PROVING COMMON FIXED POINT THEOREMS FOR LIPSCHITZ TYPE MAPPINGS VIA ABSORBING PAIR

(COMMUNICATED BY DENNY LEUNG)

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ABSTRACT. Our aim in the present paper is three fold. Firstly, we obtain a common fixed point theorem for a pair of self mappings satisfying a Lipschitz type condition employing the property (E.A.) along with a relatively new notion of absorbing pair of maps wherein we never require conditions on the completeness of the space, containment of range of one mapping into the range of other, continuity of the mappings involved besides a set of unusual alternative conditions as utilized by Pant. Secondly, we further improve our first result by replacing the Lipschitz (or non-contractive) type condition with g-continuity of the mapping f. Thirdly, in our last result, we observe that if we restrict ourselves to non-compatibility instead of the property (E.A.) in our first result, then the maps turn out to be discontinuous at their common fixed point. We also furnish illustrative examples to demonstrate the utility of our results over related ones.

1. Introduction

In 1986, G. Jungck [4] generalized the notion of weakly commuting pair by introducing compatible maps and also showed that compatible maps commute at their coincidence points. Since then many interesting fixed point theorems for compatible maps satisfying contractive type conditions have been obtained by various authors. However, the question of the study of common fixed points of non-compatible maps remain unnoticed for quite sometimes until Pant [10,11] wherein he initiated the study of non-compatible maps employing the notion of pointwise R-weakly commuting maps. Using this concept Pant [11-13] proved some interesting fixed points theorems for maps satisfying non-contractive as well as Lipschitz type conditions. In recent years, these results of Pant [13] were generalized and improved by Sastry et al. [15], Singh et al. [16] and also by V. Pant [12] using the notion of (E.A.) property. Recently Jungck and Rhoades [6,7] introduced the concept of occasionally weakly compatible maps for those pairs which do have at least one coincidence points (see also [2]) and obtain fixed point theorems for such maps which includes almost all results concerning metric fixed point theory under this restricted setting.

In the present note, our first result is a common fixed point theorem for a pair of self-mappings satisfying a Lipschitz type condition wherein the notion of property (E.A.) and a newly introduced notion of absorbing maps are employed. Our next result is a common fixed point theorem for a pair of self mappings (f, g) which is obtained by replacing the Lipschitz type condition with g-continuity of f (a notion due Sastry et al. [15]). In the last result of this note, we notice that the involved mappings turn out to be discontinuous at their common fixed points if we replace the property (E.A.) with non-compatibility of the pair. Thus, we come across with a non-contractive type condition which is strong enough to ensure the existence of common fixed points without requiring the continuity of the maps even at their common fixed points. This is in conformity to Pant's (cf. [10]) answer to Rhoades (cf.[14]) question. With a view to substantiate the realized improvements in this paper, we present examples to highlight the utility of pointwise absorbing pairs in respect of producing common fixed points for maps satisfying non-contractive or Lipschitz type conditions and at the same time exhibit the non applicability of certain commuting type conditions employed in some related results (e.g. [10-13, 15,16]).

2. Preliminaries

In this section, we collect the relevant definitions and results to make our presentation as self-contained as possible.

Definition 2.1[9]. A pair of self- mappings (f, g) defined on a metric space (X, d) is said to be R-weakly commuting if there exists some real number $R > 0$ such that $d(fgx, gfx) \leq Rd(fx, gx)$ for all x in X.

Definition 2.2[4]. A pair of self-mappings (f, g) defined on a metric space (X, d) is said to be compatible if $\lim_{n \to \infty} d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = t$ for some $t \in X$.

Definition 2.3[9]. A pair of self- mappings (f, g) defined on a metric space (X, d) is said to be pointwise R- weakly commuting if for some x in X, there exists some $R > 0$ such that $d(fgx, gfx) \leq Rd(fx, gx)$.

The notion of pointwise R-weakly commuting pair (cf.[9]), weakly compatible pair (cf.[11])and partially commuting pair (cf.[15]) have equivalent nomenclature. It is also obvious that Definition 2.1 $\to$ Definition 2.2 $\to$ Definition 2.3, but converse implications are not true in general. For such justifications, one can consult [9, 11].

Definition 2.4[10]. A pair of self- mappings (f, g) defined on a metric space (X, d) is said to be non compatible if there exists a sequence $\{x_n\}$ in X such that $\lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = t$ for some $t \in X$ but $\lim_{n \to \infty} d(fgx_n, gfx_n)$ is either non zero or nonexistent.

Definition 2.5[1]. A pair of self- mappings (f, g) defined on a metric space (X, d) is said to have property (E.A.) if there exists a sequence $\{x_n\}$ in X such that $\lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = t$ for some $t \in X$.

Recall that the notion of tangential mappings given in [15] is equivalent to property (E.A.). Clearly, the pairs of compatible as well as non compatible mappings of a metric space (X, d) satisfy property (E.A.) but not conversely (see [1]).

Definition 2.6[15]. Let f and g be two self maps defined on a metric space (X, d). Then f is said to be g-continuous if $gx_n \to gx \implies fx_n \to fx$, whenever $\{x_n\}$ is a sequence in X and $x \in X$.

Definition 2.7[6]. Let f and g be two self mappings defined on a nonempty set X. A point x in X is called coincidence point of (f, g) iff $fx = gx$.

Definition 2.8[6]. A pair of self-mappings (f, g) defined on a metric space (X, d) is said to be occasionally weakly compatible iff there is a coincidence point x of the pair (f, g) at which f and g commute.

A pair of self-mappings without coincidence points is weakly compatible as requirements of the definition are vacuously satisfied but such pairs are not interesting in common fixed point considerations. On the other hand, a pair of weakly compatible mappings with at least one coincidence point often furnishes a common fixed point in presence of suitable contractive conditions. But there exist situations when a pair does not commute at a coincidence point (e.g.[Example 1.1, 3]). In such situations, the notion of absorbing pair can be utilized to prove common fixed point theorems.

Definition 2.9[3]. A pair of self- mappings (f, g) defined on a metric space (X, d) is called g-absorbing if there exists some real number $R > 0$ such that $d(gx, gfx) \leq Rd(fx, gx)$ for all x in X. Analogously, (f, g) will be called f-absorbing if there exists some real number $R > 0$ such that $d(fx, fgx) \leq Rd(fx, gx)$ for all x in X. Also, a pair of self-mappings (f, g) is called absorbing if it is both g-absorbing as well as f-absorbing.

Definition 2.10[3]. A pair of self-mappings (f, g) defined on a metric space (X, d) is called pointwise g-absorbing if for a given x in X, there exists some $R > 0$ such that $d(gx, gfx) \leq Rd(fx, gx)$. On similar lines, we can define pointwise f-absorbing map. In particular, if we take g to be the identity map on X, then f is trivially I-absorbing. Similarly, I is also f-absorbing in respect of f.

It has been shown in [3] that a pair of compatible or R-weakly commuting pair need not be g-absorbing or f-absorbing as g-absorbing or f-absorbing pair of maps (individually) need not commute on the set of coincidence points. In respect of such inter-relations, the following observations are straight forward.

Proposition 2.11. *A pair of self- mappings (f, g) defined on a metric space (X, d) is R-weakly commuting (resp. pointwise R-weakly commuting) if the pair is g as well as f-absorbing (resp. pointwise g as well as pointwise f-absorbing).*

Proof. Notice that

$$d(fgx, gfx) \leq d(fx, fgx) + d(fx, gx) + d(gx, gfx)$$

$$\leq rd(fx, gx) + d(fx, gx) + sd(gx, fx) \leq (r + 1 + s)d(fx, gx) = Rd(fx, gx),$$

where $R > 0$. $\square$

The converse of Proposition 2.11 is not true in general as substantiated by the following example.

Example 2.12. Consider $X = [0, 1]$ equipped with natural metric. Define $f, g : X \to X$ by $fx = 1 - x$ and $gx = (1 - x)^2$ for all x in X. One can easily verify that the pair (f, g) is neither g-absorbing nor f-absorbing but it is R-weakly commuting.

Proposition 2.13. *A pair of self- mappings (f, g) defined on a metric space (X, d) is g-absorbing (resp. f-absorbing) if the pair is R-weakly commuting and f-absorbing (resp. g-absorbing).*

Proof. Notice that

$$d(gx, gfx) \le d(fx, gx) + d(fx, fgx) + d(fgx, gfx)$$

$$\le d(fx, gx) + rd(fx, gx) + sd(gx, fx) \le (r + 1 + s)d(fx, gx) = Rd(fx, gx),$$

where $R > 0$. $\qquad \square$

Proposition 2.14. *A pair of self- mappings (f, g) defined on a metric space (X, d) is pointwise g-absorbing (resp. pointwise f-absorbing) if the pair is pointwise R-weakly commuting and pointwise f-absorbing (resp. pointwise g-absorbing).*

Proof. It is similar to that of Proposition 2.13. $\qquad \square$

Proposition 2.15. *Let f and g be two self maps defined on a metric space (X, d) which satisfy the following Lipschitz type condition*

$$d(fx, fy) \le Kd(gx, gy) + a \max\{d(fx, gx) + d(fy, gy), d(fx, gy) + d(fy, gx)\}$$

where $K \ge 0$. Then the pair (f, g) is pointwise absorbing iff the pair is pointwise g-absorbing.

Proof. To prove the if part, suppose that the pair (f, g) is pointwise g-absorbing. Then in order to prove the pair to be pointwise absorbing, we distinguish two cases.
Case I. If (for some $x \in X$) $fx \ne gx$, and the pair (f, g) is g-absorbing, then we can define $R = \frac{d(fx, fgx)}{d(fx, gx)}$, such that $d(fx, fgx) \le Rd(fx, gx)$ i.e. the pair (f, g) is pointwise absorbing.
Case II. If (for some $x \in X$) $fx = gx$, then employing pointwise g-absorbing property of the pair (f, g), we have $fx = gx = gfx$, which in turn yields $gfx = ggx = fx = gx$. Now using given Lipschitz condition with $x = x$ and $y = gx$, we get $fx = fgx$ which shows that the pair (f, g) is pointwise absorbing.
The proof of the converse part is obvious by definition. $\qquad \square$

For other properties and related results of absorbing maps, one can consult [3].

In 1999, Pant [11] proved an interesting fixed point theorem for Lipschitz type mappings which has been generalized in several ways by various authors. To mention a few, we recall Sastry and Murthy [15], Pant and Pant [13], V. Pant [12], Singh and Kumar [16], Gopal et al. [3] and others. The purpose of this note is to present yet another extension of the main theorem contained in Pant [11] which in turn generalizes several previously known results mentioned earlier.

3. Main Results

The following is our main result:

Theorem 3.1. *Let (f, g) be a pair of self-mappings defined on a metric space (X, d) such that*

   (a) *the pair (f, g) satisfies the property (E.A),*
   (b) *$d(fx, fy) \leq Kd(gx, gy) + a \max\{d(fx, gx) + d(fy, gy), d(fx, gy) + d(fy, gx)\}$*
       for all $x, y \in X$, where $K > 0, a < 1$
   (c) *$g(X)$ is a closed subset of X.*

Then

   (d) *the pair (f, g) has a coincidence point,*
   (e) *the pair (f, g) has a common fixed point provided the pair is pointwise g-absorbing.*

Proof. In view of (a), there exists a sequence $\{x_n\}$ in X such that $\lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = t \in X$. By (c), $\exists\ u \in X$ such that $t = gu$. Using (b), we get

$$d(fu, fx_n) \leq Kd(gu, gx_n) + a \max\{d(fu, gu) + d(fx_n, gx_n), d(fu, gx_n) + d(fx_n, gu,)\}$$

On letting $n \to \infty$, we get

$$d(gu, fu) \leq ad(fu, gu),$$

a contradiction as $a < 1$. Thus $fu = gu$ which shows that u is a coincidence point of the maps f and g.

Employing pointwise g-absorbing property of the pair, one can write $fu = gu = gfu$ which in turn yields $gfu = ggu = fu = gu$ so that (in view of Proposition 2.15) $fu = fgu$ and in all $fu = fgu = ffu$. Again using (b) with $x = u$ and $y = fu$, we get $fu = ffu$ so that $fu = ffu = gfu$ which shows that fu is a common fixed point of f and g. This concludes the proof. $\qquad\square$

We now furnish an example to illustrate Theorem 3.1.

Example 3.2. Let X be the set of reals R equipped with usual metric d. Define $f, g : X \to X$ as follows:

$$fx = \begin{cases} 2 , & \text{if} \quad 0 \leq x \leq 2, \quad \text{or} \quad x > 5, \quad x \neq 10 \\[2mm] 6 , & \text{if} \quad 2 < x \leq 5 \\[2mm] 10 , & \text{if } x = 10 \text{ and} \end{cases}$$

$$gx = \begin{cases} 2 , & \text{if} \quad 0 \leq x \leq 2 \quad \text{or} \quad x > \frac{11}{2}, \quad x \neq 10 \\[2mm] 4, & \text{if} \quad 2 < x \leq 5 \\[2mm] \dfrac{x + 1}{3} , & \text{if} \quad x \in (5, \dfrac{11}{2}] \\[2mm] 10 , & \text{if } x = 10. \end{cases}$$

Then f and g satisfy all the conditions of Theorem 3.1 with $K = 24/11$ and have two common fixed points $x = 2$ and $x = 10$. However, it can be pointed out that $\overline{f(X)} = \{2, 6, 10\} \not\subset [2, \frac{13}{6}] \cup \{4, 10\} = g(X)$ while $g(X)$ is closed subset of R which demonstrates the utility of Theorem 3.1 over the corresponding theorem of Pant

[11]. On the other hand, notice that at $x = 4$, f and g do not satisfy any of the following conditions employed in Pant [11,12,13] and Sastry et al. [15]:

(i) $d(fx, ffx) < d(fx, gx) + d(gx, gfx) + d(gfx, ffx)$,

(ii) $d(x, fx) < \max\{d(x, gx), d(fx, gx)\}$

    or

(iii) $d(fx, ffx) \neq \max\{d(gx, gfx), d(fx, gx), d(ffx, gfx), d(fx, gfx), d(gx, ffx)\}$

whenever the right hand side of all the preceeding inequalities is non zero.

Recall that one of the foregoing inequalities (i), (ii) or (iii) was a prime requirement of Theorem 2.2 of Sastry et al. [15] (see also [6,16]). Thus, use of absorbing pair is more adequate to ascertain the existence of common fixed points as compared to a condition such as (i), (ii) or (iii) as utilized in [11,12,13,15].

The following example shows that condition (b) is necessary in Theorem 3.1.

Example 3.3. Consider $X = [0, 1]$ equipped with usual metric. Define f and g on X by $fx = 1$ if $x \neq 1$, $f1 = 0$ and $gx = 1$ for all x. One can easily verify that all the conditions of Theorem 3.1 except condition (b). Notice that with $x = 0$ and y= 1, the condition (b) gives rise

    $1 \leq a.1$

which is a contradiction as $a < 1$.

Our next example shows that pointwise g-absorbing property can not be replaced by pointwise f-absorbing in Theorem 3.1.

Example 3.4. If we interchange the roles of f and g in Example 3.3, then the conditions (a), (b) and (c) of Theorem 3.1 are satisfied and all the points of the set $[0, 1)$ are the coincidence point of the pair (f, g). Also the pair (f, g) is pointwise f-absorbing but not pointwise g-absorbing. Notice that this pair has no common fixed point. Thus, this example shows that pointwise g-absorbing property can not be replaced by pointwise f-absorbing in Theorem 3.1.

As a corollary of Theorem 3.1, we derive a sharpened version of the only result contained in Pant [11] and Theorem 1 of V. Pant [12] which runs as follows.

Corollary 3.5. *Let f and g be self mappings of a metric space (X, d) such that*

(h) *the pair (f, g) satisfies the property (E.A),*

(j) $d(fx, fy) \leq Kd(gx, gy), \quad K \geq 0,$ *for all $x, y \in X$,*

(k) $g(X)$ *is a closed subset of X.*

Then

(l) *the pair (f, g) has a coincidence point,*

(m) *the pair (f, g) has a common fixed point provided it is pointwise g-absorbing.*

Our next theorem is proved for g-continuous pair which is essentially more general then Lipschitz type mappings.

Theorem 3.6. *Let f and g be self mappings of a metric space (X, d) such that*

(n) *the pair (f, g) satisfies the property (E.A),*

(o) f *is g-continuous,*

(p) $g(X)$ *is a closed subset of X.*

Then

(q) *the pair (f, g) has a coincidence point,*

(r) *the pair (f, g) has a common fixed point provided it is pointwise absorbing.*

Proof. By (n), there exists a sequence $\{x_n\}$ in X such that $\lim_n fx_n = \lim_n gx_n = t \in X$. By (p), $\exists\, u \in X$ such that $t = gu$ and hence $gx_n \to gu$. By (o), we get $fx_n \to fu$ and hence $fu = gu$ which shows that u is a coincidence point of the pair (f, g). Using pointwise absorbing property, one gets $fu = gu = gfu$ and $fu = fgu$. Therefore, $ffu = fgu = fu = gfu$ and hence fu is a common fixed point of f and g. $\hfill\square$

To illustrate Theorem 3.6, we furnish the following example.

Example 3.7. Let $X = [2, 20]$ and d be the usual metric on X. Define $f, g : X \to X$ as

$$fx = 2 \;\; \text{if} \;\; x = 2 \;\; \text{or} \;\; x > 5, \qquad\qquad fx = 7 \quad \text{if} \;\; 2 < x \le 5$$

$$g2 = 2, \;\; gx = 7 \quad \text{if} \quad 2 < x \le 5, \qquad\qquad gx = (x+1)/3 \quad \text{if} \;\; x > 5$$

Then f and g satisfy all the conditions of Theorem 3.6 and have a common fixed point $x = 2$ but the pair (f, g) is not Lipschitzian whenever $x \in (2, 5]$, and $y = 20$. Further, at $x \in (2, 5]$, f and g do not satisfy the condition

$$d(fx, ffx) \ne \max\{d(gx, gfx), d(fx, gx), d(ffx, gfx), d(fx, gfx), d(gx, ffx)\}$$

whenever the right hand side is non zero. Thus Theorem 3.6 is a genuine extension of Theorem 2.2 due to Sastry et al. [15].

In the next theorem, we show that the use of non-compatibility instead of the property (E.A.) forces the mappings in the pair to turn discontinuous at their common fixed point. Thus, we provide an instance of the existence of a non-contractive definition which is strong enough to generate fixed point wherein the mappings in the pair are not continuous (see Rhoades [14]).

Theorem 3.8. *Let f and g be non-compatible pointwise g-absorbing self mappings of a metric space (X, d) satisfying the conditions;*

(s) *$d(fx, fy) \le Kd(gx, gy) + a \max\{d(fx, gx) + d(fy, gy), d(fx, gy) + d(fy, gx)\}$ for all $x, y \in X$, where $K \ge 0, a < 1$.*

(t) *$g(X)$ is a closed subset of X.*

(u) *Then f and g have a common fixed point and the fixed point is a point of discontinuity for both the maps in pair.*

Proof. Since the pair (f, g) is non-compatible as well as g-absorbing, there exists a sequence $\{x_n\}$ in X such that

(3.3.1) $\lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = t$

for some t in X but either $\lim_{n \to \infty} d(fgx_n, gfx_n) \ne 0$ or the limit does not exist. Now from (t), $\exists\, u \in X$ such that $t = gu$ where $t = \lim_{n \to \infty} gx_n$. By (s), we get

$$d(fu, fx_n) \le Kd(gu, gx_n) + a \max\{d(fu, gu) + d(fx_n, gx_n), d(fx_n, gu) + d(fu, gx_n)\}$$

On letting $n \to \infty$ we get,

$$d(gu, fu) \le ad(fu, gu)$$

a contradiction (since $a < 1$). Thus $fu = gu$ which shows that u is a coincidence point of the pair (f, g). Since the pair (f, g) is pointwise g-absorbing at $x = u$ implies $fu = gu = gfu$, therefore $gfu = ggu = fu = gu$. Again applying (s)

and using Proposition 2.15 for $x = u$ and $y = fu$ we get $fu = ffu$. Hence $fu = ffu = gfu$ and hence fu is a common fixed point of f and g.

We now show that f and g are discontinuous at the common fixed point $t = fu = gu$. If possible, suppose f is continuous. Then consider a sequence $\{x_n\}$ as assumed in (3.3.1), we have $\lim_{n\to\infty} ffx_n = ft = t$ and $\lim_{n\to\infty} fgx_n = ft = t$. Since the g-absorbing property of the pair (f, g) implies that $d(gx_n, gfx_n) \leq Rd(fx_n, gx_n)$ which on letting $n \to \infty$ gives $\lim_{n\to\infty} gfx_n = \lim_{n\to\infty} gx_n = t = ft$ yielding thereby $\lim_{n\to\infty} d(fgx_n, gfx_n) = d(ft, t) = 0$. This contradicts the fact that $\lim_{n\to\infty} d(fgx_n, gfx_n)$ is either nonzero or nonexistent for the sequence $\{x_n\}$ satisfying (3.3.1). Hence f is discontinuous at the fixed point.

Next, suppose g is continuous. Then for the sequence $\{x_n\}$ satisfying (3.3.1), we get $\lim_{n\to\infty} gfx_n = gt = t$ and $\lim_{n\to\infty} ggx_n = gt =$t. In view of these limits, the inequality

$$d(fgx_n, fx_n) \leq Kd(ggx_n, gx_n) + a \max\{d(fgx_n, ggx_n) + d(fx_n, gx_n),$$

$$d(fgx_n, , gx_n) + d(fx_n, ggx_n)\},$$

yields a contradiction unless $\lim_{n\to\infty} fgx_n = t = gt$. But $\lim_{n\to\infty} fgx_n = t = gt$ and $\lim_{n\to\infty} gfx_n = t = gt$ contradict the fact that $\lim_{n\to\infty} d(fgx_n, gfx_n)$ is either nonzero or nonexistent. Thus both f and g are discontinuous at the common fixed point. This completes the proof of the theorem. $\square$

Example 3.9. Again consider Example 3.2 wherein f and g satisfy all the conditions of Theorem 3.8 with $K = 24/11$ and have two common fixed points (2 and 10) at which the functions f and g are discontinuous. It can also be verified that the pair (f, g) is non-compatible as well as g-absorbing. To substantiate this claim, consider the sequence $x_n = 5 + 1/2n : n > 1$, then $\lim_{n\to\infty} fx_n = 2 = \lim_{n\to\infty} gx_n$ but $\lim_{n\to\infty} fgx_n = 6$ and $\lim_{n\to\infty} gfx_n = 2$ which shows that f and g are non-compatible. The verification of pointwise g-absorbing property is straight forward.

Finally, we present an example which shows that the requirement of pointwise g-absorbing property is necessary for producing common fixed points of mappings satisfying non-contractive or Lipschitz type conditions besides exhibiting the limitations of commuting properties of the pairs utilized in earlier related results (e.g.[11-12,15,16]).

Example 3.10. Let $X = [2, 20]$ endowed with the usual metric and define $f, g : X \to X$ by

$$fx = 6 \text{ if } 2 \leq x < 6 \text{ or } x > 6, \ f6 = 13/2,$$

$$gx = 5 \text{ if } 2 \leq x \leq 5, \ gx = (x + 7)/2 \quad \text{if} \quad 5 < x \leq 6$$

$$gx = 10 \text{ if } 6 < x < 13/2 \text{ or } x > 13/2 \ \& \ g(13/2) = 6$$

Then by a routine calculation, it can be verified that $\overline{f(X)} \subseteq g(X)$ and $d(fx, fy) \leq Kd(gx, gy)$ for all $x, y \in X$, where $K \geq 0$. Also, f and g are non-compatible pointwise R-weakly commuting pair. In order to show that (f, g) is non-compatible, the sequence $x_n = 5 + 1/n; n > 1, n \in N$ satisfies the requirements. Also, it is straight forward to verify that the pair (f, g) is not pointwise g-absorbing in respect of $x = 6$. On the other hand, at $x = 6$, it can be verified that the mappings f and

g do not satisfy any one of the conditions described by (i), (ii) or (iii) mentioned earlier. Notice that the esteemed pair has no common fixed point.

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