

ON QUASI EINSTEIN MANIFOLDS ADMITTING A RICCI
QUARTERSYMMETRIC METRIC CONNECTION

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ABSTRACT. The object of the present paper is to study quasi Einstein manifolds admitting a Ricci quartersymmetric metric connection satisfying certain curvature conditions.

1. INTRODUCTION

It is well known that a Riemannian manifold or a semi-Riemannian manifold M^n , $n = \dim M > 2$, is said to be an *Einstein manifold* if its Ricci tensor R_{kj} of type $(0, 2)$ is of the form

$$R_{kj} = \frac{R}{n} g_{kj} \quad (1.1)$$

where R is the scalar curvature of the manifold. According to [1], (1.1) is called the Einstein metric condition.

In 2000, the notion of a quasi Einstein manifold was introduced by Chaki and Maity [2]. A non-flat Riemannian manifold M^n is called a *quasi Einstein manifold* if its Ricci tensor R_{kj} of type $(0, 2)$ is not identically zero and satisfies the following:

$$R_{kj} = ag_{kj} + bp_k p_j \quad (1.2)$$

where a and b are scalar functions and p_k is a non-zero covariant vector. It is obvious that if $b = 0$, then this manifold reduces to an Einstein manifold if $a = \frac{R}{n}$, namely, Einstein manifolds form a subclass of quasi Einstein manifolds. Quasi Einstein manifolds arose in the study of exact solutions of the Einstein field equations and in the study of quasi-umbilical hypersurfaces of conformally flat spaces. This manifold has many applications in physics, especially general relativity, e.g, a quasi Einstein manifold can be taken as a model of perfect fluid spacetime in general relativity [7]. This manifold has also been studied by De and Ghosh [6] and De and De [5] and many others.

It is to be noted that Chen and Yano [3] introduced the notion of a manifold of quasi-constant curvature. A conformally flat Riemannian manifold M^n is called a

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manifold of quasi-constant curvature if its curvature tensor R_{ikjm} of type $(0, 4)$ is of the form

$$R_{ikjm} = \beta [g_{kj}g_{im} - g_{ij}g_{km}] + \gamma [g_{kj}\eta_i\eta_m - g_{ij}\eta_k\eta_m + g_{im}\eta_k\eta_j - g_{km}\eta_i\eta_j] \quad (1.3)$$

where β, γ are scalar functions, η is a 1-form. In particular, if $\gamma = 0$, then it reduces to the manifold of constant curvature.

In 1970, Yano [12] studied some curvature and derivational conditions for semi-symmetric connections in Riemannian manifolds. In 1975, Golab [8] introduced the idea of a quarter-symmetric connection in differentiable manifolds. In 1980, Misra and Pandey [10] introduced a quarter-symmetric F-connection in Riemannian manifolds with F-structures; especially, they considered the case of Kaehlerian structure. In 1982, Yano and Imai [13] studied some curvature conditions for quarter-symmetric metric connections in Riemannian, Hermitian and Kaehlerian manifolds.

A linear connection $\bar{\nabla}$ in a Riemannian manifold M^n is called a *semi-symmetric connection* if its torsion tensor T satisfies the relation [12]

$$T(X, Y) = \pi(Y)X - \pi(X)Y$$

where π is a 1-form. In [4], De and De studied some properties of this connection. A linear connection $\bar{\nabla}$ in M^n is called a *quarter symmetric connection* if its torsion tensor T satisfies the relation

$$T(X, Y) = \pi(Y)LX - \pi(X)LY \quad (1.4)$$

where π has the meaning already stated, L is a tensor field of type $(1, 1)$ [13].

If there is a Riemannian metric g on M^n such that

$$\bar{\nabla}g = 0$$

then the connection $\bar{\nabla}$ is called a *metric connection* [12].

In (1.4), if a tensor field L is a Ricci tensor of a Riemannian manifold M^n , then the connection $\bar{\nabla}$ of the manifold M^n is called a *Ricci quartersymmetric connection* [10]. In local coordinates, the torsion tensor T of the connection $\bar{\nabla}$ is the following form

$$T_{ik}^l = R_i^l\pi_k - R_k^l\pi_i \quad (1.5)$$

where π is a 1-form associated with the torsion tensor of the connection $\bar{\nabla}$ and $R_i^l = R_{it}g^{tl}$. From (1.5), it is clear that $\bar{\nabla}$ is a metric connection. Some properties of this connection has been studied by Kamilya and De [9].

Thus, the relation between Γ_{ik}^l and $\left\{ \begin{smallmatrix} l \\ ik \end{smallmatrix} \right\}$ is given by [13], [10],

$$\Gamma_{ik}^l = \left\{ \begin{smallmatrix} l \\ ik \end{smallmatrix} \right\} + R_i^l\pi_k - R_{ik}\pi^l \quad (1.6)$$

where Γ_{ik}^l and $\left\{ \begin{smallmatrix} l \\ ik \end{smallmatrix} \right\}$ are the connection coefficients of the Ricci quartersymmetric metric connection $\bar{\nabla}$ and the Levi-Civita connection ∇ , respectively and $\pi^l = \pi_t g^{tl}$, π^l being contravariant components of π_t .

The paper is organized as follows: Some theorems about a Riemannian manifold with a Ricci quartersymmetric metric connection and whose Ricci tensor R_{kj} is Codazzi type are proved. In section 4, it is obtained necessary condition that the Ricci quartersymmetric metric connection reduces to the semi symmetric metric

connection and under this condition, some properties of this manifold are also studied.

2. PRELIMINARIES

Let \bar{R}_{ikj}^l and R_{ikj}^l be denoted the curvature tensor of the connection $\bar{\nabla}$ and ∇ respectively. The curvature tensor \bar{R}_{ikj}^l is defined by

$$\bar{R}_{ikj}^l = \frac{\partial \Gamma_{kj}^l}{\partial x^i} - \frac{\partial \Gamma_{ij}^l}{\partial x^k} + \Gamma_{it}^l \Gamma_{kj}^t - \Gamma_{kt}^l \Gamma_{ij}^t \quad (2.1)$$

Then substituting (1.6) in (2.1), it is obtained the following equation for the curvature tensor of the connection $\bar{\nabla}$

$$\begin{aligned} \bar{R}_{ikjm} &= R_{ikjm} - R_{im}\pi_{kj} + R_{km}\pi_{ij} - \pi_{im}R_{kj} + \pi_{km}R_{ij} \\ &\quad + (\nabla_i R_{km} - \nabla_k R_{im})\pi_j - (\nabla_i R_{kj} - \nabla_k R_{ij})\pi_m \end{aligned} \quad (2.2)$$

where

$$\pi_{kj} = \nabla_k \pi_j - R_{kt} \pi^t \pi_j + \frac{1}{2} \pi_t \pi^t R_{kj} \quad (2.3)$$

and $\pi_j^l = \pi_{jt} g^{tl}$.

Contracting for i and m in (2.2), we get

$$\begin{aligned} \bar{R}_{kj} &= R_{kj} - R\pi_{kj} + R_k^i \pi_{ij} - \pi R_{kj} + \pi_k^i R_{ij} \\ &\quad + (\nabla_i R_k^i - \nabla_k R)\pi_j - (\nabla_i R_{kj} - \nabla_k R_{ij})\pi^i \end{aligned} \quad (2.4)$$

where $R = R_{im} g^{im}$ and $g^{im} \pi_m = \pi^i$, $\pi = \pi_{im} g^{im}$. Moreover, it follows from (1.2) that

$$R = an + bP \quad (2.5)$$

where $P = p_i p^i$.

3. SOME PROPERTIES OF A RIEMANNIAN MANIFOLD ADMITTING A RICCI QUARTERSYMMETRIC METRIC CONNECTION

In this section, we give some theorems of a Riemannian manifold admitting a Ricci quartersymmetric metric connection. We assume that the Ricci tensor R_{kj} is of Codazzi type to ∇ . Then we have

$$\bar{R}_{ikjm} = R_{ikjm} - R_{im}\pi_{kj} + R_{km}\pi_{ij} - \pi_{im}R_{kj} + \pi_{km}R_{ij} \quad (3.1)$$

By the aid of the equation (3.1), we can write

$$\begin{aligned} \bar{R}_{ikjm} - \bar{R}_{jmik} &= (\pi_{mi} - \pi_{im})R_{jk} + (\pi_{ij} - \pi_{ji})R_{mk} \\ &\quad + (\pi_{jk} - \pi_{kj})R_{im} + (\pi_{km} - \pi_{mk})R_{ij} \end{aligned} \quad (3.2)$$

If the tensor π_{kj} is symmetric, then we find

$$\bar{R}_{ikjm} = \bar{R}_{jmik} \quad (3.3)$$

In view of the equation (3.1), we get

$$\begin{aligned} \bar{R}_{ikjm} + \bar{R}_{kjim} + \bar{R}_{jikm} &= (\pi_{ij} - \pi_{ji})R_{km} \\ &\quad + (\pi_{ki} - \pi_{ik})R_{jm} + (\pi_{jk} - \pi_{kj})R_{im} \end{aligned} \quad (3.4)$$

Considering that the tensor π_{kj} is symmetric in (3.4), then we can obtain

$$\bar{R}_{ikjm} + \bar{R}_{kjim} + \bar{R}_{jikm} = 0$$

Now, let us find the condition to be the symmetry of the Ricci tensor with respect to $\bar{\nabla}$. Since R_{kj} is of Codazzi type to ∇ , it follows from (2.4) that

$$\bar{R}_{kj} = R_{kj} - R\pi_{kj} + R_k^i \pi_{ij} - \pi R_{kj} + \pi_k^i R_{ij} \quad (3.5)$$

By using (3.5), it can be written

$$\bar{R}_{jk} = R_{jk} - R\pi_{jk} + R_j^i \pi_{ik} - \pi R_{jk} + \pi_j^i R_{ik} \quad (3.6)$$

Thus, from the equations (3.5) and (3.6), it follows that

$$\bar{R}_{kj} - \bar{R}_{jk} = R(\pi_{jk} - \pi_{kj}) + R_j^t (\pi_{kt} - \pi_{tk}) + R_k^t (\pi_{tj} - \pi_{jt}) \quad (3.7)$$

Finally, if the tensor π_{kj} is symmetric, then the equation (3.7) gives

$$\bar{R}_{kj} = \bar{R}_{jk}$$

Thus, we can state the following theorem:

Theorem 3.1. *Let a Riemannian manifold M^n admit a Ricci quartersymmetric metric connection whose Ricci tensor is Codazzi type. If the tensor π_{kj} is symmetric, then the curvature tensor and the Ricci tensor of $\bar{\nabla}$ satisfy the following properties:*

- (i) $\bar{R}_{ikjm} = \bar{R}_{jmik}$,
- (ii) $\bar{R}_{ikjm} + \bar{R}_{kjim} + \bar{R}_{jikm} = 0$,
- (iii) $\bar{R}_{kj} = \bar{R}_{jk}$.

After that, using the equation (3.1), we can write

$$\begin{aligned} \nabla_l \bar{R}_{ikjm} &= \nabla_l R_{ikjm} - \nabla_l (R_{im} \pi_{kj}) + \nabla_l (R_{km} \pi_{ij}) \\ &\quad - \nabla_l (\pi_{im} R_{kj}) + \nabla_l (\pi_{km} R_{ij}) \end{aligned} \quad (3.8)$$

Considering that the Ricci tensor R_{kj} of ∇ is Codazzi type and using (3.8), it follows that

$$\begin{aligned} \nabla_l \bar{R}_{ikjm} + \nabla_i \bar{R}_{kljm} + \nabla_k \bar{R}_{lijm} &= R_{km} [\nabla_l \pi_{ij} - \nabla_i \pi_{lj}] \\ &\quad + R_{lm} [\nabla_i \pi_{kj} - \nabla_k \pi_{ij}] \\ &\quad + R_{im} [\nabla_k \pi_{lj} - \nabla_l \pi_{kj}] \\ &\quad + R_{lj} [\nabla_k \pi_{im} - \nabla_i \pi_{km}] \\ &\quad + R_{ij} [\nabla_l \pi_{km} - \nabla_k \pi_{lm}] \\ &\quad + R_{kj} [\nabla_i \pi_{lm} - \nabla_l \pi_{im}] \end{aligned} \quad (3.9)$$

If the tensor π_{kj} is of Codazzi type to ∇ , from (3.9), we get

$$\nabla_l \bar{R}_{ikjm} + \nabla_i \bar{R}_{kljm} + \nabla_k \bar{R}_{lijm} = 0 \quad (3.10)$$

In view of this, we can state the following:

Theorem 3.2. *Let a Riemannian manifold M^n admit a Ricci quartersymmetric metric connection whose Ricci tensor is Codazzi type. If the tensor π_{kj} which is of Codazzi type to ∇ is symmetric, then the curvature tensor \bar{R}_{ikjm} satisfies the second Bianchi Identity to the connection ∇ .*

4. SECTIONAL CURVATURE OF A QUASI EINSTEIN MANIFOLD WITH A SPECIAL RICCI QUARTERSYMMETRIC METRIC CONNECTION

In this section, necessary condition that the Ricci quartersymmetric metric connection reduces to the semi-symmetric metric connection is obtained by proving the following theorem. After that, under this condition, we study some properties of this manifold.

By the use of (1.2) and (1.6), we get

$$\Gamma_{ik}^l = \left\{ \begin{matrix} l \\ ik \end{matrix} \right\} + a\delta_i^l\pi_k - ag_{ik}\pi^l + bp_i(p^l\pi_k - p_k\pi^l) \quad (4.1)$$

If p_i and π_i are collinear, we have

$$\pi_k = cp_k \quad (4.2)$$

where c is a non-zero constant. From (4.1) and (4.2), we get

$$\Gamma_{ik}^l = \left\{ \begin{matrix} l \\ ik \end{matrix} \right\} + ac\delta_i^l p_k - acp^l g_{ik} \quad (4.3)$$

From (2.1) and (4.3), the curvature tensor of a Riemannian manifold admitting a Ricci quartersymmetric metric connection is the following form

$$\bar{R}_{ikjm} = R_{ikjm} - g_{im}\pi_{kj} + g_{km}\pi_{ij} - \pi_{im}g_{kj} + \pi_{km}g_{ij} \quad (4.4)$$

where

$$\pi_{kj} = (ac)\nabla_k p_j - (ac)^2 p_k p_j + \frac{(ac)^2}{2} g_{kj} P \quad (4.5)$$

and a is a non-zero constant. Therefore, we can state the following:

Theorem 4.1. *Let M^n be a quasi Einstein manifold admitting a Ricci quartersymmetric metric connection. If p_i and π_i are collinear, then the Ricci quartersymmetric metric connection reduces to a semi-symmetric metric connection.*

From (4.5), it is clear that

$$\pi_{kj} - \pi_{jk} = (ac)(\nabla_k p_j - \nabla_j p_k) \quad (4.6)$$

Using (1.2) and (2.3), we get

$$\pi_{kj} = c\nabla_k p_j - c^2 \left(a + \frac{bP}{2} \right) p_k p_j + \frac{ac^2}{2} g_{kj} P \quad (4.7)$$

Thus, it follows from (4.5) and (4.7) that

$$(1-a)\nabla_k p_j - c \left(a - a^2 + \frac{bP}{2} \right) p_k p_j + \frac{ac(1-a)}{2} g_{kj} P = 0 \quad (4.8)$$

Hence, we can state the following:

Theorem 4.2. *Let a Riemannian manifold M^n admit a Ricci quartersymmetric metric connection whose Ricci tensor satisfies the condition (1.2). If p_k and π_k are collinear then p_k satisfies the equation (4.8)*

Now, we consider the conformal curvature tensor C_{ikjm} defined by

$$\begin{aligned} C_{ikjm} &= R_{ikjm} - \frac{1}{(n-2)}(R_{kj}g_{im} - R_{ij}g_{km} + g_{kj}R_{im} - g_{ij}R_{km}) \\ &\quad + \frac{R}{(n-1)(n-2)}(g_{im}g_{kj} - g_{ij}g_{km}) \end{aligned} \quad (4.9)$$

Suppose that the sectional curvature of a linear connection $\bar{\nabla}$ is independent from the orientation chosen. Using Theorem 4.1 and remembering that Riemannian manifold with a semi-symmetric connection is conformally flat and the curvature tensor of $\bar{\nabla}$ is zero, if the sectional curvature of this manifold is independent from the orientation chosen [14].

Thus, from the equation (4.9), we obtain

$$R_{ikjm} = \frac{1}{(n-2)} (R_{kj}g_{im} - R_{ij}g_{km} + g_{kj}R_{im} - g_{ij}R_{km}) - \frac{R}{(n-1)(n-2)} (g_{im}g_{kj} - g_{ij}g_{km}) \quad (4.10)$$

Substituting the equation (1.2) in (4.10), we get

$$R_{ikjm} = \left[\frac{2a}{(n-2)} - \frac{R}{(n-1)(n-2)} \right] (g_{im}g_{kj} - g_{ij}g_{km}) + \frac{b}{(n-2)} [p_k p_j g_{im} - p_i p_j g_{km} + g_{kj} p_i p_m - g_{ij} p_k p_m] \quad (4.11)$$

Hence, by virtue of (1.3), we can state the following:

Theorem 4.3. *Suppose that a quasi Einstein manifold admits a Ricci quarter-symmetric metric connection whose 1-form π_k is a non-zero constant multiple of a covariant vector p_k and this manifold has a sectional curvature of the connection $\bar{\nabla}$ independent from the orientation chosen, then the manifold is of a quasi constant curvature.*

In a conformally flat Riemannian manifold, we know that

$$\nabla_i R_{kj} - \nabla_j R_{ki} = \frac{1}{2(n-2)} [(\nabla_i R) g_{kj} - (\nabla_j R) g_{ki}] \quad (4.12)$$

Multiplying (4.4) by g^{im} , we get

$$\bar{R}_{kj} = R_{kj} - (n-2)\pi_{kj} - \pi g_{kj} \quad (4.13)$$

Since the sectional curvature of this manifold is independent from the orientation chosen, the curvature tensor is zero. By using (4.13), we can say that π_{kj} is symmetric. From (4.6), p_j is gradient.

If b is chosen as a constant, using (1.2) in the equation (4.12), we have

$$b [p_j (\nabla_i p_k) - p_i (\nabla_j p_k)] = \frac{1}{2(n-2)} [(\nabla_i R) g_{kj} - (\nabla_j R) g_{ki}] \quad (4.14)$$

Transvecting (4.14) with g^{kj} , we find

$$b [p^k (\nabla_i p_k) - p_i (\nabla_k p^k)] = \frac{(n-1)}{2(n-2)} \nabla_i R \quad (4.15)$$

where $p^k = g^{km} p_m$. As we know, the following property is satisfied in a Riemannian manifold.

$$\nabla_m R_j^m = \frac{\nabla_j R}{2} \quad (4.16)$$

Thus, from (1.2) and (4.16), we obtain

$$b \nabla_m (p_j p^m) = \frac{\nabla_j R}{2} \quad (4.17)$$

Hence, we get

$$b(p_i \nabla_k p^k) = \frac{\nabla_i R}{2} - b p^k \nabla_k p_i \quad (4.18)$$

In this case, substituting (4.18) in the equation (4.15) and since p_k is a gradient vector, we have

$$4b(p^k \nabla_i p_k) = \frac{(2n-3)}{(n-2)} \nabla_i R \quad (4.19)$$

For a component of tangent vector field of a geodesic line, we have [11]

$$p^k \nabla_k p^i = 0 \quad (4.20)$$

Thus, from the equations (4.19) and (4.20), we have

$$\nabla_i R = 0 \quad (4.21)$$

We see that the scalar curvature R is a constant. Moreover, from (4.12) and (4.21) we get

$$\nabla_i R_{kj} - \nabla_j R_{ki} = 0 \quad (4.22)$$

Therefore, the Ricci tensor R_{kj} is of Codazzi type to ∇ .

This leads to the following theorem:

Theorem 4.4. *Let a conformally flat quasi Einstein manifold, ($b \equiv \text{constant}$), admit a Ricci quartersymmetric metric connection whose a covariant vector π_k is a non zero constant multiple of a covariant vector p_k . If p_k is a component of the tangent vector field of the geodesic line, then the Ricci tensor R_{kj} is of Codazzi type to ∇ and the scalar curvature R is also a constant.*

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