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THE SOLUTION OF ONE-DIMENSIONAL DIFFUSION EQUATION BY USING THE HERMITE WAVELET TRANSFORM METHOD

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Abstract. The diffusion equation, also known as the heat equation, governs the spatial and temporal domain of the evolution of physical quantities such as heat, mass, or concentration under diffusive transport. In this article, we solve the one-dimensional diffusion equation using the Hermite wavelet transform method. First, we convert the given diffusion equation with the corresponding initial and boundary conditions into a series of Hermite wavelet basis functions. We obtained an algebraic system of equations solved by Matlab software, and we checked the accuracy of the Hermite solution over the exact solution and analyzed the result graphically.

1. Introduction

The wavelet transform is a powerful mathematical tool for analyzing non-stationary signals across multiple scales, offering both time and frequency localization unlike the classical Fourier transform [18]. By decomposing a signal into scaled and shifted wavelets, it effectively captures transient, localized features as well as global trends [9]. This multi resolution analysis allows accurate representation of signals with abrupt changes or singularities. Due to these properties, wavelet transforms are particularly suitable for signal and image processing tasks. They have been extensively applied in compression, denoising, and feature extraction [13]. Furthermore, wavelet-based filter banks provide efficient and compact signal representations [6]. Consequently, wavelet theory has become a foundational tool in modern scientific and engineering research [3, 4].

The diffusion equation is a fundamental partial differential equation that combines processes such as heat or chemical reaction that spreads out over space and time. It is derived from conservation laws combined with constitutive relations like Ficks law, linking flux to concentration gradients [10]. The equation characterizes

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how an initial distribution spreads over time due to random molecular motion. Analytical solutions exist for simple geometries and boundary conditions, providing insight into diffusion dynamics [7]. For complex domains, numerical methods such as finite difference and finite element schemes are commonly employed. The diffusion equation plays a central role in heat transfer, fluid mechanics, and material science [25]. Several approaches have been proposed to solve the one-dimensional diffusion equation using analytical and numerical techniques. Suga [24] applied a multi-level finite difference scheme derived from the Lattice Boltzmann Method, demonstrating improved accuracy and stability for 1D diffusion problems. A high-accuracy Chebyshev pseudospectral method was developed by Bazan [5] for 1D convectiondiffusion equations, discretizing space with spectral collocation and integrating in time using a fourth-order RungeKutta scheme, outperforming classical finite-difference methods. Pereira da Silva [27] developed a numerical solution for wood drying using the 1D diffusion equation, comparing results with higher-dimensional models to show the efficiency of reduced-dimensional approaches. Ali et al. [8] employed Lucas and Fibonacci polynomial-based methods to solve 1D advection-diffusion-reaction equations, providing flexible handling of complex boundary conditions. Dagdug [11] presented finite-difference methods for confined diffusion, emphasizing stability and convergence in practical applications. Khan [2] proposed a wavelet hybrid scheme combining Haar wavelets with numerical integration to solve one- and two-dimensional advection-diffusion initial boundary value problems, achieving high accuracy and efficiency for problems with complex boundary conditions. Verma [1] utilized Lie symmetry and differential quadrature techniques for nonlinear diffusion equations, enabling both analytical insight and numerical approximation. Zhang [14] proposed a compact finite difference method for reactiondiffusion problems, achieving high-order accuracy with reduced computational effort. Tsegaye and Simonand Purnachandra Rao [23] used various finite difference schemes for solving the one dimensional diffusion equation. Linge and Langtangen [17] provided a comprehensive treatment of classical 1D diffusion problems, illustrating detailed implementations of the Forward Euler and Backward Euler schemes. Nova [12] employed spectral collocation methods based on Lagranges basis polynomials to solve one-dimensional space-fractional diffusion equations, systematically comparing different collocation points to achieve high accuracy and improved convergence for fractional-order problems.

The paper is summarized as follows: In Section 2, the basic details and notations of the Hermite wavelet transform method are discussed. In Section 3, the method for solving the one-dimensional diffusion equation is presented. In Section 4, numerical examples are given along with graphical analysis. In Section 5, a conclusion is given.

2. Hermite wavelet

The family of functions constructed from the translation and dilation parameters is called a wavelet [21]. Moreover, the continuous wavelet transform with the dilation (scale) parameter a, and the shift (translation) parameter b is given by,

$$\mathcal{W}_{a,b}(x) = \frac{1}{\sqrt{|a|}}\mathcal{W}\left(\frac{x-b}{a}\right), \quad x, b \in \mathbb{R},\ a \neq 0.$$

To obtain a discrete wavelet transform, we restrict $a = a_0^k$, $b = nb_0a_0^k$, $k, n > 1$, $a_0 > 1$, $b_0 > 0$, we get

$$\mathcal{W}_{k,n}(x) = \frac{1}{\sqrt{a_0^k}} \mathcal{W}\left(\frac{x - nb_0 a_0^k}{a_0^k}\right), \quad x \in \mathbb{R},\ k \in \mathbb{Z},\ a_0 \neq 0,$$

where $\{\mathcal{W}_{k,n}(x)\}$ forms a wavelet basis for $L^2(\mathbb{R})$.

Hermite wavelets [15, 20] are defined as

$$H_{n,m}(x) = \begin{cases} \frac{2^{\frac{k+1}{2}}}{\sqrt{\pi}}\, h_m(2^k x - 2n + 1), & \frac{n-1}{2^{k-1}} \leq x < \frac{n}{2^{k-1}}, \\ 0, & \text{otherwise} \end{cases}$$

where $n = 0, 1, \ldots, 2^{k-1}$, and $m = 0, 1, 2, \ldots, M-1$, k is a positive integer. Here, $h_m(x)$ are the polynomials of degree m with respect to the weight function $W(x) = \sqrt{1-x^2}$.

The Hermite polynomials of degree m can be defined using the following recurrence formula:

$$h_{m+2}(x) = 2x\, h_{m+1}(x) - 2(m+1)\, h_m(x),$$

where $m = 0, 1, 2, \ldots$

Function Approximation of Hermite Wavelet [19, 21, 26]**:** If $\zeta(x,t)$ is a solution of the one-dimensional diffusion equation, we can approximate it by the Hermite wavelet as follows:

$$\zeta(x,t) = H^T(x) A\, H(t)$$

where, $H^T(x) = [H_{1,0}(x), H_{1,1}(x) \ldots, H_{1,M-1}(x), H_{2,0}(x), H_{2,2}(x) \ldots, H_{2,M-1}(x), \ldots,$
$H_{2^{k-1},0}(x), H_{2^{k-1},1}(x) \ldots, H_{2^{k-1},M-1}(x)]$,
$H(t) = [H_{1,0}(t), H_{1,1}(t) \ldots, H_{1,M-1}(t), H_{2,0}(t), H_{2,2}(t) \ldots, H_{2,M-1}(t), \ldots, H_{2^{k-1},0}(t), H_{2^{k-1},1}(t) \ldots,$
$H_{2^{k-1},M-1}(t)]^T$ and $A = [a_{i,j}]$ is a $2^{k-1}M \times 2^{k-1}M$ matrix such that $i = 1, 2, \ldots, 2^{k-1}$ and $j = 0, 1, \ldots, M-1$.

Functional matrix of integration [16]**:**

The Hermite wavelet basis functions are orthogonal, continuous, and compactly supported. Here, we consider ten Hermite wavelet basis functions on $[0,1]$ given by,

$$
\begin{aligned}
H_{1,0}(x) &= \frac{2}{\sqrt{\pi}} \\
H_{1,1}(x) &= \frac{1}{\sqrt{\pi}}(-4 + 8x) \\
H_{1,2}(x) &= \frac{1}{\sqrt{\pi}}(4 - 32x + 32x^2) \\
H_{1,3}(x) &= \frac{1}{\sqrt{\pi}}(8 + 48x - 192x^2 + 128x^3) \\
H_{1,4}(x) &= \frac{1}{\sqrt{\pi}}(-40 + 128x + 384x^2 - 1024x^3 + 512x^4) \\
H_{1,5}(x) &= \frac{1}{\sqrt{\pi}}(16 - 800x + 1280x^2 + 2560x^3 - 5120x^4 + 2048x^5) \\
H_{1,6}(x) &= \frac{1}{\sqrt{\pi}}(368 + 384x - 9600x^2 + 10240x^3 + 13560x^4 - 24576x^5 + 8192x^6) \\
H_{1,7}(x) &= \frac{1}{\sqrt{\pi}}(-928 + 10304x + 5776x^2 - 89600x^3 + 71680x^4 + 86016x^5 - 114688x^6 + 32768x^7) \\
H_{1,8}(x) &= \frac{1}{\sqrt{\pi}}(-3296 - 29696x + 164864x^2 + 57344x^3 - 716800x^4 + 458752x^5 + 458752x^6 - 524288x^7 \\
&\quad + 131072x^8) \\
H_{1,9}(x) &= \frac{1}{\sqrt{\pi}}(21440 - 118656x - 534528x^2 + 1978368x^3 + 510096x^4 - 5109606x^5 + 2752512x^6 \\
&\quad + 2359296x^7 - 2359296x^8 + 524288x^9) \\
H_{1,10}(x) &= \frac{1}{\sqrt{\pi}}(16448 + 857600x + 2373120x^2 - 7170240x^3 + 19783680x^4 + 4128768x^5 - 34406400x^6 \\
&\quad + 15728640x^7 + 11796480x^8 - 10485760x^9 + 2097152x^{10})
\end{aligned}
$$

where $H_9(x) = [H_{1,0}(x), H_{1,1}(x), H_{1,2}(x), H_{1,3}(x), H_{1,4}(x), H_{1,5}(x), H_{1,6}(x), H_{1,7}(x), H_{1,8}(x)]^T$.

Now, integrating the above Hermite wavelet basis functions with limit 0 to x and with respect to x and expressing it in matrix form, we get,

$$
\begin{aligned}
\int_0^x H_{1,0}(x)\,dx &= \frac{2}{\sqrt{\pi}}x \\
&= \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} H_9(x)
\end{aligned}
$$

$$
\begin{aligned}
\int_0^x H_{1,1}(x)\,dx &= \frac{4x^2 - 4x}{\sqrt{\pi}} \\
&= \begin{bmatrix} -\frac{1}{4} & 0 & \frac{1}{8} & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} H_9(x)
\end{aligned}
$$

$$\int_0^x H_{1,2}(x)\,dx = \frac{32x^3-16x^2+4x}{\sqrt{\pi}}$$
$$= \left[-\frac{1}{3} \quad 0 \quad 0 \quad \frac{1}{12} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\right] H_9(x)$$

$$\int_0^x H_{1,3}(x)\,dx = \frac{32x^4-64x+36x^2-4x}{\sqrt{\pi}}$$
$$= \left[\frac{5}{4} \quad 0 \quad 0 \quad 0 \quad \frac{1}{16} \quad 0 \quad 0 \quad 0 \quad 0\right] H_9(x)$$

$$\int_0^x H_{1,4}(x)\,dx = \frac{\frac{512}{5}x^5-256x^5+\frac{640}{3}x^3-64x^2+4x}{\sqrt{\pi}}$$
$$= \left[-\frac{2}{5} \quad 0 \quad 0 \quad 0 \quad 0 \quad \frac{1}{20} \quad 0 \quad 0 \quad 0\right] H_9(x)$$

$$\int_0^x H_{1,5}(x)\,dx = \frac{\frac{1024}{3}x^6-1024x^5+1120x^4-\frac{1600}{6}+100x^2-4x}{\sqrt{\pi}}$$
$$= \left[-\frac{23}{3} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad \frac{1}{24} \quad 0 \quad 0\right] H_9(x)$$

$$\int_0^x H_{1,6}(x)\,dx = \frac{16x\left(512x^6-1792x^5+1344x^4+1120x^3-1400x^2+84x+161\right)}{7\sqrt{\pi}}$$
$$= \left[\frac{116}{7} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad \frac{1}{28} \quad 0\right] H_9(x)$$

$$\int_0^x H_{1,7}(x)\,dx = \frac{32x\left(128x^7-512x^6+448x^5+448x^4-700x^3+56x^2+161x-29\right)}{\sqrt{\pi}}$$
$$= \left[\frac{103}{2} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad \frac{1}{32}\right] H_9(x)$$

$$\int_0^x H_{1,8}(x)\,dx = \frac{8x}{9\sqrt{\pi}}\Big(16384x^8-73728x^7+73728x^6+86016x^5$$
$$-161280x^4+16128x^3+61824x^2-16857x-3708\Big)$$
$$= \left[-\frac{2680}{9} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad \frac{1}{36}\right] H_9(x) + \frac{1}{36}H_{1,9}(x)$$

Hence,

$$\int_0^x H(x)\,dx = H_{9\times 9}\,H_9(x) + \tilde{H}_9(x).$$

Now, by taking the double integration of the above-mentioned nine Hermite wavelet bases, is given by,

$$\int_0^x \int_0^t H_{1,0}(x)\,dx\,dt = \frac{x^2}{\sqrt{\pi}} = \begin{bmatrix} \frac{3}{16} & \frac{1}{8} & \frac{1}{32} & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} H_9(x)$$

$$\int_0^x \int_0^t H_{1,1}(x)\,dx\,dt = \frac{\frac{4}{3}x^3 - 2x^2}{\sqrt{\pi}} = \begin{bmatrix} -\frac{1}{6} & -\frac{1}{16} & 0 & \frac{1}{96} & 0 & 0 & 0 & 0 & 0 \end{bmatrix} H_9(x)$$

$$\int_0^x \int_0^t H_{1,2}(x)\,dx\,dt = \frac{\frac{8}{3}x^4 - \frac{16}{3}x^3 + 2x^2}{\sqrt{\pi}} = \begin{bmatrix} -\frac{1}{16} & -\frac{1}{12} & 0 & 0 & \frac{1}{192} & 0 & 0 & 0 & 0 \end{bmatrix} H_9(x)$$

$$\int_0^x \int_0^t H_{1,3}(x)\,dx\,dt = \frac{\frac{32}{5}x^5 - 16x^4 + 12x^3 - 2x^2}{\sqrt{\pi}} = \begin{bmatrix} \frac{3}{5} & \frac{5}{16} & 0 & 0 & 0 & \frac{1}{320} & 0 & 0 & 0 \end{bmatrix} H_9(x)$$

$$\int_0^x \int_0^t H_{1,4}(x)\,dx\,dt = \frac{\frac{256}{15}x^6 - \frac{256}{5}x^5 + \frac{160}{3}x^4 - \frac{64}{3}x^3 + 2x^2}{\sqrt{\pi}} = \begin{bmatrix} -\frac{7}{12} & -\frac{1}{10} & 0 & 0 & 0 & 0 & \frac{1}{480} & 0 & 0 \end{bmatrix} H_9(x)$$

$$\int_0^x \int_0^t H_{1,5}(x)\,dx\,dt = \frac{\frac{1024}{21}x^7 - \frac{512}{3}x^6 + 224x^5 - \frac{400}{3}x^4 + \frac{100}{3}x^3 - 2x^2}{\sqrt{\pi}} = \begin{bmatrix} -\frac{22}{7} & -\frac{23}{12} & 0 & 0 & 0 & 0 & 0 & \frac{1}{672} & 0 \end{bmatrix} H_9(x)$$

$$\int_0^x \int_0^t H_{1,6}(x)\,dx\,dt = \frac{8x^2\left(128x^6 - 512x^5 + 448x^4 + 448x^3 - 700x^2 + 56x + 161\right)}{7\sqrt{\pi}} = \begin{bmatrix} \frac{81}{8} & \frac{29}{7} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{896} \end{bmatrix} H_9(x)$$

$$\int_0^x \int_0^t H_{1,7}(x)\,dx\,dt = \frac{16x^2\left(256x^7 - 1152x^6 + 1152x^5 + 1344x^4 - 2520x^3 + 252x^2 + 966x - 261\right)}{9\sqrt{\pi}} = \begin{bmatrix} \frac{148}{9} & \frac{103}{8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} H_9(x) + \frac{1}{1152}H_{1,9}(x)$$

$$\begin{aligned}\int_0^x \int_0^t H_{1,8}(x)\,dx\,dt &= \frac{8x^2}{45\sqrt{\pi}}\Big(8192x^8 - 40960x^7 + 46080x^6 + 61440x^5 \\ &\quad - 134400x^4 + 16128x^3 + 77280x^2 - 28095x - 9270\Big) \\ &= \begin{bmatrix} -\frac{773}{5} & -\frac{670}{9} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} H_9(x) + \frac{1}{1440}H_{1,10}(x)\end{aligned}$$

$$\int_0^x \int_0^t H(x)\,dx\,dt = H'_{9\times 9}\,H_9(x) + \tilde{\tilde{H}}_9(x).$$

$$\mathcal{P}_{9\times 9} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{4} & 0 & \frac{1}{8} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{3} & 0 & 0 & \frac{1}{12} & 0 & 0 & 0 & 0 & 0 \\ \frac{5}{4} & 0 & 0 & 0 & \frac{1}{16} & 0 & 0 & 0 & 0 \\ -\frac{2}{5} & 0 & 0 & 0 & 0 & \frac{1}{20} & 0 & 0 & 0 \\ -\frac{23}{3} & 0 & 0 & 0 & 0 & 0 & \frac{1}{24} & 0 & 0 \\ \frac{116}{7} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{28} & 0 \\ \frac{103}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{32} \\ -\frac{2680}{9} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \tilde{H}_9(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{36}H_{1,9}(x) \end{bmatrix},$$

$$\mathcal{P}'_{9\times 9} = \begin{bmatrix} \frac{3}{16} & \frac{1}{8} & \frac{1}{32} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{6} & -\frac{1}{16} & 0 & \frac{1}{96} & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{16} & -\frac{1}{12} & 0 & 0 & \frac{1}{192} & 0 & 0 & 0 & 0 \\ \frac{3}{5} & \frac{5}{16} & 0 & 0 & 0 & \frac{1}{320} & 0 & 0 & 0 \\ -\frac{7}{12} & -\frac{1}{10} & 0 & 0 & 0 & 0 & \frac{1}{480} & 0 & 0 \\ -\frac{22}{7} & -\frac{23}{12} & 0 & 0 & 0 & 0 & 0 & \frac{1}{672} & 0 \\ \frac{81}{8} & \frac{29}{7} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{896} \\ \frac{148}{9} & \frac{103}{8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{773}{8} & -\frac{670}{9} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \tilde{H}'_9(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{1152}H_{1,9}(x) \\ \frac{1}{1440}H_{1,10}(x) \end{bmatrix}.$$

Similarly, we can generate matrices for our suitability.

Theorem 2.1. *If a bounded continuous function $\zeta(x,t)$ is defined in $L^2([0,1]^2)$, then the Hermite wavelet expansion of $\zeta(x,t)$ is uniformly convergence to it.*

Proof. The function $\zeta(x,t)$ can be approximated in the form of a Hermite wavelet basis as follows,

$$\zeta(x,t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} H_{n,m}H_{n,m}(x)H_{n,m}(t), \tag{2.1}$$

where $H_{n,m} = \langle \zeta(x,t), H_{n,m}(x)H_{n,m}(t)\rangle$ and $\langle -,-\rangle$ represents the inner product.

Now, the Hermite wavelet coefficients of bounded continuous function $\zeta(x,t)$ are defined by

$$\begin{aligned} H_{n,m} &= \int_0^1\int_0^1 \zeta(x,t)H_{n,m}(x)H_{n,m}(t)dxdt \\ &= \int_0^1\int_I \zeta(x,t)\frac{2^{\frac{k+1}{2}}}{\sqrt{\pi}}h_m(2^kx-2n+1)H_{n,m}(t)dxdt \end{aligned} \tag{2.2}$$

where

$$I = \left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right).$$

Replace $2^kx-2n+1$ by u, we obtain

$$
\begin{aligned}
H_{n,m} &= \frac{2^{\frac{k+1}{2}}}{\sqrt{\pi}} \int_0^1 \left[\int_{-1}^1 \zeta\left(\frac{u-1+2n}{2^k}, t\right) h_m(u) \frac{du}{2^k}\right] H_{n,m}(t)dt \\
&= \frac{2^{\frac{-k+1}{2}}}{\sqrt{\pi}} \int_0^1 \left[\int_{-1}^1 \zeta\left(\frac{u-1+2n}{2^k}, t\right) h_m(u) du\right] H_{n,m}(t)dt. \qquad (2.3)
\end{aligned}
$$

Using the generalized mean value theorem (GMVT) for integrals, we get

$$H_{n,m} = \frac{2^{\frac{-k+1}{2}}}{\sqrt{\pi}} \int_0^1 \zeta\left(\frac{\alpha-1+2n}{2^k}, t\right) H_{n,m}(t)dt \left[\int_{-1}^1 h_m(u)du\right] \qquad (2.4)$$

where $\alpha \in (-1,1)$.
Because of continuity and integrability of $h_m(x)$ on $(-1,1)$, we can take $\int_{-1}^1 h_m(u)du = T$, such that

$$
\begin{aligned}
H_{n,m} &= T\frac{2^{\frac{-k+1}{2}}}{\sqrt{\pi}} \int_0^1 \zeta\left(\frac{\alpha-1+2n}{2^k}, t\right) H_{n,m}(t)dt \qquad (2.5)\\
&= T\frac{2^{\frac{-k+1}{2}}}{\sqrt{\pi}} \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \zeta\left(\frac{\alpha-1+2n}{2^k}, t\right) \frac{2^{\frac{k+1}{2}}}{\sqrt{\pi}} h_m(2^k t-2n+1)dt. \qquad (2.6)
\end{aligned}
$$

Now, replicating the above step, i.e. substitute $2^k t - 2n + 1 = v$, we get

$$
\begin{aligned}
H_{n,m} &= \frac{2T}{(\sqrt{\pi})^2} \int_{-1}^1 \zeta\left(\frac{\alpha-1+2n}{2^k}, \frac{v-1+2n}{2^k}\right) h_m(v)\frac{dv}{2^k} \qquad (2.7)\\
&= \frac{A2^{-k+1}}{\pi} \int_{-1}^1 \zeta\left(\frac{\alpha-1+2n}{2^k}, \frac{v-1+2n}{2^k}\right) h_m(v)dv \qquad (2.8)
\end{aligned}
$$

Now, by GMVT for integrals,

$$H_{n,m} = \frac{A2^{-k+1}}{\pi} \zeta\left(\frac{\alpha-1+2n}{2^k}, \frac{\beta-1+2n}{2^k}\right) \int_{-1}^1 h_m(v)dv \qquad (2.9)$$

where $\beta \in (-1,1)$.
As of continuity and integrability of $h_m(x)$ on $(-1,1)$
Choose, $\int_{-1}^1 L_P(v)dv = U$, The above equation becomes

$$H_{n,m} = \frac{TU2^{-k+1}}{(\sqrt{\pi})^2} \zeta\left(\frac{\alpha-1+2n}{2^k}, \frac{\beta-1+2n}{2^k}\right) \quad \alpha, \beta \in (-1,1). \qquad (2.10)$$

Taking the modulus,

$$|H_{n,m}| = \left|\frac{TU2^{-k+1}}{\pi}\right| \left|\zeta\left(\frac{\alpha-1+2n}{2^k}, \frac{\beta-1+2n}{2^k}\right)\right|, \alpha, \beta \in (-1,1)$$

$$|H_{n,m}| \leq \left|\frac{TU\,2^{-\,k+1}}{\pi}\right| M =: C,$$

where C is independent of n and m. Hence, the double series

$$\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}|H_{n,m}|$$

is dominated by a convergent numerical series and therefore, is absolutely convergent. By the Weierstrass M–test, absolute convergence of a series of bounded functions implies uniform convergence on the domain.
Therefore,

$$\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}H_{n,m} \quad \text{is convergent.}$$

Hence $\zeta(x,t)$ is uniformly convergent. □

3. Method of solution

Consider the one-dimensional diffusion equation of the form

$$\frac{\partial\zeta(x,t)}{\partial t}=\frac{\partial^2\zeta(x,t)}{\partial x^2}, \quad a<x<b, \quad t>0. \tag{3.1}$$

First, we assume that

$$\frac{\partial^4\zeta(x,t)}{\partial x^3\partial t}=H^T(x)AH(t) \tag{3.2}$$

where, $H^T(x)=[H_{1,0}(x),\dots,H_{1,M-1}(x),\dots,H_{2k-1,0}(x),\dots,H_{2k-1,M-1}(x)]$, $H(t)=[H_{1,0}(t),H_{1,1}(t)\dots,H_{1,M-1}(t),H_{2,0}(t),H_{2,2}(t)\dots,H_{2,M-1}(t),\dots,H_{2^{k-1},0}(t),H_{2^{k-1},1}(t)\dots,H_{2^{k-1},M-1}(t)]^T$, and $A=[a_{i,j}]$ be $2^{k-1}M\times 2^{k-1}M$ a matrix such that $i=1,\dots,2^{k-1}$ and $j=0,1,\dots,M-1$.

Now, integrating equation (3.2) from 0 to t yields,

$$\frac{\partial^3\zeta(x,t)}{\partial x^3}=\frac{\partial^3\zeta(x,0)}{\partial x^3}+\int_0^t H^T(x)AH(t)\,dt$$
$$\frac{\partial^3\zeta(x,t)}{\partial x^3}=\frac{\partial^3\zeta(x,0)}{\partial x^3}+H^T(x)A\left[\mathcal{P}H(t)+\bar{H}(t)\right]. \tag{3.3}$$

Now, integrating equation (3.3) from 0 to x results in the following:

$$\frac{\partial^2\zeta(x,t)}{\partial x^2}=\frac{\partial^2\zeta(0,t)}{\partial x^2}+\frac{\partial^2\zeta(x,0)}{\partial x^2}-\frac{\partial^2\zeta(0,0)}{\partial x^2}+\int_0^x H^T(x)A\left[\mathcal{P}H(t)+\bar{H}(t)\right]dx$$
$$\frac{\partial^2\zeta(x,t)}{\partial x^2}=\frac{\partial^2\zeta(0,t)}{\partial x^2}+\frac{\partial^2\zeta(x,0)}{\partial x^2}-\frac{\partial^2\zeta(0,0)}{\partial x^2}$$
$$+\left[\mathcal{P}H(x)+\bar{H}(x)\right]^T A\left[\mathcal{P}H(t)+\bar{H}(t)\right]. \tag{3.4}$$

Now, integrating equation (3.4) from 0 to x is

$$\frac{\partial\zeta(x,t)}{\partial x} = \frac{\partial\zeta(0,t)}{\partial x} + x\left[\frac{\partial^2\zeta(0,t)}{\partial x^2} - \frac{\partial^2\zeta(0,0)}{\partial x^2}\right] + \frac{\partial\zeta(x,0)}{\partial x} - \frac{\partial\zeta(0,0)}{\partial x} + \int_0^x \left[\mathcal{P}H(x) + \bar{H}(x)\right]^T A\left[\mathcal{P}H(t) + \bar{H}(t)\right] dx$$

$$\frac{\partial\zeta(x,t)}{\partial x} = \frac{\partial\zeta(0,t)}{\partial x} + x\left[\frac{\partial^2\zeta(0,t)}{\partial x^2} - \frac{\partial^2\zeta(0,0)}{\partial x^2}\right] + \left[\frac{\partial\zeta(x,0)}{\partial x} - \frac{\partial\zeta(0,0)}{\partial x}\right] + A\left[\mathcal{P}H(t) + \bar{H}(t)\right]\left[\mathcal{P}'H(x) + \bar{H}'(x)\right]^T \tag{3.5}$$

$$\zeta(x,t) = \zeta(0,t) + x\left[\frac{\partial\zeta(0,t)}{\partial x} - \frac{\partial\zeta(0,0)}{\partial x}\right] + \frac{x^2}{2}\left[\frac{\partial^2\zeta(0,t)}{\partial x^2} - \frac{\partial^2\zeta(0,0)}{\partial x^2}\right] + \zeta(x,0) - \zeta(0,0) + A\left[\mathcal{P}H(t) + \bar{H}(t)\right]\left[\mathcal{P}''H(x) + \bar{H}''(x)\right]^T. \tag{3.6}$$

Next, differentiate equation (3.6) with respect to t.

$$\frac{\partial\zeta(x,t)}{\partial t} = \frac{\partial\zeta(0,t)}{\partial t} + x\left[\frac{\partial^2\zeta(0,t)}{\partial x\partial t}\right] + \frac{x^2}{2}\left[\frac{\partial^3\zeta(0,t)}{\partial x^2\partial t}\right] + A\left[\mathcal{P}''H(x) + \bar{H}''(x)\right]^T \frac{d}{dt}\left[\mathcal{P}H(t) + \bar{H}(t)\right]. \tag{3.7}$$

Substitute (3.7), (3.6), (3.5), (3.4), and (3.3) into (3.1), using the given physical conditions. We get a system of algebraic equations by collocating this equation by following the grid points;

$$x_i = t_i = \frac{2i-1}{2[2^{k-1}M]^2}, \quad i = 1, 2, \ldots, [2^{k-1}M]^2.$$

Solving this system by Newtons method yields the values of the Hermite wavelet coefficients. Substituting these coefficients into (3.6) gives a Hermite wavelets based numerical solution of (3.1).

4. Numerical Examples

Example 4.1. *Consider the one-dimensional diffusion equation.*

$$\frac{\partial\zeta(x,t)}{\partial t} = \frac{\partial^2\zeta(x,t)}{\partial x^2}, \quad 0 < x < 1, \quad t > 0,$$

subject to initial and boundary conditions

$$\zeta(x,0) = sin(\pi x), \quad \zeta(0,t) = \zeta(1,t) = 0, \quad t > 0,$$

with the exact solution

$$\zeta(x,t) = e^{-\pi^2 t} sin(\pi x).$$

The computational work is performed by Matlab R2024.

Error Metrics:

- Mean Absolute Error (MAE): 1.28453×10^{-9}.
- Root Mean Squared Error (RMSE): 3.58404×10^{-5}.

Table 1. Comparison between the exact solution and Hermite (Approximate) solution of Example (1).

x	t	Exact solution	Hermite (Approximate) solution	Absolute Error
-2.000000	0.000000	2.4502969098e-16	2.45029690982e-16	3.699758e-28
0.222222	0.000000	6.4278760969e-01	6.42787609586e-01	1.750489e-12
2.000000	0.000000	-2.4502969098e-16	-2.45029690987e-16	5.735019e-27
-2.000000	0.011111	2.7342891022e-16	2.73428910217e-16	2.922286e-27
0.222222	0.011111	7.1728742307e-01	7.17287423064e-01	4.107492e-12
2.000000	0.011111	-2.7342891022e-16	-2.73428910217e-16	2.499259e-27
-2.000000	0.022222	3.0511963119e-16	3.05119631195e-16	6.789134e-29
0.222222	0.022222	8.0042184936e-01	8.00421849365e-01	3.182343e-12
2.000000	0.022222	-3.0511963119e-16	-3.05119631193e-16	2.072880e-27
-2.000000	0.033333	3.4048334269e-16	3.40483342697e-16	5.370072e-27
0.222222	0.033333	8.9319165000e-01	8.93191650010e-01	9.137580e-12
2.000000	0.033333	-3.4048334269e-16	-3.40483342698e-16	5.418045e-27
-2.000000	0.044444	3.7994574848e-16	3.79945748482e-16	3.672641e-28
0.222222	0.044444	9.96713575881e-01	9.967136575804e-01	2.858047e-12
2.000000	0.044444	-3.7994574848e-16	-3.79945748486e-16	4.370240e-27
-2.000000	0.055556	4.2398181988e-16	4.239818191885e-16	1.010876e-27
0.222222	0.055556	1.1122338103e+00	1.11223381026e+00	3.192047e-11
2.000000	0.055556	-4.2398189188e-16	-4.23981891885e-16	7.848673e-28
-2.000000	0.066667	4.7312187428e-16	4.73121874276e-16	4.994476e-28
0.222222	0.066667	1.2411429710e+00	1.24114297104e+00	7.078782e-12
2.000000	0.066667	-4.7312187428e-16	-4.73121874276e-16	6.645167e-28
-2.000000	0.077778	5.2795723639e-16	5.27957236396e-16	3.969844e-27
0.222222	0.077778	1.3849928498e+00	1.38499284976e+00	3.162293e-11
2.000000	0.077778	-5.2795723639e-16	-5.27957239399e-16	6.291856e-27
-2.000000	0.088889	5.8914807920e-16	5.89148079206e-16	3.640889e-27
0.222222	0.088889	1.5455150927e+00	1.54551509273e+00	1.378808e-11
2.000000	0.088889	-5.8914807920e-16	-5.89148079206e-16	3.640889e-27
-2.000000	0.100000	6.5743101013e-16	6.57431010136e-16	3.792547e-27
0.222222	0.100000	1.7246420458e+00	1.72464204585e+00	4.386713e-11
2.000000	0.100000	-6.5743101013e-16	-6.57431010131e-16	1.259515e-27

Example 4.2. *We consider the one-dimensional diffusion equation*

$$\frac{\partial \zeta(x,t)}{\partial t} = \frac{\partial^2 \zeta(x,t)}{\partial x^2}, \qquad 0 < x < 1,\ t > 0,$$

with Neumann boundary conditions (insulated ends)

$$\frac{\partial \zeta}{\partial x}(0,t) = 0, \qquad \frac{\partial \zeta}{\partial x}(1,t) = 0,$$

and the initial condition

$$\zeta(x,0) = 2\cos(\pi x),$$

with an exact solution

$$\zeta(x,t) = 2\cos(\pi x) e^{-\pi^2 t}.$$

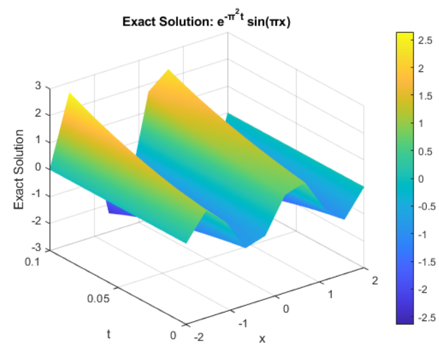

FIGURE 1. Relation between the exact solution and error of example (1).

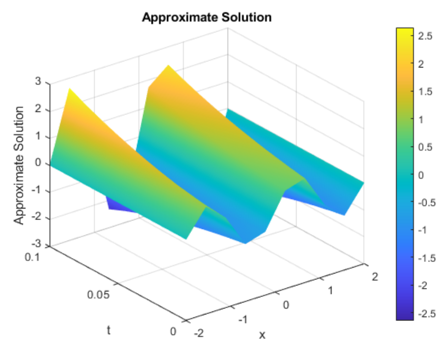

FIGURE 2. Relation between a Hermite (approximate) solution and error of example (1).

The computational work is performed by Matlab R2024.

Error Metrics:

- Mean Absolute Error (MAE): 5.40424×10^{-9}.
- Root Mean Squared Error (RMSE): 7.35135×10^{-5}.

TABLE 2. Comparison between the exact solution and a Hermite (Approximate) solution of Example (2).

x	t	Exact Solution	Hermite (Approximate) Solution	Absolute Error
-2.000000	0.000000	2.00000000000	2.00000000000	0.000000
0.222222	0.000000	1.532088886238	1.532088886241	3.274270e-12
2.000000	0.000000	2.00000000000	2.000000000000	0.000000e+00
-2.000000	0.011111	2.231802269549	2.231802269546	2.832401e-12
0.222222	0.011111	1.709659726728	1.709659726715	1.388378e-11
2.000000	0.011111	2.231802269549	2.231802269523	2.600231e-11
-2.000000	0.022222	2.490470685182	2.490470685137	4.581047e-11
0.222222	0.022222	1.90781129135	1.907811229187	5.280687e-11
2.000000	0.022222	2.490470685182	2.490470685135	4.686074e-11
-2.000000	0.033333	2.779119063718	2.779119063794	7.633139e-11
0.222222	0.033333	2.128928715527	2.128928715555	2.776135e-11
2.000000	0.033333	2.779119063718	2.779119063764	4.634160e-11
-2.000000	0.044444	3.101222116876	3.101222116844	3.263345e-11
0.222222	0.044444	2.375673969511	2.375673969512	1.588507e-12
2.000000	0.044444	3.101222116876	3.101222116880	3.286260e-12
-2.000000	0.055556	3.460657279410	3.460657279436	2.612310e-11
0.222222	0.055556	2.651017278431	2.651017278441	9.829915e-12
2.000000	0.055556	3.460657279410	3.460657279421	1.068301e-11
-2.000000	0.066667	3.861751385160	3.861751385176	1.579137e-11
0.222222	0.066667	2.958273189309	2.958273189369	6.024514e-11
2.000000	0.066667	3.861751385160	3.861751385146	1.344880e-11
-2.000000	0.077778	4.3093332752917	4.3093332752996	7.900702e-11
0.222222	0.077778	3.301140408923	3.301140408996	7.364775e-11
2.000000	0.077778	4.309332752917	4.309332752975	5.749712e-11
-2.000000	0.088889	4.808789309101	4.808789309198	9.617640e-11
0.222222	0.088889	3.683746328367	3.683746328648	2.804978e-10
2.000000	0.088889	4.808789309101	4.808789309165	6.332712e-11
-2.000000	0.100000	5.366133446918	5.366133446934	1.601563e-11
0.222222	0.100000	4.110696708047	4.11069708063	1.665157e-11
2.000000	0.100000	5.366133446918	5.366133446964	4.601564e-11

5. Conclusion

In this article, a numerical technique based on the Hermite wavelet transform method is employed to solve the one-dimensional diffusion equation. The proposed method is tested on selected problems, and the obtained results are compared with the exact solution in both tabular and graphical form. The Hermite wavelet transform method offers several advantages, including ease of implementation and suitability for problems defined on finite domains.

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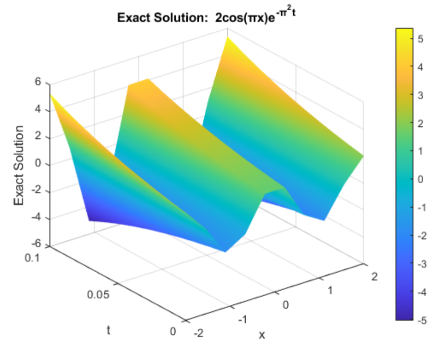

FIGURE 3. Relation between the exact solution and error of example (2).

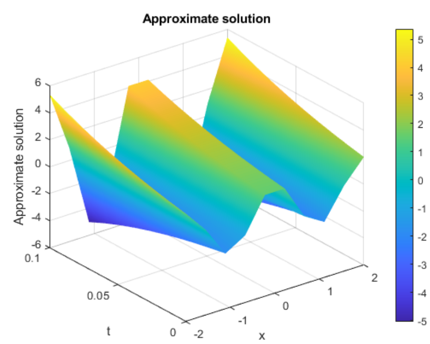

FIGURE 4. Relation between a Hermite (approximate) solution and error of example (2).

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