

SHARPNESS OF NEGOI'S INEQUALITY FOR THE EULER-MASCHERONI CONSTANT

(COMMUNICATED BY ARMEND SHABANI)

CHAO-PING CHEN

ABSTRACT. We present new estimates for the Euler-Mascheroni constant, which improve a result of Negoi.

1. INTRODUCTION

The Euler-Mascheroni constant $\gamma = 0.577215664\dots$ is defined as the limit of the sequence

$$D_n = \sum_{k=1}^n \frac{1}{k} - \ln n \quad (n \in \mathbb{N} := \{1, 2, 3, \dots\}).$$

Several bounds for $D_n - \gamma$ have been given in the literature [3, 4, 19, 22, 23, 24, 27] (see also [6, 20, 21]). For example, the following bounds for $D_n - \gamma$ were established in [19, 27]:

$$\frac{1}{2(n+1)} < D_n - \gamma < \frac{1}{2n} \quad (n \in \mathbb{N}).$$

The convergence of the sequence D_n to γ is very slow. Some quicker approximations to the Euler-Mascheroni constant were established in [5, 6, 7, 9, 8, 10, 15, 16, 18, 20, 21, 25, 26]. For example, Negoi [18] proved that the sequence

$$T_n = \sum_{k=1}^n \frac{1}{k} - \ln \left(n + \frac{1}{2} + \frac{1}{24n} \right) \quad (1.1)$$

is strictly increasing and convergent to γ . Moreover, the author proved that

$$\frac{1}{48(n+1)^3} < \gamma - T_n < \frac{1}{48n^3}. \quad (1.2)$$

The main objective of this work is to establish closer bounds for $\gamma - T_n$.

2000 *Mathematics Subject Classification.* Primary 11Y60; Secondary 40A05.

Key words and phrases. Euler-Mascheroni constant; Inequality; Rate of convergence; Digamma function.

©2011 Universiteti i Prishtinës, Prishtinë, Kosovë.

Submitted January 5, 2011. Published January 17, 2011.

2. LEMMAS

Before stating and proving the main theorems, we first include here some preliminary results.

The constant γ is deeply related to the gamma function $\Gamma(x)$ thanks to the Weierstrass formula [1, p. 255]:

$$\Gamma(x) = \frac{e^{-\gamma x}}{x} \prod_{k=1}^{\infty} \left\{ \left(1 + \frac{x}{k}\right)^{-1} e^{x/k} \right\}$$

for any real number x , except on the negative integers $\{0, -1, -2, \dots\}$. The logarithmic derivative of the gamma function:

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$$

is known as the psi (or digamma) function.

The following recurrence and asymptotic formulas are well known for the psi function:

$$\psi(x+1) = \psi(x) + \frac{1}{x} \quad (2.1)$$

(see [1, p.258]), and

$$\psi(x) \sim \ln x - \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^6} + \dots \quad (x \rightarrow \infty) \quad (2.2)$$

(see [1, p.259]). From (2.1) and (2.2), we get

$$\psi(x+1) \sim \ln x + \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^6} + \dots \quad (x \rightarrow \infty). \quad (2.3)$$

It is also known [1, p.258] that

$$\psi(n+1) = -\gamma + \sum_{k=1}^n \frac{1}{k}. \quad (2.4)$$

The following lemmas are also needed in our present investigation.

Lemma 2.1. *If the sequence $(\lambda_n)_{n \in \mathbb{N}}$ converges to zero and if there exists the following limit:*

$$\lim_{n \rightarrow \infty} n^k (\lambda_n - \lambda_{n+1}) = l \in \mathbb{R} \quad (k > 1),$$

then

$$\lim_{n \rightarrow \infty} n^{k-1} \lambda_n = \frac{l}{k-1} \quad (k > 1).$$

This lemma is suitable for accelerating some convergences, or in constructing some asymptotic expansions. For proofs and other details, see, e.g. [11, 12, 13, 14, 15, 16, 17].

Lemma 2.2 ([2, Theorem 9]). *Let $k \geq 1$ and $n \geq 0$ be integers. Then for all real numbers $x > 0$:*

$$S_k(2n; x) < (-1)^{k+1} \psi^{(k)}(x) < S_k(2n+1; x), \quad (2.5)$$

where

$$S_k(p; x) = \frac{(k-1)!}{x^k} + \frac{k!}{2x^{k+1}} + \sum_{i=1}^p \left[B_{2i} \prod_{j=1}^{k-1} (2i+j) \right] \frac{1}{x^{2i+k}},$$

and B_i ($i = 0, 1, 2, \dots$) are Bernoulli numbers, defined by

$$\frac{t}{e^t - 1} = \sum_{i=0}^{\infty} B_i \frac{t^i}{i!}$$

(see [1, p. 804]).

In particular, it follows from (2.5) that

$$\begin{aligned} \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7} - \frac{1}{30x^9} &< \psi'(x) \\ &< \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7} - \frac{1}{30x^9} + \frac{5}{66x^{11}}, \quad x > 0. \end{aligned} \quad (2.6)$$

From (2.1) and (2.6), we obtain

$$\begin{aligned} \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7} - \frac{1}{30x^9} &< \psi'(x+1) \\ &< \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7}, \quad x > 0. \end{aligned} \quad (2.7)$$

3. MAIN RESULTS

3.1. We define the sequence $(u_n)_{n \in \mathbb{N}}$ by

$$u_n = \ln \left(n + \frac{1}{2} + \frac{1}{24n} \right) - \psi(n+1) - \frac{a}{\left(n + b + \frac{c}{n+d} \right)^3}. \quad (3.1)$$

We are interested in finding the values of the parameters a , b , c and d such that $(u_n)_{n \in \mathbb{N}}$ is the *fastest* sequence which would converge to zero. This provides the best approximations of the form:

$$\psi(n+1) \approx \ln \left(n + \frac{1}{2} + \frac{1}{24n} \right) - \frac{a}{\left(n + b + \frac{c}{n+d} \right)^3}. \quad (3.2)$$

Our study is based on the above Lemma 2.1.

Theorem 3.1. *Let the sequence $(u_n)_{n \in \mathbb{N}}$ be defined by (3.1). Then for*

$$a = \frac{1}{48}, \quad b = \frac{83}{360}, \quad c = \frac{4909}{64800}, \quad d = \frac{11976997}{37112040}, \quad (3.3)$$

we have

$$\lim_{n \rightarrow \infty} n^8 (u_n - u_{n+1}) = \frac{1763157528883853}{83111968235520000} \quad (3.4)$$

and

$$\lim_{n \rightarrow \infty} n^7 u_n = \frac{1763157528883853}{581783777648640000}. \quad (3.5)$$

The speed of convergence of the sequence $(u_n)_{n \in \mathbb{N}}$ is given by the order estimate $O(n^{-7})$.

Proof. First of all, we write the difference $u_n - u_{n+1}$ as the following power series in n^{-1} :

$$\begin{aligned}
u_n - u_{n+1} = & \frac{1 - 48a}{16n^4} + \frac{-263 + 8640a + 17280ab}{1440n^5} \\
& + \frac{139 - 3840a - 11520ab - 11520ab^2 + 5760ac}{384n^6} \\
& + \left(90720a + 362880ab + 362880ab^3 + 544320ab^2 - 272160ac \right. \\
& \quad \left. - 435456acb - 108864acd - 3685 \right) \frac{1}{6048n^7} \\
& + \left(-193536a - 967680ab + 774144abcd + 1935360ab^2 + 193536acd^2 \right. \\
& \quad \left. + 8663 + 2322432acb - 387072ac^2 - 1935360ab^3 - 967680ab^4 \right. \\
& \quad \left. + 580608acd + 967680ac - 1935360ab^2 \right) \frac{1}{9216n^8} + O\left(\frac{1}{n^9}\right). \tag{3.6}
\end{aligned}$$

The fastest sequence $(u_n)_{n \in \mathbb{N}}$ is obtained when the first four coefficients of this power series vanish. In this case

$$a = \frac{1}{48}, \quad b = \frac{83}{360}, \quad c = \frac{4909}{64800}, \quad d = \frac{11976997}{37112040},$$

we have

$$u_n - u_{n+1} = \frac{1763157528883853}{83111968235520000n^8} + O\left(\frac{1}{n^9}\right). \tag{3.7}$$

Finally, by using Lemma 2.1, we obtain assertions (3.4) and (3.5) of Theorem 3.1. \square

Solution (3.1) provides the best approximation of type (3.2):

$$\psi(n+1) \approx \ln\left(n + \frac{1}{2} + \frac{1}{24n}\right) - \frac{\frac{1}{48}}{\left(n + \frac{83}{360} + \frac{\frac{4909}{64800}}{n + \frac{11976997}{37112040}}\right)^3}. \tag{3.8}$$

Motivated by approximation (3.8), we establish Theorem 3.2 below, which provides closer bounds for $\gamma - T_n$.

Theorem 3.2. *For $n \geq 1$, then*

$$\frac{\frac{1}{48}}{\left(n + \frac{83}{360} + \frac{\frac{4909}{64800}}{n + \frac{11976997}{37112040}}\right)^3} < \gamma - T_n < \frac{\frac{1}{48}}{\left(n + \frac{83}{360}\right)^3}. \tag{3.9}$$

Proof. We only prove the right-hand inequality in (3.9). The proof of the left-hand inequality in (3.9) is similar. The inequality (3.9) can be written for $n \geq 1$ as

$$\frac{\frac{1}{48}}{\left(n + \frac{83}{360} + \frac{\frac{4909}{64800}}{n + \frac{11976997}{37112040}}\right)^3} < \ln\left(n + \frac{1}{2} + \frac{1}{24n}\right) - \psi(n+1) < \frac{\frac{1}{48}}{\left(n + \frac{83}{360}\right)^3}. \tag{3.10}$$

The upper bound of (3.9) is obtained by considering the function $f(x)$ which is defined, for $x > 0$, by

$$f(x) = \ln \left(x + \frac{1}{2} + \frac{1}{24x} \right) - \psi(x+1) - \frac{\frac{1}{48}}{\left(x + \frac{83}{360}\right)^3}.$$

We conclude from the asymptotic formula (2.3) that

$$\lim_{x \rightarrow \infty} f(x) = 0.$$

Differentiating $f(x)$ and applying the second inequality in (2.7) yields,

$$\begin{aligned} f'(x) &= \frac{24x^2 - 1}{x(24x^2 + 12x + 1)} - \psi'(x+1) + \frac{1049760000}{(360x + 83)^4} \\ &> \frac{24x^2 - 1}{x(24x^2 + 12x + 1)} - \left(\frac{1}{x} - \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7} \right) + \frac{1049760000}{(360x + 83)^4} \\ &= \frac{p(x)}{210x^7(24x^2 + 12x + 1)(360x + 83)^4}, \end{aligned}$$

where

$$\begin{aligned} p(x) &= 147550579398783 + 637562673548352(x-2) + 1095096221221183(x-2)^2 \\ &\quad + 997896029835428(x-2)^3 + 528831825356263(x-2)^4 \\ &\quad + 164401992148725(x-2)^5 + 27912981996000(x-2)^6 \\ &\quad + 2004050160000(x-2)^7 > 0 \quad \text{for } x \geq 2. \end{aligned}$$

Therefore, $f'(x) > 0$ for $x \geq 2$.

Direct computation would yield

$$\begin{aligned} f(1) &= \gamma + \ln \left(\frac{37}{24} \right) - \frac{87910307}{86938307} = -0.00110059 \dots, \\ f(2) &= \gamma + \ln \left(\frac{121}{48} \right) - \frac{1555288881}{1035563254} = -0.000072039 \dots \end{aligned}$$

Consequently, the sequence $(f(n))_{n \in \mathbb{N}}$ is strictly increasing. This leads us to

$$f(n) < \lim_{n \rightarrow \infty} f(n) = 0, \quad n \geq 1,$$

which means that the upper bound in assertion (3.9) of Theorem 3.2 holds true for all $n \in \mathbb{N}$. The proof of Theorem 3.2 is thus completed. \square

Remark 1. In fact, the following inequality holds true:

$$\gamma - T_n < \frac{\frac{1}{48}}{\left(n + \frac{83}{360} + \frac{\frac{4909}{64800}}{n + \frac{11976997}{37112040} + \frac{\frac{1763157528883853}{2754607025923200}}{n + \frac{2160995763710564441795}{13086874547647741578024}}} \right)^3} \quad (3.11)$$

for $n \in \mathbb{N}$.

3.2. We now define the sequence $(v_n)_{n \in \mathbb{N}}$ by

$$v_n = \ln \left(n + \frac{1}{2} + \frac{1}{24n} \right) - \psi(n+1) - \frac{1}{a_1 n^3 + b_1 n^2 + c_1 n + d_1}. \quad (3.12)$$

where $a_1, b_1, c_1, d_1 \in \mathbb{R}$. Following the same method used in the proof of Theorem 3.1, we find that for

$$a_1 = 48, \quad b_1 = \frac{166}{5}, \quad c_1 = \frac{5569}{300}, \quad d_1 = \frac{58741}{28000}, \quad (3.13)$$

we have

$$\lim_{n \rightarrow \infty} n^8 (v_n - v_{n+1}) = \frac{183358033}{9953280000} \quad \text{and} \quad \lim_{n \rightarrow \infty} n^7 v_n = \frac{183358033}{69672960000}. \quad (3.14)$$

The speed of convergence of the sequence $(v_n)_{n \in \mathbb{N}}$ is given by the order estimate $O(n^{-7})$.

Theorem 3.3. For $n \geq 1$, then

$$\frac{1}{48n^3 + \frac{166}{5}n^2 + \frac{5569}{300}n + \frac{58741}{28000}} < \gamma - T_n. \quad (3.15)$$

Proof. The inequality (3.15) can be written for $n \geq 1$ as

$$\frac{1}{48n^3 + \frac{166}{5}n^2 + \frac{5569}{300}n + \frac{58741}{28000}} < \ln \left(n + \frac{1}{2} + \frac{1}{24n} \right) - \psi(n+1). \quad (3.16)$$

We consider the function $F(x)$ defined for $x > 0$ by

$$F(x) = \ln \left(x + \frac{1}{2} + \frac{1}{24x} \right) - \psi(x+1) - \frac{1}{48x^3 + \frac{166}{5}x^2 + \frac{5569}{300}x + \frac{58741}{28000}}.$$

We conclude from the asymptotic formula (2.3) that

$$\lim_{x \rightarrow \infty} F(x) = 0.$$

Differentiating $F(x)$ and applying the first inequality in (2.7) yields,

$$\begin{aligned} F'(x) &= \frac{24x^2 - 1}{x(24x^2 + 12x + 1)} - \psi'(x+1) \\ &\quad + \frac{23520000(43200x^2 + 19920x + 5569)}{(4032000x^3 + 2788800x^2 + 1559320x + 176223)^2} \\ &< \frac{24x^2 - 1}{x(24x^2 + 12x + 1)} - \left(\frac{1}{x} - \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7} - \frac{1}{30x^9} \right) \\ &\quad + \frac{23520000(43200x^2 + 19920x + 5569)}{(4032000x^3 + 2788800x^2 + 1559320x + 176223)^2} \\ &= - \frac{q(x)}{210x^9(24x^2 + 12x + 1)(4032000x^3 + 2788800x^2 + 1559320x + 176223)^2}, \end{aligned}$$

where

$$\begin{aligned} q(x) &= 18130487257947165687 + 63552993678839537457(x-3) \\ &\quad + 94471229612034347921(x-3)^2 + 79408865830190450709(x-3)^3 \\ &\quad + 41975644888778012717(x-3)^4 + 14553520724815257633(x-3)^5 \\ &\quad + 3322393272176291138(x-3)^6 + 482867798807968875(x-3)^7 \\ &\quad + 40622141576265200(x-3)^8 + 1509403327656000(x-3)^9 > 0 \quad \text{for } x \geq 3. \end{aligned}$$

Therefore, $F'(x) < 0$ for $x \geq 3$.

Direct computation would yield

$$\begin{aligned} F(1) &= -\frac{8640343}{8556343} + \gamma + \ln 37 - 3 \ln 2 - \ln 3 = 0.000262469 \dots, \\ F(2) &= -\frac{140286189}{93412126} + \gamma + 2 \ln 11 - 4 \ln 2 - \ln 3 = 0.000006718 \dots, \\ F(3) &= -\frac{509165071}{277634766} + \gamma + \ln 11 + \ln 23 - 3 \ln 2 - 2 \ln 3 = 0.000000589 \dots. \end{aligned}$$

Consequently, the sequence $(F(n))_{n \in \mathbb{N}}$ is strictly decreasing. This leads us to

$$F(n) > \lim_{n \rightarrow \infty} F(n) = 0, \quad n \geq 1,$$

which means that inequality (3.15) holds true for all $n \in \mathbb{N}$. \square

Remark 2. *The lower bound in (3.15) is sharper than one in (3.9).*

Remark 3. *In fact, the following inequality holds true:*

$$\gamma - T_n < \frac{1}{48n^3 + \frac{166}{5}n^2 + \frac{5569}{300}n + \frac{58741}{28000} - \frac{183358033}{30240000n}}. \quad (3.17)$$

for $n \in \mathbb{N}$.

Remark 4. *The numerical calculations presented in this work were performed by using the Maple software for symbolic computations.*

Acknowledgments. The author would like to thank the anonymous referee for his/her comments that helped us improve this article.

REFERENCES

- [1] M. Abramowitz and I. A. Stegun (Editors), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Applied Mathematics Series **55**, Ninth printing, National Bureau of Standards, Washington, D.C., 1972.
- [2] H. Alzer, *On some inequalities for the gamma and psi functions*, Math. Comp. **66** (1997), 373–389.
- [3] H. Alzer, *Inequalities for the gamma and polygamma functions*, Abh. Math. Sem. Univ. Hamburg **68** (1998), 363–372.
- [4] G. D. Anderson, R. W. Barnard, K. C. Richards, M. K. Vamanamurthy and M. Vuorinen, *Inequalities for zero-balanced hypergeometric functions*, Trans. Amer. Math. Soc. **347** (1995), 1713–1723.
- [5] E. Cesàro, *Sur la serie harmonique*, Nouvelles Annales de Mathématiques **4** (1885), 295–296.
- [6] C.-P. Chen, *Inequalities for the Euler-Mascheroni constant*, Appl. Math. Lett. **23** (2010), 161–164.
- [7] C.-P. Chen, *Inequalities and monotonicity properties for some special functions*, J. Math. Inequal. **3** (2009), 79–91.
- [8] C.-P. Chen, *Monotonicity properties of functions related to the psi function*, Appl. Math. Comput. **217** (2010), 2905–2911.
- [9] C.-P. Chen, *The Best Bounds in Vernescu's Inequalities for the Euler's Constant*, RGMIA Res. Rep. Coll. **12** (2009), no.3, Article 11. Available online at <http://ajmaa.org/RGMIA/v12n3.php>.
- [10] D. W. DeTemple, *A quicker convergence to Euler's constant*, Amer. Math. Monthly **100** (1993), 468–470.
- [11] C. Mortici, *New bounds for a convergence by DeTemple*, J. Sci. Arts **10** (2010), 239–242.
- [12] C. Mortici, *Product approximations via asymptotic integration*, Amer. Math. Monthly **117** (2010), 434–441.
- [13] C. Mortici, *The proof of Muqattash-Yahdi conjecture*, Math. Comput. Modelling **51** (2010), 1154–1159.

- [14] C. Mortici, *Best estimates of the generalized Stirling formula*, Appl. Math. Comp. **215** (2010), 4044-4048.
- [15] C. Mortici, *On new sequences converging towards the Euler-Mascheroni constant*. Comput. Math. Appl. **59** (2010), 2610–2614.
- [16] C. Mortici, *Improved convergence towards generalized Euler-Mascheroni constant*. Appl. Math. Comput. **215** (2010), 3443–3448.
- [17] C. Mortici, *A new Stirling series as continued fraction*, Numer. Algorithms **56** (2011), 17-26.
- [18] T. Negoi, *A faster convergence to the constant of Euler*, Gazeta Matematică, seria A, **15** (1997), 111–113 (in Romanian).
- [19] P. J. Rippon, *Convergence with pictures*, Amer. Math. Monthly, **93** (1986), 476–478.
- [20] A. Sintămărian, *A generalization of Euler's constant*, Numer. Algorithms **46** (2007), 141–151.
- [21] A. Sintămărian, *Some inequalities regarding a generalization of Euler's constant*, J. Inequal. Pure Appl. Math. **9** (2008), no.2, Article 46. http://www.emis.de/journals/JIPAM/images/352_07_JIPAM/352_07.pdf.
- [22] S. R. Tims and J. A. Tyrrell, *Approximate evaluation of Euler's constant*, Math. Gaz. **55** (1971), 65–67.
- [23] L. Tóth, *Problem E3432*, Amer. Math. Monthly, **98** (1991), 264.
- [24] L. Tóth, *Problem E3432 (Solution)*, Amer. Math. Monthly, **99** (1992), 684–685.
- [25] A. Vernescu, *A new accelerate convergence to the constant of Euler*, Gaz. Mat. Ser. A (**17**) **96** (1999), 273-278 (in Romanian).
- [26] M. Villarino, *Ramanujan's harmonic number expansion into negative powers of a triangular number*, J. Inequal. Pure Appl. Math. **9** (2008), no. 3, Article 89. Available online at
- [27] R. M. Young, *Euler's Constant*, Math. Gaz. **75** (1991), 187–190.

CHAO-PING CHEN

SCHOOL OF MATHEMATICS AND INFORMATICS, HENAN POLYTECHNIC UNIVERSITY, JIAOZUO CITY
454003, HENAN PROVINCE, PEOPLE'S REPUBLIC OF CHINA

E-mail address: chenciaoping@sohu.com