Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 2 Issue 4(2010), Pages 83-86.

ON THE GENERALIZED ABSOLUTE CESÀRO SUMMABILITY

(DEDICATED IN OCCASION OF THE 70-YEARS OF PROFESSOR HARI M. SRIVASTAVA)

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Abstract. In this paper, a known theorem dealing with $\mid C, \alpha \mid$ summability factors, has been generalized for $\mid C, \alpha, \beta \mid_k$ summability factors. Some new results have also been obtained.

1. Introduction

Let $\sum a_n$ be a given infinite series with partial sums (s_n). We denote by $u_n^{\alpha,\beta}$ and $t_n^{\alpha,\beta}$ the n-th Cesàro means of order (α, β), with $\alpha + \beta > -1$, of the sequence (s_n) and (na_n), respectively, i.e., (see [2])

$$u_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} s_v \tag{1.1}$$

$$t_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v, \tag{1.2}$$

where

$$A_n^{\alpha+\beta} = O(n^{\alpha+\beta}), \quad A_0^{\alpha+\beta} = 1 \quad and \quad A_{-n}^{\alpha+\beta} = 0 \quad for \quad n > 0. \tag{1.3}$$

The series $\sum a_n$ is said to be summable $\mid C, \alpha, \beta \mid_k$, $k \geq 1$ and $\alpha + \beta > -1$, if (see [4])

$$\sum_{n=1}^{\infty} n^{k-1} \mid u_n^{\alpha,\beta} - u_{n-1}^{\alpha,\beta} \mid < \infty. \tag{1.4}$$

Since $t_n^{\alpha,\beta} = n(u_n^{\alpha,\beta} - u_{n-1}^{\alpha,\beta})$ (see [4]), condition (4) can also be written as

$$\sum_{n=1}^{\infty} \frac{1}{n} \mid t_n^{\alpha,\beta} \mid^k < \infty. \tag{1.5}$$

If we take $\beta = 0$, then $\mid C, \alpha, \beta \mid_k$ summability reduces to $\mid C, \alpha \mid_k$ summability (see [5]). It should be noted that obviously $(C, \alpha, 0)$ mean is the same as (C, α)

2000 *Mathematics Subject Classification.* 40D15, 40F05, 40G05, 40G99.
Key words and phrases. Absolute Cesàro summability, infinite series, summability factors.

Submitted September 2, 2010. Published October 23, 2010.

mean. A sequence (λ_n) is said to be convex if $\Delta^2\lambda_n \geq 0$, where $\Delta^2\lambda_n = \Delta\lambda_n - \Delta\lambda_{n+1}$.

Pati [6] has proved the following theorem dealing with $\mid C, \alpha \mid$ summability factors.

Theorem 1.1. *If (λ_n) is a convex sequence such that $\sum n^{-1}\lambda_n$ is convergent and the sequence (θ_n^α) defined by*

$$\theta_n^\alpha = \mid t_n^\alpha \mid, \ \alpha = 1, \tag{1.6}$$

$$\theta_n^\alpha = \max_{1\leq v\leq n} \mid t_v^\alpha \mid, \ 0 < \alpha < 1 \tag{1.7}$$

satisfies the condition

$$\theta_n^\alpha = O(1)(C, 1), \tag{1.8}$$

then the series $\sum a_n\lambda_n$ is summable $\mid C, \alpha \mid$ for $0 < \alpha \leq 1$.

2. The Main Result

The aim of this paper is to generalize Theorem 1.1 for $\mid C, \alpha, \beta \mid_k$ summability. We shall prove the following theorem.

Theorem 2.1. *If (λ_n) is a convex sequence such that $\sum n^{-1}\lambda_n$ is convergent and the sequence $(\theta_n^{\alpha,\beta})$ defined by*

$$\theta_n^{\alpha,\beta} = \mid t_n^{\alpha,\beta} \mid, \ \alpha = 1, \beta > -1 \tag{2.1}$$

$$\theta_n^{\alpha,\beta} = \max_{1\leq v\leq n} \mid t_v^{\alpha,\beta} \mid, \ 0 < \alpha < 1, \beta > -1 \tag{2.2}$$

satisfies the condition

$$(\theta_n^{\alpha,\beta})^k = O(1)(C, 1), \tag{2.3}$$

then the series $\sum a_n\lambda_n$ is summable $\mid C, \alpha, \beta \mid_k$ for $0 < \alpha \leq 1$, $\beta > -1$ and $k \geq 1$.

It should be noted that if we take $k = 1$ and $\beta = 0$, then we get Theorem 1.1.

We need the following lemmas for the proof of our theorem.

Lemma 2.2. ([3]) *If (λ_n) is a convex sequence such that $\sum n^{-1}\lambda_n$ is convergent, then*

$$n\Delta\lambda_n \to 0,$$

$$\sum_{n=1}^{\infty}(n+1)\Delta^2\lambda_n$$

is convergent.

Lemma 2.3. ([1]). *If $0 < \alpha \leq 1$, $\beta > -1$ and $1 \leq v \leq n$, then*

$$\mid \sum_{p=0}^{v} A_{n-p}^{\alpha-1}A_p^\beta a_p \mid \leq \max_{1\leq m\leq v} \mid \sum_{p=0}^{m} A_{m-p}^{\alpha-1}A_p^\beta a_p \mid . \tag{2.4}$$

Proof of the theorem. Let $(T_n^{\alpha,\beta})$ be the n-th (C, α, β) mean of the sequence $(na_n\lambda_n)$. Then, by (1.2), we have

$$T_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}}\sum_{v=1}^{n} A_{n-v}^{\alpha-1}A_v^\beta v a_v\lambda_v.$$

First, applying Abel's transformation and then using Lemma 2.3, we have that

$$T_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} \Delta\lambda_v \sum_{p=1}^{v} A_{n-p}^{\alpha-1} A_p^{\beta} p a_p + \frac{\lambda_n}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v,$$

thus,

$$\begin{aligned} \mid T_n^{\alpha,\beta} \mid &\leq \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} \mid \Delta\lambda_v \mid\mid \sum_{p=1}^{v} A_{n-p}^{\alpha-1} A_p^{\beta} p a_p \mid + \frac{\mid \lambda_n \mid}{A_n^{\alpha+\beta}} \mid \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v \mid \\ &\leq \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} A_v^{\alpha} A_v^{\beta} \theta_v^{\alpha,\beta} \mid \Delta\lambda_v \mid + \mid \lambda_n \mid \theta_n^{\alpha,\beta} \\ &= T_{n,1}^{\alpha,\beta} + T_{n,2}^{\alpha,\beta}, \quad say. \end{aligned}$$

Since

$$\mid T_{n,1}^{\alpha,\beta} + T_{n,2}^{\alpha,\beta} \mid^k \leq 2^k (\mid T_{n,1}^{\alpha,\beta} \mid^k + \mid T_{n,2}^{\alpha,\beta} \mid^k),$$

in order to complete the proof of the theorem, by (5), it is sufficient to show that

$$\sum_{n=1}^{\infty} \frac{1}{n} \mid T_{n,r}^{\alpha,\beta} \mid^k < \infty \quad for \quad r = 1, 2.$$

Whenever $k > 1$, we can apply Hölder's inequality with indices k and k', where $\frac{1}{k} + \frac{1}{k'} = 1$, we get that

$$\begin{aligned} \sum_{n=2}^{m+1} \frac{1}{n} \left| T_{n,1}^{\alpha,\beta} \right|^k &\leq \sum_{n=2}^{m+1} \frac{1}{n} \left| \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} A_v^{\alpha} A_v^{\beta} \theta_v^{\alpha,\beta} \Delta\lambda_v \right|^k \\ &= O(1) \sum_{n=2}^{m+1} \frac{1}{n^{1+(\alpha+\beta)k}} \left\{ \sum_{v=1}^{n-1} v^{\alpha k} v^{\beta k} \Delta\lambda_v (\theta_v^{\alpha,\beta})^k \right\} \\ &\quad \times \left\{ \sum_{v=1}^{n-1} \Delta\lambda_v \right\}^{k-1} \\ &= O(1) \sum_{v=1}^{m} v^{(\alpha+\beta)k} \Delta\lambda_v (\theta_v^{\alpha,\beta})^k \sum_{n=v+1}^{m+1} \frac{1}{n^{1+(\alpha+\beta)k}} \\ &= O(1) \sum_{v=1}^{m} v^{(\alpha+\beta)k} \Delta\lambda_v (\theta_v^{\alpha,\beta})^k \int_v^{\infty} \frac{dx}{x^{1+(\alpha+\beta)k}} \\ &= O(1) \sum_{v=1}^{m} \Delta\lambda_v (\theta_v^{\alpha,\beta})^k \\ &= O(1) \sum_{v=1}^{m-1} \Delta(\Delta\lambda_v) \sum_{p=1}^{v} (\theta_p^{\alpha,\beta})^k + O(1) \Delta\lambda_m \sum_{v=1}^{m} (\theta_v^{\alpha,\beta})^k \\ &= O(1) \sum_{v=1}^{m} v \Delta^2 \lambda_v + O(1) m \Delta\lambda_m \\ &= O(1) \quad as \quad m \to \infty, \end{aligned}$$

in view of hypotheses of the theorem and Lemma 2.2.

Similarly , we have that

$$\begin{aligned}
\sum_{n=1}^{m}\frac{1}{n}\mid \lambda_n\theta_n^{\alpha,\beta}\mid^k &= O(1)\sum_{n=1}^{m}\frac{\lambda_n}{n}(\theta_n^{\alpha,\beta})^k \\
&= O(1)\sum_{n=1}^{m-1}\Delta(n^{-1}\lambda_n)\sum_{v=1}^{n}(\theta_v^{\alpha,\beta})^k \\
&+ O(1)\frac{\lambda_m}{m}\sum_{v=1}^{m}(\theta_v^{\alpha,\beta})^k \\
&= O(1)\sum_{n=1}^{m-1}\Delta\lambda_n + O(1)\sum_{n=1}^{m-1}\frac{\lambda_{n+1}}{n+1} + O(1)\lambda_m \\
&= O(1)(\lambda_1-\lambda_m) + O(1)\sum_{n=1}^{m-1}\frac{\lambda_{n+1}}{n+1} + O(1)\lambda_m \\
&= O(1) \quad as \quad m\to\infty.
\end{aligned}$$

Therefore, by (1.5), we get that

$$\sum_{n=1}^{\infty}\frac{1}{n}\mid T_{n,r}^{\alpha,\beta}\mid^k < \infty \quad for \quad r=1,2.$$

This completes the proof of the theorem. If we take $\beta = 0$, then we get a new result for $\mid C,\alpha\mid_k$ summability factors. Also, if we take $\beta = 0$ and $\alpha = 1$, then we get another new result for $\mid C,1\mid_k$ summability factors.

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