Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 2 Issue 4(2010), Pages 106-118.

MEROMORPHIC FUNCTIONS THAT SHARE FIXED POINTS WITH FINITE WEIGHTS

(DEDICATED IN OCCASION OF THE 70-YEARS OF PROFESSOR HARI M. SRIVASTAVA)

PULAK SAHOO

Abstract. With the aid of weighted sharing method we study the uniqueness of meromorphic (entire) functions concerning some general nonlinear differential polynomials sharing fixed points. The results of the paper improve and generalize some results due to Zhang [24] and the present author [16].

1. Introduction, Definitions and Results

In this paper, by meromorphic functions we will always mean meromorphic functions in the complex plane. We adopt the standard notations in the Nevanlinna theory of meromorphic functions as explained in [10], [20] and [22]. For a nonconstant meromorphic function h, we denote by $T(r,h)$ the Nevanlinna characteristic of h and by $S(r,h)$ any quantity satisfying $S(r,h)=o\{T(r,h)\}$ as $r\to\infty$ possibly outside a set of finite linear measure. A meromorphic function $a(z)(\not\equiv\infty)$ is called a small function with respect to f, provided that $T(r,a)=S(r,f)$.

Let f and g be two nonconstant meromorphic functions, and let a be a finite value. We say that f and g share the value a CM, provided that $f-a$ and $g-a$ have the same zeros with the same multiplicities. Similarly, we say that f and g share a IM, provided that $f-a$ and $g-a$ have the same zeros ignoring multiplicities. In addition, we say that f and g share ∞ CM, if $\frac{1}{f}$ and $\frac{1}{g}$ share 0 CM, and we say that f and g share ∞ IM, if $\frac{1}{f}$ and $\frac{1}{g}$ share 0 IM (see[22]). A finite value z_0 is a fixed point of $f(z)$ if $f(z_0)=z_0$ and we define

$$E_f=\{z\in\mathbb{C}:f(z)=z, counting\ multiplicities\}.$$

In 1959, Hayman (see [9], Corollary of Theorem 9) proved the following theorem.

Theorem A. *Let f be a transcendental meromorphic function and $n(\geq 3)$ is an integer. Then $f^nf'=1$ has infinitely many solutions.*

2000 *Mathematics Subject Classification.* 30D35.

Key words and phrases. Meromorphic function, Differential Polynomial, Weighted Sharing, Fixed Point.

Submitted May 14, 2010, 2010. Published October 20, 2010.

Corresponding to which, the following result was obtained by Yang and Hua [19] and by Fang and Hua [7] respectively.

Theorem B. *Let f and g be two nonconstant meromorphic (entire) functions, $n \geq 11$ $(n \geq 6)$ be a positive integer. If $f^n f'$ and $g^n g'$ share 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(c_1 c_2)^{n+1} c^2 = -1$ or $f \equiv tg$ for a constant t such that $t^{n+1} = 1$.*

In 2000, Fang [5] proved the following result.

Theorem C. *Let f be a transcendental meromorphic function, and let n be a positive integer. Then $f^n f' - z$ has infinitely many solutions.*

Corresponding to Theorem C, Fang and Qiu [8] proved the following result.

Theorem D. *Let f and g be two nonconstant meromorphic functions, and let $n \geq 11$ be a positive integer. If $f^n f' - z$ and $g^n g' - z$ share 0 CM, then either $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, where c_1, c_2 and c are three nonzero complex numbers satisfying $4(c_1 c_2)^{n+1} c^2 = -1$ or $f = tg$ for a complex number t such that $t^{n+1} = 1$.*

Considering k th derivative instead of first derivative, Hennekemper-Hennekemper [11], Chen [4] and Wang [17] proved the following theorem.

Theorem E. *Let f be a transcendental entire function and n, k be two positive integers with $n \geq k + 1$. Then $(f^n)^{(k)} = 1$ has infinitely many solutions.*

Corresponding to Theorem E Fang [6] proved the following theorem.

Theorem F. *Let f and g be two nonconstant entire functions, and let n, k be two positive integers with $n \geq 2k + 8$. If $[f^n(f - 1)]^{(k)}$ and $[g^n(g - 1)]^{(k)}$ share 1 CM, then $f \equiv g$.*

In 2008, Zhang [24] extended Theorem F by using the idea of sharing fixed points and obtained the following theorems.

Theorem G. *Let f and g be two nonconstant entire functions, and n, k be two positive integers with $n \geq 2k + 6$. If $E_{(f^n(f-1))^{(k)}} = E_{(g^n(g-1))^{(k)}}$, then $f \equiv g$.*

Naturally one may ask the following question.

Question 1. Is it really possible in any way to relax the nature of sharing the fixed point in Theorem G without increasing the lower bound of n ?

To state the next result we need the following definition known as weighted sharing of values introduced by I. Lahiri [12, 13] which measure how close a shared value is to being shared CM or to being shared IM.

Definition 1. *Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.*

The definition implies that if f, g share a value a with weight k, then z_0 is an a-point of f with multiplicity $m(\leq k)$ if and only if it is an a-point of g with multiplicity $m(\leq k)$ and z_0 is an a-point of f with multiplicity $m(> k)$ if and only if it is an a-point of g with multiplicity $n(> k)$, where m is not necessarily equal to n.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ and (a, ∞) respectively.

Using the idea of weighted sharing of values, recently the present author [16] proved the following uniqueness theorem for some nonlinear differential polynomials sharing 1-points.

Theorem H. *Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions, and let $n(\geq 1)$, $k(\geq 1)$, $m(\geq 1)$ and $l(\geq 0)$ be four integers. Let $P(z) = a_m z^m + ... + a_1 z + a_0$, where $a_0(\neq 0)$, a_1, ... , $a_m(\neq 0)$ are complex constants. Let $[f^n P(f)]^{(k)}$ and $[g^n P(g)]^{(k)}$ share $(1, l)$ and one of the following conditions holds:*
(a) $l \geq 2$ and $n > 3k + m + 8$;
(b) $l = 1$ and $n > 4k + 3m/2 + 9$;
(c) $l = 0$ and $n > 9k + 4m + 14$.
Then either $[f^n P(f)]^{(k)}[g^n P(g)]^{(k)} \equiv 1$ or $f(z) \equiv tg(z)$ for a constant t such that $t^d = 1$, where $d = \gcd\{n + m, ..., n + m - i, ..., n + 1, n\}$, $a_{m-i} \neq 0$ for some $i = 0, 1, ..., m$ or f and g satisfy the algebraic equation $R(f, g) = 0$, where $R(x, y) = x^m(a_m x^m + ... + a_1 x + a_0) - y^m(a_m y^m + ... + a_1 y + a_0)$.
The possibility $[f^n P(f)]^{(k)}[g^n P(g)]^{(k)} \equiv 1$ does not occur for $k = 1$.

Natural question arises:
Question 2. Can one replace the shared value by shared fixed points in Theorem H ?
In the paper, we will prove two theorems second of which will not only improve Theorem G by relaxing the nature of sharing the fixed point and at the same time provide a supplementary and generalized result of Theorem H. Moreover, Theorem 2 deal with question 1 and question 2. We now state the main results of the paper.

Theorem 1.1. *Let f be a transcendental meromorphic function and n, k, m be three positive integers such that $n \geq k + 3$. Let $P(z)$ be defined as in Theorem H. Then $(f^n P(f))^{(k)}$ has infinitely many fixed points.*

Theorem 1.2. *Let f and g be two transcendental meromorphic functions, and let n, k and m be three positive integers. Let $P(z)$ be defined as in Theorem H. Let $[f^n P(f)]^{(k)}$ and $[g^n P(g)]^{(k)}$ share (z, l) where $l(\geq 0)$ is an integer; f, g share $(\infty, 0)$ and one of the following conditions holds:*
(i) $l \geq 2$ and $n > 3k + m + 7$;
(ii) $l = 1$ and $n > 4k + 3m/2 + 8$;
(iii) $l = 0$ and $n > 9k + 4m + 13$.
Then either $f(z) = tg(z)$ for a constant t such that $t^d = 1$, where $d = (n+m, ..., n+m-i, ..., n)$, $a_{m-i} \neq 0$ for some $i = 0, 1, 2, ..., m$ or f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where

$$R(x, y) = x^n(a_m x^m + a_{m-1}x^{m-1} + ... + a_0) - y^n(a_m y^m + a_{m-1}y^{m-1} + ... + a_0).$$

Corollary 1.3. *Let f and g be two transcendental entire functions, and let n, k and m be three positive integers. Let $P(z)$ be defined as in Theorem H. Let $[f^n P(f)]^{(k)}$ and $[g^n P(g)]^{(k)}$ share (z, l) where $l(\geq 0)$ is an integer and one of the following conditions holds:*
(i) $l \geq 2$ and $n > 2k + m + 4$;

(ii) $l = 1$ *and* $n > \frac{5k+3m+9}{2}$;
(iii) $l = 0$ *and* $n > 5k + 4m + 7$.
Then the conclusions of Theorem 1.2 holds.

Remark. *Corollary 1.3 is an improvement of Theorem G.*

Remark. *If* $[f^n P(f)]^{(k)}$ *and* $[g^n P(g)]^{(k)}$ *share* $(1, l)$ *where* $l(\geq 0)$ *is an integer and* f, g *share* $(\infty, 0)$, *then the conclusions of Theorem 1.2 holds in each of the cases (i) - (iii) of Theorem 1.2.*

We now explain some definitions and notations which are used in the paper.

Definition 2. [14] *For* $a \in \mathbb{C} \cup \{\infty\}$ *we denote by* $N(r, a; f \mid= 1)$ *the counting functions of simple* a*-points of* f. *For a positive integer* p *we denote by* $N(r, a; f \mid\leq p)$ *the counting function of those* a*-points of* f *(counted with proper multiplicities) whose multiplicities are not greater than* p. *By* $\overline{N}(r, a; f \mid\leq p)$ *we denote the corresponding reduced counting function.*

In an analogous manner we define $N(r, a; f \mid\geq p)$ *and* $\overline{N}(r, a; f \mid\geq p)$.

Definition 3. [13] *Let* k *be a positive integer or infinity. We denote by* $N_k(r, a; f)$ *the counting function of* a*-points of* f, *where an* a*-point of multiplicity* m *is counted* m *times if* $m \leq k$ *and* k *times if* $m > k$. *Then*

$$N_k(r, a; f) = \overline{N}(r, a; f) + \overline{N}(r, a; f \mid\geq 2) + ... + \overline{N}(r, a; f \mid\geq k).$$

Clearly $N_1(r, a; f) = \overline{N}(r, a; f)$.

Definition 4. [1, 2] *Let* f *and* g *be two nonconstant meromorphic functions such that* f *and* g *share the value* 1 *IM. Let* z_0 *be a* 1*-point of* f *with multiplicity* p *and also a* 1*-point of* g *with multiplicity* q. *We denote by* $\overline{N}_L(r, 1; f)$ *the counting function of those* 1*-points of* f *and* g, *where* $p > q$, *by* $N_E^{1)}(r, 1; f)$ *the counting function of those* 1*-points of* f *and* g, *where* $p = q = 1$, *by* $N_E^{(k}(r, 1; f)$ $(k \geq 2$ *is an integer*$)$ *the counting function of those* 1*-points of* f *and* g, *where* $p = q \geq k$, *where each point in these counting functions is counted only once. In the same manner we can define* $\overline{N}_L(r, 1; g)$, $N_E^{1)}(r, 1; g)$ *and* $N_E^{(k}(r, 1; g)$.

Definition 5. [1, 2] *Let* f *and* g *be two nonconstant meromorphic functions such that* f *and* g *share the value* 1 *IM. Let* z_0 *be a* 1*-point of* f *with multiplicity* p *and also a* 1*-point of* g *with multiplicity* q. *For a positive integer* k, $\overline{N}_{f>k}(r, 1; g)$ *denotes the reduced counting function of those* 1*-points of* f *and* g *such that* $p > q = k$. *In an analogous way we can define* $\overline{N}_{g>k}(r, 1; f)$.

Definition 6. [12, 13] *Let* f *and* g *be two nonconstant meromorphic functions such that* f *and* g *share the value* a *IM. We denote by* $\overline{N}_*(r, a; f, g)$ *the reduced counting function of those* a*-points of* f *whose multiplicities differ from the multiplicities of the corresponding* a*-points of* g. *Clearly* $\overline{N}_*(r, a; f, g) = \overline{N}_*(r, a; g, f)$ *and* $\overline{N}_*(r, a; f, g) = \overline{N}_L(r, a; f) + \overline{N}_L(r, a; g)$.

2. LEMMAS

In this section we present some lemmas which will be needed in the sequel. Let F and G be two nonconstant meromorphic functions defined in $\mathbb{C}$. We shall denote by H the following function:

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right).$$

Lemma 2.1. [21, 23] *If F, G share $(1,0)$ and $H \not\equiv 0$ then*

$$N_E^{1)}(r,1;F) \le N(r,\infty;H) + S(r,F) + S(r,G).$$

Lemma 2.2. [15] *If F, G share $(1,0)$, $(\infty,0)$ and $H \not\equiv 0$ then*

$$N(r,\infty;H) \le \overline{N}(r,0;F \mid\ge 2) + \overline{N}(r,0;G \mid\ge 2) + \overline{N}_*(r,\infty;F,G) + \overline{N}_*(r,1;F,G) + \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G'),$$

where $\overline{N}_0(r,0;F')$ is the reduced counting function of those zeros of F' which are not the zeros of $F(F-1)$, and $\overline{N}_0(r,0;G')$ is similarly defined.

Lemma 2.3. [18] *Let f be a nonconstant meromorphic function and let $a_n(z)(\not\equiv 0)$, $a_{n-1}(z)$, ... , $a_0(z)$ be meromorphic functions such that $T(r,a_i(z)) = S(r,f)$ for $i = 0,1,2,...,n$. Then*

$$T(r, a_n f^n + a_{n-1} f^{n-1} + ... + a_1 f + a_0) = nT(r,f) + S(r,f).$$

Lemma 2.4. [25] *Let f be a nonconstant meromorphic function, and p, k be positive integers. Then*

$$N_p\left(r,0;f^{(k)}\right) \le T\left(r,f^{(k)}\right) - T(r,f) + N_{p+k}(r,0;f) + S(r,f), \tag{2.1}$$

$$N_p\left(r,0;f^{(k)}\right) \le k\overline{N}(r,\infty;f) + N_{p+k}(r,0;f) + S(r,f). \tag{2.2}$$

Lemma 2.5. [1] *Let f and g be two nonconstant meromorphic functions that share $(1,1)$. Then*

$$2\overline{N}_L(r,1;f) + 2\overline{N}_L(r,1;g) + N_E^{(2}(r,1;f) - \overline{N}_{f>2}(r,1;g) \le N(r,1;g) - \overline{N}(r,1;g).$$

Lemma 2.6. [2] *Let f and g be two nonconstant meromorphic functions that share $(1,1)$. Then*

$$\overline{N}_{f>2}(r,1;g) \le \frac{1}{2}\overline{N}(r,0;f) + \frac{1}{2}\overline{N}(r,\infty;f) - \frac{1}{2}N_{\oplus}(r,0;f') + S(r,f),$$

where $N_{\oplus}(r,0;f')$ denotes the counting function of those zeros of f' which are not zeros of $f(f-1)$.

Lemma 2.7. [2] *Let f and g be two nonconstant meromorphic functions that share $(1,0)$. Then*
$\overline{N}_L(r,1;f) + 2\overline{N}_L(r,1;g) + N_E^{(2}(r,1;f) - \overline{N}_{f>1}(r,1;g) - \overline{N}_{g>1}(r,1;f)$
$\le N(r,1;g) - \overline{N}(r,1;g)$.

Lemma 2.8. [21] *Let f and g be two nonconstant meromorphic functions sharing $(1,0)$. Then*

$$\overline{N}_L(r,1;f) \le \overline{N}(r,0;f) + \overline{N}(r,\infty;f) + S(r,f).$$

Lemma 2.9. [2] *Let f and g be two nonconstant meromorphic functions that share $(1,0)$. Then*
(i) $\overline{N}_{f>1}(r,1;g) \le \overline{N}(r,0;f) + \overline{N}(r,\infty;f) - N_{\oplus}(r,0;f') + S(r,f)$;
(ii) $\overline{N}_{g>1}(r,1;f) \le \overline{N}(r,0;g) + \overline{N}(r,\infty;g) - N_{\oplus}(r,0;g') + S(r,g)$,
where $N_{\oplus}(r,0;f')$ and $N_{\oplus}(r,0;g')$ are defined as in Lemma 2.6.

Lemma 2.10. [3] *Let F, G be two nonconstant meromorphic functions sharing $(1,2)$, $(\infty,0)$ and $H\not\equiv 0$. Then*
(i) $T(r,F)\le N_2(r,0;F)+N_2(r,0;G)+\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}_*(r,\infty;F,G)-m(r,1;G)-N_E^{(3}(r,1;F)-\overline{N}_L(r,1;G)+S(r,F)+S(r,G)$;
(ii) $T(r,G)\le N_2(r,0;F)+N_2(r,0;G)+\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}_*(r,\infty;F,G)-m(r,1;F)-N_E^{(3}(r,1;G)-\overline{N}_L(r,1;F)+S(r,F)+S(r,G)$.

Lemma 2.11. *Let F, G be two nonconstant meromorphic functions sharing $(1,1)$, $(\infty,0)$ and $H\not\equiv 0$. Then*
(i) $T(r,F)\le N_2(r,0;F)+N_2(r,0;G)+\frac{3}{2}\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}_*(r,\infty;F,G)+\frac{1}{2}\overline{N}(r,0;F)+S(r,F)+S(r,G)$.
(ii) $T(r,G)\le N_2(r,0;F)+N_2(r,0;G)+\overline{N}(r,\infty;F)+\frac{3}{2}\overline{N}(r,\infty;G)+\overline{N}_*(r,\infty;F,G)+\frac{1}{2}\overline{N}(r,0;G)+S(r,F)+S(r,G)$;

Proof. We prove (i) only since the proof of (ii) is similar. Since F, G share $(1,1)$, $N_E^{1)}(r,1;F)=N(r,1;F\mid=1)$. By the second fundamental theorem of Nevanlinna we have

$$T(r,F)\le\overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,1;F)-\overline{N}_0(r,0;F')+S(r,F)\tag{2.3}$$

and

$$T(r,G)\le\overline{N}(r,0;G)+\overline{N}(r,\infty;G)+\overline{N}(r,1;G)-\overline{N}_0(r,0;G')+S(r,G),\tag{2.4}$$

where $\overline{N}_0(r,0;F')$ and $\overline{N}_0(r,0;G')$ are defined as in Lemma 2.2. Since

$$\begin{aligned}\overline{N}(r,1;F)+\overline{N}(r,1;G)\le{}&N(r,1;F\mid=1)+N_E^{(2}(r,1;F)+\overline{N}_L(r,1;F)\\&+\overline{N}_L(r,1;G)+\overline{N}(r,1;G),\end{aligned}$$

using Lemmas 2.1, 2.2, 2.5 and 2.6 we obtain

$$\begin{aligned}\overline{N}(r,1;F)+\overline{N}(r,1;G)\le{}&\overline{N}(r,0;F\mid\ge 2)+\overline{N}(r,0;G\mid\ge 2)+\overline{N}_*(r,\infty;F,G)\\&+2\overline{N}_*(r,1;F,G)+N_E^{(2}(r,1;F)+\overline{N}(r,1;G)\\&+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')\\\le{}&\overline{N}(r,0;F\mid\ge 2)+\overline{N}(r,0;G\mid\ge 2)+\overline{N}_*(r,\infty;F,G)\\&+\overline{N}_{F>2}(r,1;G)+T(r,G)+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')\\\le{}&\overline{N}(r,0;F\mid\ge 2)+\overline{N}(r,0;G\mid\ge 2)+\overline{N}_*(r,\infty;F,G)\\&+\frac{1}{2}\overline{N}(r,0;F)+\frac{1}{2}\overline{N}(r,\infty;F)+T(r,G)\\&+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G').\end{aligned}$$

Using (2.3) and (2.4) we obtain from above

$$\begin{aligned}T(r,F)\le{}&N_2(r,0;F)+N_2(r,0;G)+\frac{3}{2}\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)\\&+\overline{N}_*(r,\infty;F,G)+\frac{1}{2}\overline{N}(r,0;F)+S(r,F)+S(r,G).\end{aligned}$$

This completes the proof of the lemma. □

Lemma 2.12. *Let F, G be two nonconstant meromorphic functions sharing $(1,0)$, $(\infty,0)$ and $H\not\equiv 0$. Then*

(i) $T(r,F) \le N_2(r,0;F)+N_2(r,0;G)+3\overline{N}(r,\infty;F)+2\overline{N}(r,\infty;G)+\overline{N}_*(r,\infty;F,G)+2\overline{N}(r,0;F)+\overline{N}(r,0;G)+S(r,F)+S(r,G)$.
(ii) $T(r,G) \le N_2(r,0;F)+N_2(r,0;G)+2\overline{N}(r,\infty;F)+3\overline{N}(r,\infty;G)+\overline{N}_*(r,\infty;F,G)+\overline{N}(r,0;F)+2\overline{N}(r,0;G)+S(r,F)+S(r,G)$;

Proof. We prove (i) only since the proof of (ii) is similar. Since

$$\begin{aligned}\overline{N}(r,1;F)+\overline{N}(r,1;G) \le N_E^{1)}(r,1;F)+N_E^{(2}(r,1;F)+\overline{N}_L(r,1;F)\\ +\overline{N}_L(r,1;G)+\overline{N}(r,1;G),\end{aligned}$$

using Lemmas 2.1, 2.2, 2.7, 2.8 and 2.9 we obtain

$$\begin{aligned}\overline{N}(r,1;F)+\overline{N}(r,1;G) &\le \overline{N}(r,0;F\mid\ge 2)+\overline{N}(r,0;G\mid\ge 2)+\overline{N}_*(r,\infty;F,G)\\ &+2\overline{N}_*(r,1;F,G)+N_E^{(2}(r,1;F)+\overline{N}(r,1;G)\\ &+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')\\ &\le \overline{N}(r,0;F\mid\ge 2)+\overline{N}(r,0;G\mid\ge 2)+\overline{N}_*(r,\infty;F,G)\\ &+\overline{N}_{F>1}(r,1;G)+\overline{N}_{G>1}(r,1;F)+\overline{N}_L(r,1;F)+T(r,G)\\ &+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')\\ &\le \overline{N}(r,0;F\mid\ge 2)+\overline{N}(r,0;G\mid\ge 2)+\overline{N}_*(r,\infty;F,G)\\ &+2\overline{N}(r,0;F)+\overline{N}(r,0;G)+2\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)\\ &+T(r,G)+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G').\end{aligned}$$

Using (2.3) and (2.4) we obtain from above

$$\begin{aligned}T(r,F) &\le N_2(r,0;F)+N_2(r,0;G)+3\overline{N}(r,\infty;F)+2\overline{N}(r,\infty;G)+\overline{N}_*(r,\infty;F,G)\\ &+2\overline{N}(r,0;F)+\overline{N}(r,0;G)+S(r,F)+S(r,G).\end{aligned}$$

This completes the proof of the lemma. □

Lemma 2.13. [10, 20] *Let f be a transcendental meromorphic function, and let $a_1(z)$, $a_2(z)$ be two distinct meromorphic functions such that $T(r,a_i(z))=S(r,f)$, i=1,2. Then*

$$T(r,f) \le \overline{N}(r,\infty;f)+\overline{N}(r,a_1;f)+\overline{N}(r,a_2;f)+S(r,f).$$

Lemma 2.14. *Let f and g be two transcendental meromorphic (entire) functions and let n, k be two positive integers. Suppose that $F_1 = \frac{(f^nP(f))^{(k)}}{z}$ and $G_1 = \frac{(g^nP(g))^{(k)}}{z}$ where $P(z)$ be defined as in Theorem H. If there exist two nonzero constants c_1 and c_2 such that $\overline{N}(r,c_1;F_1)=\overline{N}(r,0;G_1)$ and $\overline{N}(r,c_2;G_1)=\overline{N}(r,0;F_1)$, then $n \le 3k+m+3$ ($n \le 2k+m+2$).*

Proof. We prove the case when f and g are two nonconstant meromorphic functions. The case when f and g are two nonconstant entire functions can be proved similarly. By the second fundamental theorem of Nevanlinna we have

$$\begin{aligned}T(r,F_1) &\le \overline{N}(r,0;F_1)+\overline{N}(r,\infty;F_1)+\overline{N}(r,c_1;F_1)+S(r,F_1)\\ &\le \overline{N}(r,0;F_1)+\overline{N}(r,0;G_1)+\overline{N}(r,\infty;F_1)+S(r,F_1).\end{aligned} \tag{2.5}$$

By (2.1), (2.2), (2.5) and Lemma 2.3 we obtain

$$\begin{aligned}(n+m)T(r,f) &\leq T(r,F_1) - \overline{N}(r,0;F_1) + N_{k+1}(r,0;f^nP(f)) + O\{\log r\} + S(r,f)\\ &\leq \overline{N}(r,0;G_1) + N_{k+1}(r,0;f^nP(f)) + \overline{N}(r,\infty;f) + O\{\log r\} + S(r,f)\\ &\leq N_{k+1}(r,0;f^nP(f)) + N_{k+1}(r,0;g^nP(g)) + \overline{N}(r,\infty;f)\\ &+ k\overline{N}(r,\infty;g) + O\{\log r\} + S(r,f) + S(r,g)\\ &\leq (k+m+2)T(r,f) + (2k+m+1)T(r,g)\\ &+ O\{\log r\} + S(r,f) + S(r,g).\end{aligned} \tag{2.6}$$

Similarly we obtain

$$\begin{aligned}(n+m)T(r,g) \leq (k+m+2)T(r,g) + (2k+m+1)T(r,f)\\ + O\{\log r\} + S(r,f) + S(r,g).\end{aligned} \tag{2.7}$$

Since f and g are transcendental meromorphic functions, we have

$$T(r,z) = o\{T(r,f)\} \text{ and } T(r,z) = o\{T(r,g)\}. \tag{2.8}$$

Hence from (2.6) and (2.7) we get

$$(n-3k-m-3)\{T(r,f)+T(r,g)\} \leq S(r,f) + S(r,g),$$

which gives $n \leq 3k+m+3$. This completes the proof of the lemma. □

Proceeding as in the proof of Lemma 2.14 we get

Lemma 2.15. *Let f and g be two transcendental meromorphic (entire) functions and let n, k be two positive integers. Suppose that $F_2 = (f^nP(f))^{(k)}$ and $G_2 = (g^nP(g))^{(k)}$ where $P(z)$ be defined as in Theorem H. If there exist two nonzero constants d_1 and d_2 such that $\overline{N}(r,d_1;F_2) = \overline{N}(r,0;G_2)$ and $\overline{N}(r,d_2;G_2) = \overline{N}(r,0;F_2)$, then $n \leq 3k+m+3$ ($n \leq 2k+m+2$).*

3. Proof of the Theorem

Proof of Theorem 1.1. We consider $F(z) = f^nP(f)$ and $G(z) = g^nP(g)$. Then by Lemma 2.13 we have

$$T\left(r,F^{(k)}\right) \leq \overline{N}\left(r,0;F^{(k)}\right) + \overline{N}\left(r,\infty;F^{(k)}\right) + \overline{N}\left(r,z;F^{(k)}\right) + S(r,F).$$

Using (2.1) and the above inequality we obtain

$$\begin{aligned}(n+m)T(r,f) &\leq T\left(r,F^{(k)}\right) - \overline{N}\left(r,0;F^{(k)}\right) + N_{k+1}(r,0;F) + S(r,f)\\ &\leq \overline{N}\left(r,\infty;F^{(k)}\right) + \overline{N}\left(r,z;F^{(k)}\right) + N_{k+1}(r,0;F) + S(r,f)\\ &\leq (k+m+2)T(r,f) + \overline{N}\left(r,z;F^{(k)}\right) + S(r,f).\end{aligned}$$

Since $n \geq k+3$, from this we can say that $F^{(k)} = (f^nP(f))^{(k)}$ has infinitely many fixed points. □

Proof of Theorem 1.2. We consider $F(z) = \frac{(f^nP(f))^{(k)}}{z}$ and $G(z) = \frac{(g^nP(g))^{(k)}}{z}$. Then $F(z)$, $G(z)$ are transcendental meromorphic functions that share $(1,l)$ and f, g share $(\infty,0)$. We assume that $H \not\equiv 0$. Then from Lemma 2.3 and (2.1) we obtain

$$\begin{aligned} N_2(r,0;F) &\leq N_2\left(r,0;(f^nP(f))^{(k)}\right) + S(r,f) \\ &\leq T\left(r,(f^nP(f))^{(k)}\right) - (n+m)T(r,f) + N_{k+2}(r,0;f^nP(f)) + S(r,f) \\ &\leq T(r,F) - (n+m)T(r,f) + N_{k+2}(r,0;f^nP(f)) + O\{\log r\} + S(r,f). \end{aligned} \tag{3.1}$$

In a similar way we obtain

$$N_2(r,0;G) \leq T(r,G) - (n+m)T(r,g) + N_{k+2}(r,0;g^nP(g)) + O\{\log r\} + S(r,g). \tag{3.2}$$

Again by (2.2) we have

$$N_2(r,0;F) \leq k\overline{N}(r,\infty;f) + N_{k+2}(r,0;f^nP(f)) + S(r,f). \tag{3.3}$$

$$N_2(r,0;G) \leq k\overline{N}(r,\infty;g) + N_{k+2}(r,0;g^nP(g)) + S(r,g). \tag{3.4}$$

From (3.1) and (3.2) we get

$$\begin{aligned} (n+m)\{T(r,f)+T(r,g)\} &\leq T(r,F) + T(r,G) + N_{k+2}(r,0;f^nP(f)) \\ &\quad + N_{k+2}(r,0;g^nP(g)) - N_2(r,0;F) - N_2(r,0;G) \\ &\quad + O\{\log r\} + S(r,f) + S(r,g). \end{aligned} \tag{3.5}$$

Now we consider the following three cases.

Case 1. *Let $l \geq 2$. Then using Lemma 2.10, (3.3) and (3.4) we obtain from (3.5)*

$$\begin{aligned} (n+m)\{T(r,f)+T(r,g)\} &\leq N_2(r,0;F) + N_2(r,0;G) + 2\overline{N}(r,\infty;F) + 2\overline{N}(r,\infty;G) \\ &\quad + 2\overline{N}_*(r,\infty;F,G) + N_{k+2}(r,0;f^nP(f)) \\ &\quad + N_{k+2}(r,0;g^nP(g)) + O\{\log r\} + S(r,f) + S(r,g) \\ &\leq 2N_{k+2}(r,0;f^nP(f)) + 2N_{k+2}(r,0;g^nP(g)) \\ &\quad + (k+2)\overline{N}(r,\infty;f) + (k+2)\overline{N}(r,\infty;g) + 2\overline{N}_L(r,\infty;F) \\ &\quad + 2\overline{N}_L(r,\infty;G) + O\{\log r\} + S(r,f) + S(r,g) \\ &\leq 2(k+m+2)\{T(r,f)+T(r,g)\} + (k+2)(\overline{N}(r,\infty;f) \\ &\quad + \overline{N}(r,\infty;g)) + 2(\overline{N}_L(r,\infty;f) + \overline{N}_L(r,\infty;g)) \\ &\quad + O\{\log r\} + S(r,f) + S(r,g). \end{aligned}$$

Using (2.8) and noting that

$$\overline{N}_L(r,\infty;f) + \overline{N}_L(r,\infty;g) \leq \overline{N}(r,\infty;f) = \overline{N}(r,\infty;g),$$

we obtain from above

$$(n-2k-m-4)\{T(r,f)+T(r,g)\} \leq (k+3)(\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g))+S(r,f)+S(r,g),$$

which leads to a contradiction as $n > 3k+m+7$.

Case 2. *Let $l=1$. Using Lemma 2.11, (3.3) and (3.4) we obtain from (3.5)*

$$\begin{aligned}(n+m)\{T(r,f)+T(r,g)\} &\le N_2(r,0;F)+N_2(r,0;G)+\frac{5}{2}\overline{N}(r,\infty;F)\\ &+\frac{5}{2}\overline{N}(r,\infty;G)+2\overline{N}_*(r,\infty;F,G)+\frac{1}{2}\overline{N}(r,0;F)\\ &+\frac{1}{2}\overline{N}(r,0;G)+N_{k+2}(r,0;f^nP(f))+N_{k+2}(r,0;g^nP(g))\\ &+O\{\log r\}+S(r,f)+S(r,g)\\ &\le 2N_{k+2}(r,0;f^nP(f))+2N_{k+2}(r,0;g^nP(g))\\ &+\frac{1}{2}N_{k+1}(r,0;f^nP(f))+\frac{1}{2}N_{k+1}(r,0;g^nP(g))\\ &+\left(\frac{3k}{2}+\frac{5}{2}\right)(\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g))+2(\overline{N}_L(r,\infty;F)\\ &+\overline{N}_L(r,\infty;G))+O\{\log r\}+S(r,f)+S(r,g)\\ &\le\left(\frac{5k+5m+9}{2}\right)\{T(r,f)+T(r,g)\}+\left(\frac{3k}{2}+\frac{7}{2}\right)\\ &(\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g))+O\{\log r\}+S(r,f)+S(r,g).\end{aligned}$$

Using (2.8) we obtain

$$\begin{aligned}\left(n-\frac{5k+3m+9}{2}\right)\{T(r,f)+T(r,g)\} &\le\left(\frac{3k}{2}+\frac{7}{2}\right)(\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g))\\ &+S(r,f)+S(r,g),\end{aligned}$$

which contradicts our assumption that $n>4k+3m/2+8$.

Case 3. *Let $l=0$. Using Lemma 2.12, (3.3) and (3.4) we obtain from (3.5)*

$$\begin{aligned}(n+m)\{T(r,f)+T(r,g)\} &\le N_2(r,0;F)+N_2(r,0;G)+5\overline{N}(r,\infty;F)+5\overline{N}(r,\infty;G)\\ &+2\overline{N}_*(r,\infty;F,G)+3\overline{N}(r,0;F)+3\overline{N}(r,0;G)\\ &+N_{k+2}(r,0;f^nP(f))+N_{k+2}(r,0;g^nP(g))+O\{\log r\}\\ &+S(r,f)+S(r,g)\\ &\le 2N_{k+2}(r,0;f^nP(f))+2N_{k+2}(r,0;g^nP(g))\\ &+3N_{k+1}(r,0;f^nP(f))+3N_{k+1}(r,0;g^nP(g))+(4k+5)\\ &(\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g))+2(\overline{N}_L(r,\infty;F)+\overline{N}_L(r,\infty;G))\\ &+O\{\log r\}+S(r,f)+S(r,g)\\ &\le(5k+5m+7)\{T(r,f)+T(r,g)\}+(4k+6)(\overline{N}(r,\infty;f)\\ &+\overline{N}(r,\infty;g))+O\{\log r\}+S(r,f)+S(r,g).\end{aligned}$$

This gives by (2.8)

$$\begin{aligned}(n-5k-4m-7)\{T(r,f)+T(r,g)\} &\le(4k+6)(\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g))\\ &+S(r,f)+S(r,g),\end{aligned}$$

contradicting the fact that $n>9k+4m+13$.

We now assume that $H \equiv 0$. That is

$$\left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right) = 0.$$

Integrating both sides of the above equality twice we get

$$\frac{1}{F-1} = \frac{A}{G-1} + B, \tag{3.6}$$

where $A(\neq 0)$ and B are constants.

Now we consider the following three subcases.

Subcase (i) Let $B \neq 0$ and $A = B$. Then from (3.6) we get

$$\frac{1}{F-1} = \frac{BG}{G-1}. \tag{3.7}$$

If $B = -1$, then from (3.7) we obtain

$$FG = 1,$$

i.e.,

$$(f^n P(f))^{(k)}(g^n P(g))^{(k)} = z^2.$$

From our assumption it is clear that $f \neq 0$ and $f \neq \infty$. Let $f(z) = e^{\beta}$, where β is a nonconstant entire function. Then by induction we get

$$(a_m f^{n+m})^{(k)} = t_m(\beta', \beta'', ..., \beta^{(k)})e^{(n+m)\beta}, \tag{3.8}$$

$$(a_0 f^n)^{(k)} = t_0(\beta', \beta'', ..., \beta^{(k)})e^{n\beta}, \tag{3.9}$$

where $t_i(\beta', \beta'', ..., \beta^{(k)})$ $(i = 0, 1, ..., m)$ are differential polynomials in β', β'', . . . , $\beta^{(k)}$. Obviously

$$t_i(\beta', \beta'', ..., \beta^{(k)}) \not\equiv 0$$

for $i = 0, 1, 2, ..., m$, and

$$(f^n P(f))^{(k)} \not\equiv 0.$$

From (3.8) and (3.9) we obtain

$$\overline{N}(r, 0; t_m(\beta', \beta'', ..., \beta^{(k)})e^{m\beta(z)} + ... + t_0(\beta', \beta'', ..., \beta^{(k)})) \leq N(r, 0; z^2) = S(r, f). \tag{3.10}$$

Since β is an entire function, we obtain $T(r, \beta^{(j)}) = S(r, f)$ for $j = 1, 2, ..., k$. Hence $T(r, t_i) = S(r, f)$ for $i = 0, 1, 2, ..., m$.

So from (3.10), Lemmas 2.3 and 2.13 we obtain

$$\begin{aligned} mT(r, f) &= T(r, t_m e^{m\beta} + ... + t_1 e^{\beta}) + S(r, f) \\ &\leq \overline{N}(r, 0; t_m e^{m\beta} + ... + t_1 e^{\beta}) + \overline{N}(r, 0; t_m e^{m\beta} + ... + t_1 e^{\beta} + t_0) + S(r, f) \\ &\leq \overline{N}(r, 0; t_m e^{(m-1)\beta} + ... + t_1) + S(r, f) \\ &\leq (m-1)T(r, f) + S(r, f), \end{aligned}$$

which is a contradiction.

If $B \neq -1$, from (3.7), we have $\frac{1}{F} = \frac{BG}{(1+B)G-1}$ and so $\overline{N}(r, \frac{1}{1+B}; G) = \overline{N}(r, 0; F)$. Now from the second fundamental theorem of Nevanlinna, we get

$$\begin{aligned} T(r, G) &\leq \overline{N}(r, 0; G) + \overline{N}\left(r, \frac{1}{1+B}; G\right) + \overline{N}(r, \infty; G) + S(r, G) \\ &\leq \overline{N}(r, 0; F) + \overline{N}(r, 0; G) + \overline{N}(r, \infty; G) + S(r, G). \end{aligned}$$

Using (2.1 and (2.2) we obtain from above inequality

$$T(r,G) \leq N_{k+1}(r,0;f^nP(f)) + k\overline{N}(r,\infty;f) + T(r,G) + N_{k+1}(r,0;g^nP(g)) - (n+m)T(r,g) + \overline{N}(r,\infty;g) + O\{\log r\} + S(r,g).$$

Using (2.8) we obtain

$$(n+m)T(r,g) \leq (2k+m+1)T(r,f) + (k+m+2)T(r,g) + S(r,g).$$

Thus we obtain

$$(n-3k-m-3)\{T(r,f)+T(r,g)\} \leq S(r,f) + S(r,g),$$

a contradiction as $n > 3k+m+7$.

Subcase (ii) Let $B \neq 0$ and $A \neq B$. Then from (3.6) we get $F = \frac{(B+1)G-(B-A+1)}{BG+(A-B)}$ and so $\overline{N}(r, \frac{B-A+1}{B+1}; G) = \overline{N}(r,0;F)$. Proceeding as in **Subcase (i)** we obtain a contradiction.

Subcase (iii) Let $B = 0$ and $A \neq 0$. Then from (3.6) $F = \frac{G+A-1}{A}$ and $G = AF-(A-1)$. If $A \neq 1$, we have $\overline{N}(r, \frac{A-1}{A}; F) = \overline{N}(r,0;G)$ and $\overline{N}(r, 1-A; G) = \overline{N}(r,0;F)$. So by Lemma 2.14 we have $n \leq 3k+m+3$, a contradiction. Thus $A = 1$ and hence $F = G$. That is

$$[f^n(a_mf^m+a_{m-1}f^{m-1}+...+a_1f+a_0)]^{(k)} = [g^n(a_mg^m+a_{m-1}g^{m-1}+...+a_1g+a_0)]^{(k)}.$$

Integrating we get

$$[f^n(a_mf^m+a_{m-1}f^{m-1}+...+a_1f+a_0)]^{(k-1)} = [g^n(a_mg^m+a_{m-1}g^{m-1}+... + a_1g+a_0)]^{(k-1)} + c_{k-1},$$

where c_{k-1} is a constant. If $c_{k-1} \neq 0$, from Lemma 2.15 we obtain $n \leq 3k+m$, a contradiction. Hence $c_{k-1} = 0$. Repeating k-times, we obtain

$$f^n(a_mf^m+a_{m-1}f^{m-1}+...+a_1f+a_0) = g^n(a_mg^m+a_{m-1}g^{m-1}+...+a_1g+a_0). \tag{3.11}$$

Let $h = \frac{f}{g}$. If h is a constant, by putting $f = gh$ in (3.11) we get

$$a_mg^{n+m}(h^{n+m}-1) + a_{m-1}g^{n+m-1}(h^{n+m-1}-1) + ... + a_0g^n(h^n-1) = 0,$$

which implies $h^d = 1$, where $d = (n+m, ..., n+m-i, ..., n+1, n)$, $a_{m-i} \neq 0$ for some $i = 0,1,...,m$. Thus $f(z) = tg(z)$ for a constant t such that $t^d = 1$, $d = (n+m,...,n+m-i,...,n+1,n)$, $a_{m-i} \neq 0$ for some $i = 0,1,...,m$.

If h is not a constant, then from (3.11) we can say that f and g satisfy the algebraic equation $R(f,g) = 0$, where

$$R(x,y) = x^n(a_mx^m+a_{m-1}x^{m-1}+...+a_0) - y^n(a_my^m+a_{m-1}y^{m-1}+...+a_0).$$

This completes the proof of Theorem 1.2.

Using the arguments similar to the proof of Theorem 1.2, we can prove Corollary 1.3. □

Acknowledgements. The author is grateful to the referee for his/her valuable suggestions and comments towards the improvement of the paper.

References

[1] T.C. Alzahary and H.X. Yi, *Weighted value sharing and a question of I. Lahiri*, Complex Var. Theory Appl., **49** **15**(2004), 1063-1078.

[2] A. Banerjee, *Meromorphic functions sharing one value*, Int. J. Math. Math. Sci., **22**(2005), 3587-3598.

[3] A. Banerjee, *Uniqueness of meromorphic functions sharing two sets with finite weight*, Portugal. Math. (N.S.) **65** **1**(2008), 81-93.

[4] H.H. Chen, *Yoshida functions and Picard values of integral functions and their derivatives*, Bull. Austral. Math. Soc., **54**(1996), 373-381.

[5] M.L. Fang, *A note on a problemma of Hayman*, Analysis (Munich), **20** **1** (2000), 45-49.

[6] M.L. Fang, *Uniqueness and value sharing of entire functions*, Comput. Math. Appl., **44**(2002), 828-831.

[7] M.L. Fang and X.H. Hua, *Entire functions that share one value*, J. Nanjing Univ. Math. Biquarterly, **13** **1**(1996), 44-48.

[8] M.L. Fang and H.L. Qiu, *Meromorphic functions that share fixed points*, J. Math. Anal. Appl., **268** (2002), 426-439.

[9] W.K. Hayman, *Picard values of meromorphic Functions and their derivatives*, Ann. of Math., **70** (1959), 9-42.

[10] W.K. Hayman, *Meromorphic Functions*, The Clarendon Press, Oxford (1964).

[11] G. Hennekemper and W. Hennekemper, *Picard Ausnahmewerte von Ableitungen gewisser meromorpher Funktionen*, Complex Var. Theory Appl., **5** (1985), 87-93.

[12] I. Lahiri, *Weighted sharing and uniqueness of meromorphic functions*, Nagoya Math. J., **161**(2001), 193-206.

[13] I. Lahiri, *Weighted value sharing and uniqueness of meromorphic functions*, Complex Var. Theory Appl., **46**(2001),241-253.

[14] I. Lahiri, *Value distribution of certain differential polynomials*, Int. J. Math. Math. Sc., **28** **2** (2001), 83-91.

[15] I. Lahiri and A. Banerjee, *Weighted sharing of two sets*, Kyungpook Math. J., **46**(2006), 79-87.

[16] P. Sahoo, *Uniqueness and weighted sharing of meromorphic functions*, Ann. Polon. Math., (to appear).

[17] Y.F. Wang, *On Mues conjecture and Picard values*, Sci. China, **36** **1**(1993), 28-35.

[18] C.C. Yang, *On deficiencies of differential polynomials II*, Math. Z., **125** (1972), 107-112.

[19] C.C. Yang and X.H. Hua, *Uniqueness and value sharing of meromorphic functions*, Ann. Acad. Sci. Fenn. Math. **22** **2**(1997), 395-406.

[20] L.Yang, *Value Distribution Theory*, Springer- Verlag, Berlin, 1993.

[21] H.X. Yi, *Meromorphic functions that shares one or two values II*, Kodai Math. J., **22** **2**(1999), 264-272.

[22] H.X. Yi and C.C. Yang, *Uniqueness Theory of Meromorphic Functions*, Science Press, Beijing, 1995.

[23] H.X. Yi and L.Z. Yang, *Meromorphic functions that share two sets*, Kodai Math. J., **20** **2** (1997), 127-134.

[24] J.L. Zhang, *Uniqueness theorems for entire functions concerning fixed points*, Comput. Math. Appl., **56**(2008), 3079-3087.

[25] J.L. Zhang and L.Z. Yang, *Some results related to a conjecture of R. Bruck*, J. Inequal. Pure Appl. Math., **8** **1**(2007), Art. 18.

Department of Mathematics, Silda Chandra Sekhar College, Silda, Paschim Medinipur, West Bengal 721515, India.

E-mail address: sahoopulak1@gmail.com, sahoopulak@yahoo.com