy$. Similarly, we say that f is non-increasing if for $x, y \in X$, $x \preceq y$, we have $fx \succeq fy$. For more details on fixed point theory, we refer the reader to [6].

Definition 2.1. [7]*(Mixed g-Monotone Property)*
Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$. We say that the mapping F has the mixed g-monotone property if F is monotone g-non-decreasing in its first argument and is monotone g-non-increasing in its second argument. That is, for any $x, y \in X$,

$$x_1, x_2 \in X, gx_1 \preceq gx_2 \Rightarrow F(x_1, y) \preceq F(x_2, y) \tag{1}$$

and

$$y_1, y_2 \in X, gy_1 \preceq gy_2 \Rightarrow F(x, y_1) \succeq F(x, y_2). \tag{2}$$

Definition 2.2. [7] *(Coupled Coincidence Point)*
Let $(x, y) \in X \times X$, $F : X \times X \to X$ and $g : X \to X$. We say that (x, y) is a coupled coincidence point of F and g if $F(x, y) = gx$ and $F(y, x) = gy$ for $x, y \in X$.

Definition 2.3. [4] *The mappings F and g where $F : X \times X \to X$ and $g : X \to X$, are said to be compatible if*

$$\lim_{n\to\infty} d(g(F(x_n, y_n)), F(gx_n, gy_n)) = 0$$

and

$$\lim_{n\to\infty} d(g(F(y_n, x_n)), F(gy_n, gx_n)) = 0,$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X, such that $\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} gx_n = x$ and $\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} gy_n = y$, for all $x, y \in X$.

3. **Existence of Coupled Coincidence Points**

Theorem 3.1. *Let $(X, \preceq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Suppose $F : X \times X \to X$ and $g : X \to X$ be such that F has the mixed g-monotone property . Suppose there exist non-negative real numbers α, β, γ with $\alpha + \beta < 1$ such that, for all $x, y, u, v \in X$ with $gx \succeq gu$ and $gy \preceq gv$,*

$$\begin{aligned} d(F(x, y), F(u, v)) \leq{} & \alpha d(gx, gu) + \beta d(gy, gv) \\ & + \gamma \min\{d(F(x, y), gu), d(F(u, v), gx), \\ & d(F(x, y), gx), d(F(u, v), gu)\} \end{aligned} \tag{3}$$

Suppose $F(X \times X) \subseteq g(X)$, g is continuous and monotone increasing and F and g are compatible mappings. Also suppose either

(a) *F is continuous or*
(b) *X has the following property:*

(i) if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all n, (4)

(ii) if a non-increasing sequence $\{y_n\} \to y$, then $y_n \succeq y$ for all n, (5)

If there exist two elements $x_0, y_0 \in X$ with

$$gx_0 \preceq F(x_0, y_0) \quad \textit{and} \quad gy_0 \succeq F(y_0, x_0),$$

then there exist $x, y \in X$ such that

$$F(x, y) = g(x) \quad \textit{and} \quad F(y, x) = g(y),$$

that is, F and g have a coupled coincidence point in X.

Proof. Let $x_0, y_0 \in X$ be such that $gx_0 \preceq F(x_0, y_0)$ and $gy_0 \succeq F(y_0, x_0)$. Since $F(X \times X) \subseteq g(X)$, we can choose $x_1, y_1 \in X$ such that $gx_1 = F(x_0, y_0)$ and $gy_1 = F(y_0, x_0)$. Continuing this process we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$gx_{n+1} = F(x_n, y_n) \text{ and } gy_{n+1} = F(y_n, x_n) \text{ for all } n \geq 0. \tag{6}$$

We shall use the mathematical induction to show that

$$gx_n \preceq gx_{n+1} \quad \text{for all } n \geq 0 \tag{7}$$

and

$$gy_n \succeq gy_{n+1} \quad \text{for all } n \geq 0. \tag{8}$$

Since $gx_0 \preceq F(x_0, y_0)$ and $gy_0 \succeq F(y_0, x_0)$, and as $gx_1 = F(x_0, y_0)$ and $gy_1 = F(y_0, x_0)$, we have $gx_0 \preceq gx_1)$ and $gy_0 \succeq g(y_1)$. Thus (7) and (8) hold for the case $n = 0$.

Suppose now that (7) and (8) hold for some fixed $n \geq 0$. Then, since $gx_n \preceq gx_{n+1}$ and $gy_{n+1} \preceq gy_n$, and as F has the mixed g-monotone property, we get ; from (1) and (6),

$$gx_{n+1} = F(x_n, y_n) \preceq F(x_{n+1}, y_n) \text{ and } F(y_{n+1}, x_n) \preceq F(y_n, x_n) = gy_{n+1}, \tag{9}$$

and from (2) and (6),

$$gx_{n+2} = F(x_{n+1}, y_{n+1}) \succeq F(x_{n+1}, y_n) \text{ and } F(y_{n+1}, x_n) \succeq F(y_{n+1}, x_{n+1}) = gy_{n+2}. \tag{10}$$

Now from (9) and (10) we get

$$gx_{n+1} \preceq gx_{n+2}$$

and

$$gy_{n+1} \succeq gy_{n+2}.$$

Thus we conclude that (7) and (8) hold for all $n \geq 0$ by mathematical induction.

Therefore,

$$gx_0 \preceq gx_1 \preceq gx_2 \preceq gx_3 \preceq \cdots \preceq gx_n \preceq gx_{n+1} \preceq \cdots \tag{11}$$

and

$$gy_0 \succeq gy_1 \succeq gy_2 \succeq gy_3 \succeq \cdots \succeq gy_n \succeq gy_{n+1} \succeq \cdots. \tag{12}$$

From (3), (6),(7) and (8), we have

$$\begin{aligned} d(F(x_n, y_n), F(x_{n-1}, y_{n-1}) \leq{} & \alpha d(gx_n, gx_{n-1}) + \beta d(gy_n, gy_{n-1}) \\ & + \gamma \min\{d(F(x_n, y_n), gx_{n-1}), d(F(x_{n-1}, y_{n-1}), gx_n) \\ & \quad d(F(x_n, y_n), gx_n), d(F(x_{n-1}, y_{n-1}), gx_{n-1})\}, \end{aligned}$$

or

$$d(gx_{n+1}, gx_n) \leq \alpha d(gx_n, gx_{n-1}) + \beta d(gy_n, gy_{n-1}) \tag{13}$$

Similarly, we have

$$\begin{aligned} d(F(y_{n-1}, x_{n-1}), F(y_n, x_n) &\leq \alpha d(gy_{n-1}, gy_n) + \beta d(gx_{n-1}, gx_n) \\ &+ \gamma \min\{d(F(y_{n-1}, x_{n-1}), gy_n), d(F(y_n, x_n), gy_{n-1}) \\ &\qquad d(F(y_{n-1}, x_{n-1}), gy_{n-1}), d(F(y_n, x_n), gy_n)\}, \end{aligned}$$

or

$$d(gy_n, gy_{n+1}) \leq \alpha d(gy_{n-1}, gy_n) + \beta d(gx_{n-1}, gx_n) \tag{14}$$

By (13) and (14), we get

$$d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n) \leq (\alpha + \beta)[d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1})]. \tag{15}$$

Set

$$d_n = d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n) \quad \text{and} \quad \delta = \alpha + \beta < 1,$$

we have

$$0 \leq d_n \leq \delta d_{n-1} \leq \delta^2 d_{n-2} \leq \cdots \leq \delta^n d_0$$

which implies

$$\lim_{n\to\infty} [d(gx_{n+1}, gx_n) + dg(y_{n+1}, gy_n)] = \lim_{n\to\infty} d_n = 0.$$

Therefore,

$$\lim_{n\to\infty} d(gx_{n+1}, gx_n) = \lim_{n\to\infty} d(gy_{n+1}, gy_n) = 0.$$

For each $m \geq n$, we have

$$d(gx_m, gx_n) \leq d(gx_m, gx_{m-1}) + d(gx_{m-1}, gx_{m-2}) + \cdots + d(gx_{n+1}, gx_n)$$

and

$$d(gy_m, gy_n) \leq d(gy_m, gy_{m-1}) + d(gy_{m-1}, gy_{m-2}) + \cdots + d(gy_{n+1}, gy_n).$$

Thus

$$\begin{aligned} d(gx_m, gx_n) + d(gy_m, gy_n) &\leq d_{m-1} + d_{m-2} + \cdots + d_n \\ &\leq (\delta^{m-1} + \delta^{m-2} + \cdots + \delta^n) d_0 \\ &\leq \frac{\delta^n}{1-\delta} d_0 \end{aligned} \tag{16}$$

which implies

$$\lim_{n,m\to\infty} [dg(x_m, gx_n) + dg(y_m, gy_n)] = 0.$$

Therefore, the sequences $\{gx_n\}$ and $\{gy_n\}$ are Cauchy in X. Because of the completeness of X , there exist $x, y \in X$ such that

$$\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} gx_n = x \quad \text{and} \quad \lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} gy_n = y. \tag{17}$$

Since F and g are compatible mappings, we have by (17),

$$\lim_{n\to\infty} d(g(F(x_n, y_n)), F(gx_n, gy_n)) = 0 \tag{18}$$

and

$$\lim_{n\to\infty} d(g(F(y_n, x_n)), F(gy_n, gx_n)) = 0 \tag{19}$$

We now show that $gx = F(x, y)$ and $gy = F(y, x)$. Suppose that the assumption (a) holds. For all $n \geq 0$, we have ,

$$d(gx, F(gx_n, gy_n)) \leq d(gx, g(F(x_n, y_n))) + d(g(F(x_n, y_n)), F(gx_n, gy_n).$$

Taking the limit as $n \to \infty$, using (6), (17), (18) and the fact that F and g are continuous, we have $d(gx, F(x,y)) = 0$.

Similarly, from (6), (17), (19) and the fact that F and g are continuous, we have $d(gy, F(y,x)) = 0$.

Thus

$$gx = F(x,y) \quad \textit{and} \quad gy = F(y,x).$$

Finally, suppose that (b) holds. By (7), (8) and (17), we have $\{gx_n\}$ is a non-decreasing sequence and $gx_n \to x$ and $\{gy_n\}$ is a non-increasing sequence, $gy_n \to y$ as $n \to \infty$. Then by (4) and (5) we have for all $n \geq 0$,

$$gx_n \preceq x \quad \textit{and} \quad gy_n \succeq y. \tag{20}$$

Since F and g are compatible mappings and g is continuous, by (18) and (19) we have

$$\lim_{n\to\infty} ggx_n = gx = \lim_{n\to\infty} g(F(x_n, y_n)) = \lim_{n\to\infty} F(gx_n, gy_n) \tag{21}$$

and,

$$\lim_{n\to\infty} ggy_n = gy = \lim_{n\to\infty} g(F(y_n, x_n)) = \lim_{n\to\infty} F(gy_n, gx_n). \tag{22}$$

Now we have

$$d(gx, F(x,y) \leq d(gx, ggx_{n+1}) + d(ggx_{n+1}, F(x,y)).$$

Taking $n \to \infty$ in the above inequality, using (6) and (21) we have,

$$\begin{aligned} d(gx, F(x,y)) &\leq \lim_{n\to\infty} d(gx, ggx_{n+1}) + \lim_{n\to\infty} d(g(F(x_n, y_n)), F(x,y)) \\ &\leq \lim_{n\to\infty} d(F(gx_n, gy_n)), F(x,y)) \end{aligned}$$

Since the mapping g is monotone increasing, by (3) and (20) and the above inequality, we have for all $n \geq 0$,

$$\begin{aligned} d(gx, F(x,y) \leq{}& \alpha d(ggx_n, gx) + \beta d(ggy_n, gy) \\ &+ \gamma \min\{d(F(gx_n, gy_n), gx), d(F(x,y), ggx_n) \\ &\qquad d(F(gx_n, gy_n), ggx_n), d(F(x,y), gx)\}, \end{aligned} \tag{23}$$

Using (17) and letting $n \to \infty$ in (23) we get $d(gx, F(x,y)) \leq 0$ which implies $F(x,y) = gx$. Similarly, by the virtue of (6), (17) and (22) we obtain $F(y,x) = gy$. Hence F and g have a coupled coincidence point in X. □

It is well-known that commuting maps are compatible, thus we have the following:

Corollary 3.1. *Let $(X, \preceq)$ be a partially ordered set and suppose there is a metric d on X such that (X,d) is a complete metric space. Suppose $F : X \times X \to X$ and $g : X \to X$ such that F has the mixed g-monotone property on X. Suppose there* *exist non-negative real numbers α, β, γ with $\alpha+\beta < 1$ such that, for all $x, y, u, v \in X$ with $gx \succeq gu$ and $gy \preceq gv$,*

$$\begin{aligned} d(F(x,y), F(u,v)) \leq\ & \alpha d(gx, gu) + \beta d(gy, gv) \\ & + \gamma \min\{d(F(x,y), gu), d(F(u,v), gx), \\ & d(F(x,y), gx), d(F(u,v), gu)\} \end{aligned} \tag{24}$$

Suppose $F(X \times X) \subseteq g(X)$, g is continuous and commutes with F and also suppose either

(a) *F is continuous or*
(b) *X has the following property:*
  (i) if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all n,
  (ii) if a non-increasing sequence $\{y_n\} \to y$, then $y_n \succeq y$ for all n,

If there exist two elements $x_0, y_0 \in X$ with

$$gx_0 \preceq F(x_0, y_0) \quad \textit{and} \quad gy_0 \succeq F(y_0, x_0),$$

then there exist $x, y \in X$ such that

$$F(x,y) = gx \quad \textit{and} \quad F(y,x) = gy,$$

that is, F and g have a coupled coincidence point in X.

Corollary 3.2. *Let $(X, \preceq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Suppose $F : X \times X \to X$ and $g : X \to X$ such that F has the mixed g-monotone property on X. Suppose there exist non-negative real numbers α, β, γ with $\alpha+\beta < 1$ such that, for all $x, y, u, v \in X$ with $gx \succeq gu$ and $gy \preceq gv$,*

$$d(F(x,y), F(u,v)) \leq \alpha d(gx, gu) + \beta d(gy, gv)$$

Suppose $F(X \times X) \subseteq g(X)$, g is continuous and commutes with F and also suppose either

(a) *F is continuous or*
(b) *X has the following property:*
  (i) if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all n,
  (ii) if a non-increasing sequence $\{y_n\} \to y$, then $y_n \succeq y$ for all n,

If there exist two elements $x_0, y_0 \in X$ with

$$gx_0 \preceq F(x_0, y_0) \quad \textit{and} \quad gy_0 \succeq F(y_0, x_0),$$

then there exist $x, y \in X$ such that

$$F(x,y) = gx \quad \textit{and} \quad F(y,x) = gy,$$

that is, F has a coincidence fixed point in X.

Moreover, some known results become corollaries of the above theorem.

Corollary 3.3. [8] *Let $(X, \preceq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ be a mapping having the mixed monotone property on X. Suppose there exist non-negative real numbers α, β and γ with $\alpha + \beta < 1$ such that, for all $x, y, u, v \in X$*

with $x \succeq u$ *and* $y \preceq v$,

$$d(F(x,y), F(u,v)) \leq \alpha d(x,u) + \beta d(y,v) + \gamma \min\{d(F(x,y),u), d(F(u,v),x), d(F(x,y),x), d(F(u,v),u)\} \tag{25}$$

Suppose either

(a) *F is continuous or*

(b) *X has the following property:*

(i) if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all n,

(ii) if a non-increasing sequence $\{y_n\} \to y$, then $y_n \succeq y$ for all n,

If there exist two elements $x_0, y_0 \in X$ with

$$x_0 \preceq F(x_0, y_0) \quad \text{and} \quad y_0 \succeq F(y_0, x_0),$$

then there exist $x, y \in X$ such that

$$x = F(x,y) \quad \text{and} \quad y = F(y,x),$$

that is, F has a coupled fixed point in X.

Corollary 3.4. [3] *Let $(X, \preceq)$ be a partially ordered set and suppose there is a metric d on X such that (X,d) is a complete metric space. Let $F : X \times X \to X$ be a mapping having the mixed monotone property on X. Assume that there exists a $k \in [0,1)$ with*

$$d(F(x,y), F(u,v)) \leq \frac{k}{2}[d(x,u) + d(y,v)]$$

for all $x, y, u, v \in X$ with $x \succeq u$ and $y \preceq v$, Suppose either

(a) *F is continuous or*

(b) *X has the following property:*

(i) if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all n,

(ii) if a non-increasing sequence $\{y_n\} \to y$, then $y_n \succeq y$ for all n,

If there exist two elements $x_0, y_0 \in X$ with

$$x_0 \preceq F(x_0, y_0) \quad \text{and} \quad y_0 \succeq F(y_0, x_0),$$

then there exist $x, y \in X$ such that

$$x = F(x,y) \quad \text{and} \quad y = F(y,x),$$

that is, F has a coupled fixed point in X.

4. **Uniqueness of Coupled Coincidence Point**

We shall prove the uniqueness of coupled coincidence point. Let $(X, \preceq)$ be a partially ordered set. Then we endow the product $X \times X$ with the following partial order:

$$for \quad (x,y),(u,v) \in X \times X, (x,y) \preceq (u,v) \Leftrightarrow x \preceq u, y \succeq v.$$

Theorem 4.1. *In addition to the hypotheses of Theorem 3.1, suppose that for every $(x,y),(z,t) \in X \times X$, there exists a $(u,v) \in X \times X$ such that $(F(u,v), F(v,u))$ is comparable to $(F(x,y), F(y,x))$ and $(F(z,t), F(t,z))$. Then F and g have a unique coupled coincidence point, that is, there exist a unique $(x,y) \in X \times X$ such that*

$$gx = F(x,y) \quad \text{and} \quad gy = F(y,x).$$

Proof. From Theorem 3.1, the set of coupled coincidence points is non-empty. We shall show that if (x, y) and (z, t) are coupled coincidence points, that is, if $gx = F(x, y), gy = F(y, x)$ and $gz = F(z, t), gt = F(t, z)$, then

$$gx = gz \quad \text{and} \quad gy = gt. \tag{26}$$

By hypothesis there is $(u, v) \in X \times X$ such that $(F(u, v), F(v, u))$ is comparable to $(F(x, y), F(y, x))$ and $(F(z, t), F(t, z))$. Put $u_0 = u$, $v_0 = v$ and choose $u_1, v_1 \in X$ so that $gu_1 = F(u_0, v_0)$ and $gv_1 = F(v_0, u_0)$. Then, as in the proof of Theorem 3.1, we can inductively define sequences $\{gu_n\}$ and $\{gv_n\}$ such that

$$gu_{n+1} = F(u_n, v_n) \quad \text{and} \quad gv_{n+1} = F(v_n, u_n) \text{ for all } n.$$

Further, set $x_0 = x, y_0 = y, t_0 = t, z_0 = z$ and, in the same way, define the sequences $\{gx_n\}$, $\{gy_n\}$, $\{gt_n\}$ and $\{gz_n\}$. Then it is easy to show that, for all $n \geq 1$,

$$gx_n = F(x, y), \quad gy_n = F(y, x), \quad gt_n = F(t, z) \quad \text{and} \quad gz_n = F(z, t).$$

Since $(F(x, y), F(y, x)) = (gx_1, gy_1) = (gx, gy)$ and $(F(u, v), F(v, u)) = (gu_1, gv_1)$ are comparable, therefore $gx \preceq gu_1$ and $gy \succeq gv_1$. It is easy to show that

$$(gx, gy) \succeq (gu_n, gv_n) \text{ for all } n,$$

that is, $gx \preceq gu_n$ and $gy \succeq gv_n$. Therefore, from this and (3), we have

$$\begin{aligned} d(F(x, y), F(u_n, v_n)) \leq \alpha d(gx, gu_n) + \beta d(gy, gv_n) \\ + \gamma \min\{d(F(x, y), gu_n), d(F(u_n, v_n), gx), \\ d(F(x, y), gx), d(F(u_n, v_n), gu_n)\}. \end{aligned} \tag{27}$$

or

$$d(gx, gu_{n+1}) \leq \alpha d(gx, gu_n) + \beta d(gy, gv_n). \tag{28}$$

Similarly, we have

$$d(gv_{n+1}, gy) \leq \alpha d(gv_n, gy) + \beta d(gu_n, gx). \tag{29}$$

Adding (28) and (29), we get

$$\begin{aligned} d(gx, gu_{n+1}) + d(gy, gv_{n+1}) &\leq (\alpha + \beta)[d(gx, gu_n) + d(gy, gv_n)] \\ &\leq (\alpha + \beta)^2[d(gx, gu_{n-1}) + d(gy, gv_{n-1})] \\ &\leq \cdots \\ &\leq (\alpha + \beta)^{n+1}[d(gx, gu_0) + d(gy, gv_0)]. \end{aligned} \tag{30}$$

Taking the limit as $n \to \infty$ in (30), we get

$$\lim_{n\to\infty}[d(gx, gu_{n+1}) + d(gy, gv_{n+1})] = 0.$$

Therefore,

$$\lim_{n\to\infty} d(gx, gu_{n+1}) = 0 \quad \textit{and} \quad \lim_{n\to\infty} d(gy, gv_{n+1}) = 0. \tag{31}$$

Similarly, one can prove that

$$\lim_{n\to\infty} d(gz, gu_{n+1}) = 0 \quad \textit{and} \quad \lim_{n\to\infty} d(gt, gv_{n+1}) = 0. \tag{32}$$

From (31) and (32), we get $gx = g(z)$ and $gy = gt$. Hence we proved (26). □

We improve Example 2.6 in [8] to verify our main Theorem 3.1.

5. **Example**

Example 5.1. *Let $X = [0,1]$ be endowed with the metric $d(x,y) = |x-y|$ for $x, y \in X$. On the set X, we consider the following relation:*

$$\text{for} \quad x, y \in X, x \preceq y \Leftrightarrow x, y \in \{0,1\} \text{ and } \quad x \le y,$$

where $\le$ be the usual ordering. Clearly, (X, d) is a complete metric space and $(X, \preceq)$ is a partially ordered set.

Let $g : X \to X$ be defined as

$$g(x) = x^2, \text{ for all } x \in X,$$

and let $F : X \times X \to X$ be defined as

$$F(x,y) = \begin{cases} \frac{x^2-y^2}{2}, & \text{if } x, y \in [0,1], x \ge y, \\ 0, & \text{if } x < y. \end{cases}$$

Note that F has the mixed g-monotone property.

Also, note that X satisfies conditions (4) and (5). Moreover, it is clear that F is continuous.

Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that $\lim_{n\to\infty} F(x_n, y_n) = a, \lim_{n\to\infty} gx_n = a$, $\lim_{n\to\infty} F(y_n, x_n) = b$ and $\lim_{n\to\infty} gy_n = b$ Then obviously, $a = 0$ and $b = 0$. Now, for all $n \ge 0$,

$$g(x_n) = x_n^2, \quad g(y_n) = y_n^2,$$

$$F(x_n, y_n) = \begin{cases} \frac{x_n^2-y_n^2}{2}, & \text{if }, x_n \ge y_n, \\ 0, & \text{if } x_n < y_n. \end{cases}$$

and

$$F(y_n, x_n) = \begin{cases} \frac{y_n^2-x_n^2}{2}, & \text{if }, y_n \ge x_n, \\ 0, & \text{if } y_n < x_n. \end{cases}$$

Then it follows that,

$$\lim_{n\to\infty} d(g(F(x_n, y_n)), F(gx_n, gy_n)) = 0$$

and

$$\lim_{n\to\infty} d(g(F(y_n, x_n)), F(gy_n, gx_n)) = 0,$$

Hence, the mappings F and g are compatible in X. Also, $x_0 = 0$ and $y_0 = 0$ are two points in X such that

$$g(x_0) = g(0) = 0 \preceq F(0,0) = F(x_0, y_0)$$

and

$$g(y_0) = g(0) = 0 \succeq= F(0,0) = F(y_0, x_0).$$

We next verify the contractive condition (3) with $\alpha = \frac{2}{3}$, $\beta = 0$ and $\gamma = 2$. We take $x, y, u, v, \in X$, such that $gx \succeq gu$ and $gy \preceq gv$ or $(gx, gy) \succeq (gu, gv)$.

We have the following cases:

Case 1. *$(x,y) = (u,v)$ or $(x,y) = (0,0)$, $(u,v) = (0,1)$ or $(x,y) = (1,1)$, $(u,v) = (0,1)$, we have $(d(F(x,y), F(u,v)) = 0$. Hence, (3) holds.*

Case 2. *$(x,y) = (1,0)$, $(u,v) = (0,0)$, we have*

$$d(F(x,y), F(u,v)) = d(F(1,0), F(0,0)) = \frac{1}{2} < \frac{2}{3} = \frac{2}{3}d(1,0) = \alpha d(gx, gu)$$

Hence, (3) holds.

Case 3. $(x, y) = (1, 0)$, $(u, v) = (0, 1)$, *we have*

$$d(F(x, y), F(u, v)) = d(F(1, 0), F(0, 1)) = \frac{1}{2} < \frac{2}{3} = \frac{2}{3} d(1, 0) = \alpha d(gx, gu)$$

Hence, (3) holds.

Case 4. $(x, y) = (1, 0)$, $(u, v) = (1, 1)$, *we have*

$$\begin{aligned} & \gamma \min\{d(F(x, y), gu), d(F(u, v), gx), d(F(x, y), gx), d(F(u, v), gu)\} \\ = \ & 2 \min\{d(F(1, 0), 1), d(F(1, 1), 1), d(F(1, 0), 1), d(F(1, 1), 1)\} \\ = \ & 2 \min\{\frac{1}{2}, 1\} = 1 \\ > \ & \frac{1}{2} = d(F(1, 0), F(1, 1)) \\ = \ & d(F(x, y), F(u, v). \end{aligned}$$

Hence, (3) holds.

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