Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 18 Issue 1(2026), Pages 74-86
https://doi.org/10.54671/BMAA-2026-1-5

THEOREMS IN RECTANGULAR COMPLEX-VALUED METRIC SPACES FOR FOUR MAPPINGS

FAISAL MAHMOOD, MUMTAZ ALI, ABDULLAH SHOAIB

Abstract. In this article, some theorems related to the common fixed point of four mappings in the setting of rectangular complex-valued metric spaces are proved. The main concern is the extension, generalization and improvement of some fixed point theorems in the present literature on metric fixed point theory for weakly compatible mappings.

1. Introduction

A number of authors generalized the classical Banach [7] contraction mapping theorem to obtain fixed point results in terms of various metric spaces, see [14, 19, 20, 21, 22, 26, 29, 30]. Azam et al. [4] generalized Banach contraction mapping theorem by introducing the notion of complex-valued metric and mentioned some important points for existence of common fixed points of a pair of mappings satisfying a contractive condition. Their research is further extended in different ways see, [3, 5, 8, 9, 10, 12, 17, 27, 28]. Gerald Jungck [15] introduced the idea of compatible mappings which is the generalization of the commuting mappings [32]. Their fixed point results are generalized by many authors, see [1, 2, 24, 25]. Recently, Sintunavarat et al. [33] proved a common fixed theorem for weakly compatible mappings in complex-valued metric spaces. Also for more concepts in this direction, we refer to [10, 11, 13, 18, 23] and references therein.

Recently, Azmi et al. [6] established some concepts on complex-valued controlled rectangular metric spaces, Hossaini et al. [16] introduced some results which are related to complex-valued like metric spaces and Singh et al. [31] found a few common fixed point results for two pairs of weakly compatible mapping with rational type terms in complex-valued rectangular (generalized) metric spaces that satisfy contractive condition.

This paper's objective is to continue with the concept of weakly compatible mappings and to generalize results in the available literature, especially the result of Azam et al. [4]. We demonstrate a few common fixed point theorems in the context of rectangular complex-valued metric spaces for four weakly compatible

2000 *Mathematics Subject Classification.* 54H25, 47H10.

Key words and phrases. Compatible mapping; weakly compatible mapping; common fixed point; rectangular complex-valued metric space.

Submitted August 24, 2025. Accepted February 3, 2026. Published March 2, 2026.

Communicated by N. Hussain.

mappings without using the idea of continuity. We recall some definitions that will be needed in our subsequent discussion.

2. Preliminaries

Define a partial order relation $\preceq$ on $\mathbb{C}$, here $\mathbb{C}$ is a set of complex number, then we say that $(z_1, z_2) \in\preceq$ if and only if real part of z_1 is less or equal to the real part of z_2 and imaginary part of z_1 is less or equal to the imaginary part of z_2. We denote $(z_1, z_2) \in\preceq$ by $z_1 \preceq z_2$.

Definition 2.1. [4] Let $d : X \times X \to \mathbb{C}$, X be a non-empty and $\preceq$ be the above mentioned partial order relation on $\mathbb{C}$. If the following conditions hold:

(i) $0 \preceq d(x_1, y_1)\ \forall\ x_1, y_1 \in X$ and $d(x_1, y_1) = 0$ if and only if $x_1 = y_1$;

(ii) $d(x_1, y_1) = d(y_1, x_1)\ \forall\ x_1, y_1 \in X$;

(iii) $d(x_1, y_1) \preceq d(x_1, z_1) + d(z_1, y_1)\ \forall\ x_1, y_1, z_1 \in X$.

Then d is known as a complex-valued metric on X and the triplet $(X, \preceq, d)$ is known as a complex-valued metric spaces.

Example 2.2. Let $Y = \mathbb{C}$ and $\mathbb{C}$ be a nonempty set and it is a set of complex numbers. Define $d : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ by

$$d(z_1, z_2) = |a_1 - a_2| + i|b_1 - b_2|.$$

Here $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$. The $(\mathbb{C}, d)$ is a complex-valued metric space.

Definition 2.3. [31] Let $d : X \times X \to \mathbb{C}$, X be a nonempty set and $\preceq$ be the above mentioned partial order relation on $\mathbb{C}$. If,

(i) $0 \preceq d(x_1, y_1)\ \forall\ x_1, y_1 \in X$ and $d(x_1, y_1) = 0$ if and only if $x_1 = y_1$;

(ii) $d(x_1, y_1) = d(y_1, x_1)\ \forall\ x_1, y_1 \in X$;

(iii) $d(x_1, y_1) \preceq d(x_1, z_1) + d(z_1, w_1) + d(w_1, y_1)\ \forall\ x_1, y_1 \in X$ and all distinct points $z_1, w_1 \in X - \{x_1, y_1\}$.

Then d is known as a complex-valued rectangular metric spaces on X and the triplet $(X, \preceq, d)$ is known as a rectangular complex-valued metric spaces.

Example 2.4. Let $Y = \{2 + i, -2 + i, -2 - i, 2 - i\}$. Define $d : Y \times Y \to \mathbb{C}$ as

$$\begin{aligned}
d(2+i, -2+i) &= d(-2+i, 2+i) = 3e^{i\theta};\\
d(2+i, 2+i) &= d(-2+i, -2+i) = d(-2-i, -2-i) = d(2-i, 2-i) = 0;\\
d(-2+i, -2-i) &= d(-2-i, -2+i) = d(2+i, -2-i) = d(-2-i, 2+i) = e^{i\theta};\\
d(2+i, 2-i) &= d(2-i, 2+i) = d(-2+i, 2-i) = d(2-i, -2+i)\\
&= d(-2-i, 2-i) = d(2-i, -2-i) = 4e^{i\theta}.
\end{aligned}$$

Then d is known as rectangular complex-valued metric on Y and (Y, d) is known as rectangular complex-valued metric spaces. Note that $\theta \in [0, \pi/2]$, otherwise it is not rectangular complex-valued metric spaces.

Definition 2.5. [4] A sequence in $\{w_r\}$ is called convergent to some w in a complex-valued metric spaces $(X, \preceq, d)$ if for arbitrary $z \in \mathbb{C}$ such that $0 \preceq z$ and $0 \neq z$ implies the existence of a n, where n be a natural number, such that $d(w_r, w) \preceq z$ for all $r \geq n$ or $|d(w_r, w)| \to 0$ as $r \to \infty$. $\{w_r\}$ is said to be a Cauchy sequence if for arbitrary $z \in \mathbb{C}$ such that $0 \preceq z$ and $0 \neq z$ implies the existence of a n such that $d(w_r, w_e) \preceq z$ for all r, $e \geq n$ or $|d(w_r, w_e)| \to 0$ as $r, e \to \infty$. $(X, \preceq, d)$ is complete if each Cauchy sequence is convergent to some w in X.

Definition 2.6. [15] Let $M, V : A \to A$. Then $u \in A$ is a

(i) fixed point of M if $Mu = u$.

(ii) coincidence point of M and V if $Mu = Vu$.

(iii) common fixed point of M and V if $Mu = Vu = u$.

Definition 2.7. [11] Consider $(X, \preceq, d)$ be a complex-valued metric spaces. Two self mappings h and k are said to be weakly compatible if $hkx = khx$ whenever $hx = kx$, that is, at coincidence points they commute.

Definition 2.8. Let A_1, B_1 and $S, T : Y \to Y$ be such that $A_1(Y) \subseteq T(Y)$ and $B_1(Y) \subseteq S(Y)$. Suppose for any $x_0 \in Y$, $x_1 \in Y$ is chosen so that $Tx_1 = A_1 x_0$ and for $x_1 \in Y$, x_2 is chosen such that $Sx_2 = B_1 x_1$. Continuing in this way, a sequence $\{y_n\}$ in Y is defined by the relations

$$\begin{cases} y_{2n} = A_1 x_{2n} = Tx_{2n+1} & \text{and} \\ y_{2n+1} = B_1 x_{2n+1} = Sx_{2n+2} \ \forall \ n \geq 0, \end{cases}$$

and is called $\{A_1 T B_1 S\}$ sequence of x_0 in Y.

3. Common Fixed Point Results

Here to start our main findings we require the following lemma which is an extension of the lemma proved in [4].

Lemma 3.1. Let $(W, \preceq, d)$ be a complete complex-valued rectangular metric spaces and let A, B, S and $T : W \to W$ satisfying $A(W) \subseteq T(W)$ and $B(W) \subseteq S(W)$. Suppose $\{w_n\}$ is $\{ATBS\}$ sequence of x_0 in W. If there exists $\eta \in (0,1)$ such that

$$d(w_n, w_{n+1}) \ \preceq \ \eta d(w_{n-1}, w_n) \ \text{ for all } n \in \mathbb{N}, \tag{3.1}$$

then the pairs $\{A, S\}$ and $\{B, T\}$ have coincidence points and the sequence $\{w_n\}$ in W is a Cauchy sequence. Further, if W is complete implies that the sequence $\{w_n\}$ converges to a point $u \in W$.

Proof Suppose there exists an integer n which is positive such that $w_n = w_{n+1}$ or $w_{2n-1} = w_{2n}$. Then by Definition 2.8, we have $Bx_{2n-1} = Sx_{2n} = Ax_{2n} = Tx_{2n+1}$. So the pair of mappings $\{A, S\}$ have coincidence point x_{2n} and by (3.1), we have

$$|d(w_{2n}, w_{2n+1})| \leq \eta \, |d(w_{2n-1}, w_{2n})| = 0.$$

Hence, $w_{2n} = w_{2n+1}$ implies $Ax_{2n} = Tx_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$ and the pair $\{B, T\}$ have coincidence point x_{2n+1}. Continuous repetition of process in (3.1) yields us

$$w_{2n-1} = w_{2n} = w_{2n+1} = \ldots = u \text{ (say)},$$

and so $\{w_n\}$ converges in W. Next, suppose $w_n \neq w_{n+1}$ for all $n \in \mathbb{N}$. Then by (3.1) for all $n \in \mathbb{N}$

$$d(w_n, w_{n+1}) \ \preceq \ \eta^n d(w_0, w_1).$$

This implies

$$|d(w_n, w_{n+1})| \leq \ \eta^n \, |d(w_0, w_1)| \ \text{ as } n, m \to \infty. \tag{3.2}$$

Consider $n, m \in \mathbb{N}$ having $m > n$. Now, if $m = n + 1$, then inequality (3.2) implies

$$\lim_{n,m\to\infty} |d(w_n, w_m)| \leq \ \lim_{n\to\infty} \eta^n \, |d(w_0, w_1)| = 0.$$

Hence, $\{w_n\}$ is a Cauchy sequence in this case. Now, if $m > n+2$, then

$$\begin{aligned}
d(w_n, w_m) &\preceq [d(w_n, w_{n+1}) + d(w_{n+1}, w_{n+2}) + d(w_{n+2}, w_m)] \\
&\preceq d(w_n, w_{n+1}) + d(w_{n+1}, w_{n+2}) + d(w_{n+2}, w_{n+3}) \\
&\quad + d(w_{n+3}, w_{n+4}) + d(w_{n+4}, w_m) \\
&\preceq \eta^n d(w_0, w_1) + \eta^{n+1} d(w_0, w_1) + \eta^{n+2} d(w_0, w_1) + \eta^{n+3} d(w_0, w_1) \\
&\quad + \eta^{n+4} d(w_0, w_1) + ... \\
&\preceq \eta^{n-1}\eta d(w_0, w_1) + \eta^{n-1}\eta^2 d(w_0, w_1) + \eta^{n-1}\eta^3 d(w_0, w_1) + \eta^{n-1}\eta^4 d(w_0, w_1) \\
&\quad + \eta^{n-1}\eta^5 d(w_0, w_1) + ... \\
&\preceq \eta^{n-1} d(w_0, w_1)(\eta + \eta^2 + \eta^3 + \eta^4 + ...) \\
&\preceq \eta^{n-1} d(w_0, w_1)[\frac{\eta}{1-\eta}]
\end{aligned}$$

Taking $n, m \to \infty$, we obtain

$$\lim_{r,m\to\infty} |d(w_n, w_m)| = 0.$$

Hence, $\{w_n\}$ is a Cauchy sequence in this case. Now, if $m = n+2$, then, we get

$$\begin{aligned}
d(w_n, w_{n+2}) &\preceq d(w_n, w_{n+1}) + d(w_{n+1}, w_{n+3}) + d(w_{n+3}, w_{n+2}) \\
&\preceq d(w_n, w_{n+1}) + [d(w_{n+1}, w_{n+2}) + d(w_{n+2}, w_{n+4}) + d(w_{n+4}, w_{n+3})] + d(w_{n+3}, w_{n+2}) \\
&\preceq d(w_n, w_{n+1}) + d(w_{n+1}, w_{n+2}) + [d(w_{n+2}, w_{n+3}) \\
&\quad + d(w_{n+3}, w_{n+5}) + d(w_{n+5}, w_{n+4})] + d(w_{n+4}, w_{n+3}) + d(w_{n+3}, w_{n+2}) \\
&\preceq d(w_n, w_{n+1}) + d(w_{n+1}, w_{n+2}) + d(w_{n+2}, w_{n+3}) \\
&\quad + [d(w_{n+3}, w_{n+4}) + d(w_{n+4}, w_{n+6}) + d(w_{n+6}, w_{n+5})] \\
&\quad + d(w_{n+5}, w_{n+4}) + d(w_{n+4}, w_{n+3}) + d(w_{n+3}, w_{n+2}) \\
&\preceq d(w_n, w_{n+1}) + d(w_{n+1}, w_{n+2}) + d(w_{n+2}, w_{n+3}) + d(w_{n+3}, w_{n+4}) \\
&\quad + [d(w_{n+4}, w_{n+5}) + d(w_{n+5}, w_{n+7}) + d(w_{n+7}, w_{n+6})] \\
&\quad + d(w_{n+6}, w_{n+5}) + d(w_{n+5}, w_{n+4}) + d(w_{n+4}, w_{n+3}) + d(w_{n+3}, w_{n+2}) \\
&\preceq [d(w_n, w_{n+1}) + d(w_{n+1}, w_{n+2}) + d(w_{n+2}, w_{n+3}) + d(w_{n+3}, w_{n+4}) + ...] \\
&\quad + [d(w_{n+3}, w_{n+2}) + d(w_{n+4}, w_{n+3}) + d(w_{n+5}, w_{n+4}) + d(w_{n+6}, w_{n+5}) + ...].
\end{aligned}$$

Now, we have

$$\begin{aligned}
d(w_n, w_{n+2}) &\preceq [\eta^n d(w_0, w_1) + \eta^{n+1} d(w_0, w_1) + \eta^{n+2} d(w_0, w_1) + ...] \\
&\quad + [\eta^{n+2} d(w_0, w_1) + \eta^{n+3} d(w_0, w_1) + \eta^{n+4} d(w_0, w_1) + ...] \\
&\preceq [\eta^{n-1}\eta d(w_0, w_1) + \eta^{n-1}\eta^2 d(w_0, w_1) + \eta^{n-1}\eta^3 d(w_0, w_1) + ...] \\
&\quad + [\eta^{n+1}\eta d(w_0, w_1) + \eta^{n+1}\eta^2 d(w_0, w_1) + \eta^{n+1}\eta^3 d(w_0, w_1) + ...] \\
&\preceq \eta^{n-1} d(w_0, w_1)[\eta + \eta^2 + \eta^3 + \eta^4 + ...] \\
&\quad + \eta^{n+1} d(w_0, w_1)[\eta + \eta^2 + \eta^3 + \eta^4 + ...] \\
&\preceq \eta^{n-1} d(w_0, w_1)[\frac{\eta}{1-\eta}] + \eta^{n+1} d(w_0, w_1)[\frac{\eta}{1-\eta}].
\end{aligned}$$

Letting $n, m \to \infty$, we obtain $\lim_{n,m\to\infty} |d(w_n, w_m)| = 0$. Hence, $\{w_n\}$ is a Cauchy sequence in all cases. Now, as W is complete, there exists $u \in W$, such that $w_n \to u$. Hence proof of the lemma is completed.

Theorem 3.2. Consider $(X, \preceq, d)$ be a complete rectangular complex-valued metric spaces and A, B, S and $T : X \to X$, if

(i) $A(X) \subseteq T(X), B(X) \subseteq S(X)$,

(ii) one of $A(X)$, $B(X)$, $S(X)$ or $T(X)$ is a complete subspaces of X and

(iii) for all $x, y \in X$,

$$d(Ax, By) \preceq \lambda (ABST)_{(x,y)}, \tag{3.3}$$

where $\lambda \in (0,1)$ and

$$(ABST)_{(x,y)} \in \left\{ d(Sx, Ty), d(Ax, Sx), d(By, Ty), \tfrac{d(Ax,Ty)+d(By,Sx)}{2} \right\}. \tag{3.4}$$

Then the mappings A, B, S and T have a point of coincidence which is unique. Furthermore, if $\{A, S\}$ and $\{B, T\}$ are weakly compatible, then mappings A, B, S and T have a unique common fixed point in X.

Proof Let $x_0 \in X$ be arbitrary. Since $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$, we take sequences $\{x_n\}$ and $\{y_n\}$ in X as follows

$$y_{2n+1} = Ax_{2n} = Tx_{2n+1} \text{ and } y_{2n+2} = Bx_{2n+1} = Sx_{2n+2} \ \forall \ n \geq 0,$$

By (3.3), we obtain

$$d(y_{2n+1}, y_{2n+2}) = d(Ax_{2n}, Bx_{2n+1}) \preceq \lambda (ABST)_{(x_{2n}, x_{2n+1})},$$

where

$$\begin{aligned} &(ABST)_{(x_{2n},x_{2n+1})} \in \\ &\left\{ \begin{array}{c} d(Sx_{2n}, Tx_{2n+1}), d(Ax_{2n}, Sx_{2n}), d(Bx_{2n+1}, Tx_{2n+1}), \\ \frac{d(Ax_{2n},Tx_{2n+1})+d(Bx_{2n+1},Sx_{2n})}{2} \end{array} \right\} \\ &= \left\{ \begin{array}{c} d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n}), d(y_{2n+2}, y_{2n+1}), \\ \frac{d(y_{2n+1},y_{2n+1})+d(y_{2n+2},y_{2n})}{2} \end{array} \right\}. \end{aligned} \tag{3.5}$$

From (3.5), we have the following possibilities.

(i) $(ABST)_{(x_{2n},x_{2n+1})} = d(y_{2n}, y_{2n+1})$,

(ii) $(ABST)_{(x_{2n},x_{2n+1})} = d(y_{2n+1}, y_{2n+2})$,

(iii) $(ABST)_{(x_{2n},x_{2n+1})} = \frac{d(y_{2n+2},y_{2n})}{2}$.

Now, if we have (i), then

$$d(y_{2n+1}, y_{2n+2}) \preceq \lambda d(y_{2n}, y_{2n+1}).$$

Now, if we have (ii), then

$$d(y_{2n+1}, y_{2n+2}) \preceq \lambda d(y_{2n+1}, y_{2n+2}),$$

which gives us $y_{2n+1} = y_{2n+2}$. Now, if we have (iii), then

$$d(y_{2n+1}, y_{2n+2}) \preceq \frac{\lambda}{2} d(y_{2n}, y_{2n+1}) + \frac{\lambda}{2} d(y_{2n+1}, y_{2n+2}).$$

From the last inequality, we get

$$d(y_{2n+1}, y_{2n+2}) \preceq \lambda d(y_{2n}, y_{2n+1}).$$

Similarly,

$$d(y_{2n}, y_{2n+1}) \preceq \lambda d(y_{2n-1}, y_{2n}).$$

Now, for all $n \in \mathbb{N}$, we can write

$$d(y_{n+1}, y_{n+2}) \preceq \lambda d(y_n, y_{n+1}).$$

by Lemma 3.1, we have $y \in X$ such that $\{y_n\} \to y$. Furthermore, the subsequences $\{Sx_{2n+2}\} = \{Bx_{2n+1}\} = \{y_{2n+2}\}$ and $\{Tx_{2n+1}\} = \{Ax_{2n}\} = \{y_{2n+1}\}$ of $\{y_n\}$ also converges to the same point y. Consider $T(X)$ is a complete subspaces of X, there exist $u \in X$ such that $Tu = y$. If $Bu \neq y$, so by the conditions (3.3) and (3.4), we get

$$d(Ax_{2n}, Bu) \preceq \lambda (ABST)_{(x_{2n},u)}, \tag{3.6}$$

where

$$(ABST)_{(x_{2n},u)} \in \left\{ \begin{array}{c} d(Sx_{2n}, Tu), d(Ax_{2n}, Sx_{2n}), d(Bu, Tu), \\ \frac{d(Ax_{2n},Tu)+d(Bu,Sx_{2n})}{2} \end{array} \right\}$$

Taking limit as $n \to \infty$, yields

$$\begin{aligned} \lim_{n\to\infty} (ABST)_{(x_{2n},u)} &\in \left\{ d(y,y), d(y,y), d(Bu,y), \frac{d(y,y)+d(Bu,y)}{2} \right\} \\ &= \left\{ 0, 0, d(Bu,y), \frac{d(Bu,y)}{2} \right\}. \end{aligned}$$

Now, we have the following possibilities.

(*iv*) $\lim_{n\to\infty} (ABST)_{(x_{2n},u)} = 0,$

(*v*) $\lim_{n\to\infty} (ABST)_{(x_{2n},u)} = d(Bu,y),$

(*vi*) $\lim_{n\to\infty} (ABST)_{(x_{2n},u)} = \frac{d(Bu,y)}{2}.$

By taking limit as $n \to \infty$ in (3.6) gives us $d(y,Bu) \preceq 0$ or $d(y,Bu) \preceq \lambda d(Bu,y)$ or $d(y,Bu) \preceq \frac{1}{2}\lambda d(Bu,y)$ and these all implies that $d(y,Bu) = 0$ and so $y = Bu$. This implies that $Tu = y = Bu$. Since, the pair of self mappings $\{B,T\}$ is weakly compatible, we have $BTu = TBu \Rightarrow By = Ty$. If $y \neq By$, then by (3.4), we get

$$d(Ax_{2n}, By) \preceq \lambda (ABST)_{(x_{2n},y)}, \tag{3.7}$$

where

$$(ABST)_{(x_{2n},y)} \in \left\{ \begin{array}{c} d(Tx_{2n}, Ty), d(Ax_{2n}, Tx_{2n}), d(By, Ty), \\ \frac{d(Ax_{2n},Ty)+d(By,Tx_{2n})}{2} \end{array} \right\},$$

By taking limit as $n \to \infty$, we get

$$\lim_{n\to\infty} (ABST)_{(x_{2n},y)} \in \left\{ d(y,Ty), d(y,y), d(By,Ty), \frac{d(y,Ty)+d(By,y)}{2} \right\},$$

By using $Ty = By$, we get

$$\lim_{n\to\infty} (ABST)_{(x_{2n},y)} \in \{ d(y,By), 0, 0, d(y,By) \},$$

Now, (3.7) implies after applying limit as $n \to \infty$

$$d(y,By) \preceq \lambda d(y,By) \text{ or } d(y,By) \preceq 0,$$

In either case, we have $d(y,By) = 0$ or $By = y$. Therefore, $By = Ty = y$. Since, $B(X) \subseteq S(X)$, there exists $v \in X$ such that $Sv = y$. If $Av \neq y$, then by using

(3.3) and (3.4) we get

$$\begin{aligned} d(Av,By) &\preceq \lambda (ABST)_{(v,y)}, \\ (ABST)_{(v,y)} &\in \left\{ \begin{array}{c} d(Sv,Ty), d(Av,Sv), d(By,Ty), \\ \frac{d(Av,Ty)+d(By,Sv)}{2} \end{array} \right\} \\ &= \left\{ d(y,y), d(Av,y), d(y,y), \frac{d(Av,y)+d(y,y)}{2} \right\} \\ &= \left\{ 0, d(Av,y), 0, \frac{d(Av,y)}{2} \right\} \end{aligned}$$

It follows the possibilities.

(i) $d(Av,y) \preceq 0$.

(ii) $d(Av,y) \preceq \lambda d(Av,y)$ or (iii) $d(Av,y) \preceq \frac{\lambda}{2} d(Av,y)$. Either of the above cases imply that

$$d(Av,y) = 0 \text{ or } y = Av.$$

As A and S which are weakly compatible, we obtain $ASv = SAv$ or $Ay = Sy$. Lastly, if $Ay \neq y$, then (3.3) and (3.4), gives us

$$\begin{aligned} d(Ay,y) = d(Ay,By) &\preceq \lambda (ABST)_{(y,y)}, \\ (ABST)_{(y,y)} &\in \left\{ d(Sy,Ty), d(Ay,Sy), d(By,Ty), \frac{d(Ay,Ty)+d(By,Sy)}{2} \right\}, \\ &= \left\{ d(Sy,y), d(Ay,Sy), d(y,y), \frac{d(Ay,y)+d(y,Sy)}{2} \right\}, \\ &= \left\{ d(Ay,y), d(Ay,Ay), d(y,y), \frac{d(Ay,y)+d(y,Ay)}{2} \right\}, \\ &= \{ d(Ay,y), 0, 0, d(y,Ay) \}. \end{aligned}$$

It follows that $d(Ay,y) \preceq \lambda d(Ay,y)$ or $d(Ay,y) \preceq 0$. Hence, $d(Ay,y) = 0$ and $Ay = y$. Thus, $Ay = By = Sy = Ty = y$. So, y is common fixed point of A, B,S and T

Uniqueness: For uniqueness, suppose $y_1.y_2$ are two common fixed points of A, B, S and T such that $y_1 \neq y_2$.We establish that $y_1 = y_2$. By $x = y_1$ and $y = y_2$ in (3.3) and (3.4) ,

$$\begin{aligned} d(y_1,y_2) &= d(Ay_1,By_2) \preceq \lambda (ABST)_{(y_1,y_2)}, \\ (ABST)_{(y_1,y_2)} &\in \left\{ \begin{array}{c} d(Sy_1,Ty_2), d(Ay_1,Sy_1), d(By_2,Ty_2), \\ \frac{d(Ay_1,Ty_2)+d(By_2,Sy_1)}{2} \end{array} \right\} \\ &= \{ d(y_1,y_2), d(y_1,y_1), d(y_2,y_2), d(y_1,y_2) \}. \end{aligned}$$

It follows that $d(y_1,y_2) \preceq \lambda d(y_1,y_2)$ or $d(y_1,y_2) \preceq 0$ which is a contradiction, so our supposition is wrong and $y_1 = y_2$.

Corollary 3.3. Let $(X, \preceq, d)$ be a complex-valued rectangular metric spaces and let self mappings A, B, S and $T : X \to X$ satisfying the conditions:

(i) $A(X) \subseteq T(X), B(X) \subseteq S(X)$ and for some $m, n \geq 1$,

(ii) One of $A(X)$, $B(X)$, $S(X)$ or $T(X)$ is a complete subspaces of X,

(iii) For every $x, y \in X$.

$$d(A^m x, B^n y) \preceq \lambda (A^m B^n S^m T^n)_{(x,y)},$$

where $\lambda \in (0,1)$ and

$$(A^m B^n S^m T^n)_{(x,y)} \in \left\{ \begin{array}{c} d(S^m x, T^n y), d(A^m x, S^m x), d(B^n y, T^n y), \\ \frac{d(A^m x, T^n y) + d(B^n y, S^m x)}{2} \end{array} \right\},$$

Then the point of coincidence of mappings A, B, S and T will be unique. Furthermore, if $\{A^m, S^m\}$ and $\{B^n, T^n\}$ are weakly compatible, then mappings A, B, S and T have a unique common fixed point in X..

Corollary 3.4. Consider $(X, \preceq, d)$ be a rectangular complex-valued metric spaces and let self mappings $A, B, S, T : X \to X$ such that $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$ and satisfy the inequality

$$d(Ax, By) \preceq \lambda d(Sx, Ty),$$

for $x, y \in X$ and $\lambda \in (0,1)$ is a nonnegative real number. If one of $A(X)$, $B(X)$, $S(X)$ or $T(X)$ is a complete subspaces of X and the pairs $\{A, S\}$and $\{B, T\}$ are weakly compatible. Then mappings A, B, S and T have a unique common fixed point in X.

Example 3.5. Consider $X = [0,2]$ with rectangular complex-valued metric given by $d : X : X \to C$ and is defined by $d(x,y) = i|x-y|$ for all $x, y \in X$. Then the self mappings A, B, S and T on X are given by

$$A(x) = \left\{ \begin{array}{ll} \frac{4}{3} & \text{if } 0 \le x < 2 \\ 2 & \text{if } x = 2 \end{array} \right., \quad B(x) = \left\{ \begin{array}{l} \frac{4}{3} \text{ if } 0 \le x \le \frac{4}{3} \\ 2 \text{ if } \frac{4}{3} < x \le 2 \end{array} \right.$$

$$S(x) = \left\{ \begin{array}{ll} 0 & \text{if } 0 \le x < \frac{2}{3} \\ 1 & \text{if } x = 1 \\ \frac{4}{3} & \text{if } 1 < x \le \frac{4}{3} \\ 2 & \text{if } \frac{4}{3} < x \le 2 \end{array} \right., \quad T(x) = \left\{ \begin{array}{ll} \frac{4}{3} & \text{if } 0 < x \le \frac{4}{3} \\ 2 & \text{if } x = 2 \\ \frac{9}{5} & \text{if } 2 < x \le \frac{4}{3} \\ 0 & \text{if } \frac{4}{3} < x \le 2 \end{array} \right.$$

Obviously, $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$. Further, the pairs $\{A, S\}$ and $\{B, T\}$ are weakly compatible and satisfy conditions (3.3) and (3.4) of Theorem 3.2 with $\frac{4}{3}$ as the unique common fixed point of A, B, S and T in X.

Theorem 3.6. Let $(X, \preceq, d)$ be a rectangular complex-valued metric spaces and let self mappings $A, B, S, T : X \to X$ of X into itself such that

(i) $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$,

(ii) one of $A(X), B(X), S(X)$ or $T(X)$ is a complete subspaces of X,

(v) the pairs $\{A, S\}$ and $\{B, T\}$ are weakly compatible and for all x, $y \in X$ satisfy the inequality

$$d(Ax, By) \preceq \alpha_1 d(Sx, Ty) + \alpha_2 d(Ax, Sx) + \alpha_3 d(By, Ty) + \alpha_4(d(Ax, Ty) + d(By, Sx)), \tag{3.8}$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in (0,1)$ and satisfy $\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 < 1$, then A, B, S and T have a unique common fixed point in X.

Proof Let $x_0 \in X$ be arbitrary. We construct sequences $\{x_n\}$ and $\{y_n\}$of points in X like in the proof of Theorem 3.2, we have

$$\begin{array}{c} d(y_{2n+1}, y_{2n+2}) = d(Ax_{2n}, Bx_{2n+1}) \\ \preceq \alpha_1 d(Sx_{2n}, Tx_{2n+1}) + \alpha_2 d(Ax_{2n}, Sx_{2n}) + \alpha_3 d(Bx_{2n+1}, Tx_{2n+1}) + \\ \alpha_4 (d(Ax_{2n}, Tx_{2n+1}) + d(Bx_{2n+1}, Sx_{2n})), \end{array}$$

and is equivalent to

$$\begin{aligned} d(y_{2n+1}, y_{2n+2}) &\preceq \alpha_1 d(y_{2n}, y_{2n+1}) + \alpha_2 d(y_{2n+1}, y_{2n}) + \\ &\alpha_3 d(y_{2n+2}, y_{2n+1}) + \alpha_4 (d(y_{2n+1}, y_{2n+1}) + d(y_{2n}, y_{2n+2})) \\ &\preceq \alpha_1 d(y_{2n}, y_{2n+1}) + \alpha_2 d(y_{2n}, y_{2n+1}) + \alpha_3 d(y_{2n+1}, y_{2n+2}) + \\ &\alpha_4 d(y_{2n}, y_{2n+1}) + \alpha_4 d(y_{2n+2}, y_{2n+1}) \\ (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) d(y_{2n+1}, y_{2n+2}) &\preceq \alpha_4 d(y_{2n}, y_{2n+1}), \end{aligned}$$

$$d(y_{2n+1}, y_{2n+2}) \preceq \frac{\alpha_4}{[1-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)]} d(y_{2n}, y_{2n+1}) \ \forall n = 1, 2, 3, ...$$

or

$$d(y_{2n+1}, y_{2n+2}) \preceq \eta d(y_{2n}, y_{2n+1}),$$

where $\eta = \frac{\alpha_4}{[1-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)]} < 1$. It follows from Lemma 3.1 that $\{y_n\}$ is a Cauchy sequence in X and the proof lines of the theorem are the same as Theorem 3.2. The proof of the theorem is completed.

The following example illustrate the validity of Theorem 3.6.

Example: 3.7. Let $X = [3, 18)$ and $d : X \times X \to \mathbb{C}$ defined by $d(x, y) = i|x - y|$ for every $x, y \in X$ and define the self mappings $A, B, S, T : X \to X$ by

$$A(x) = \begin{cases} 3 \text{ if } x = 3, x \geq 7 \\ 7 \text{ if } 3 < x < 7 \end{cases}, \qquad B(x) = \begin{cases} 3 \text{ if } x = 3, x \geq 7 \\ 8 \quad \text{if } 2 < x < 7 \end{cases}$$

$$S(x) = \begin{cases} 3 \text{ if } \quad x = 3 \\ 11 \text{ if } \ 3 < x < 7 \\ \frac{x-1}{2} \text{ if } \quad x \geq 7 \end{cases}, \qquad T(x) = \begin{cases} 3 \quad \text{if} \quad x = 3 \\ 15 \text{ if } \ 3 < x < 7 \\ x - 6, \quad x \geq 7 \end{cases}$$

Now, we see that the pairs $\{A, S\}$ and $\{B, T\}$ are weakly compatible. Since, $A3 = S3 = 3$ and $B3 = T3 = 3$. Hence $\{A, S\}$ and $\{B, T\}$ have coincidence point in X. Also, $AS3 = SA3 = 3$ and $AT3 = TA3 = 3$. Now, $A(X) = [3, 7] \subseteq [3, 11] \cup \{15\} = T(X)$, $B(X) = [3, 8] \subseteq [3, 8] \cup \{11\} = S(X)$. Thus, we see that these self mappings fulfil all the requirements of Theorem 3.6 with 3 as the unique common fixed point of A, B, S and T in X.

4. Application

This section is devoted to apply Theorem 3.2 to ensure the existence and uniqueness of the common solution to the integral equations given in (4.1), (4.2), (4.3) and (4.4). Let $Y = C([0, 1], \mathbb{C})$ be a space of all continuous complex-valued functions on $[0, 1]$. Consider the nonlinear integral equations

$$z(x) = \int_0^x K_1(x, y, z) dy, \tag{4.1}$$

$$r(x) = \int_0^x K_2(x, y, r) dy, \tag{4.2}$$

$$v(x) = \int_0^x K_3(x, y, v) dy, \tag{4.3}$$

$$w(x) = \int_0^x K_4(x, y, w) dy, \tag{4.4}$$

here $x, y \in [0,1]$, $z, r, v, w \in Y$, $K_1, K_2, K_3, K_4 : [0,1] \times [0,1] \times Y \to \mathbb{C}$ are continuous kernals. For $v, w \in Y$, define rectangular complete complex-valued metric $d : Y \times Y \to \mathbb{C}$ by

$$d(v,w) = i \sup_{x\in[0,1]} |v(x) - w(x)| \in i\mathbb{R} \subset \mathbb{C}.$$

Define $A, B, S, T : Y \to Y$ as

$$(Az)(x) = \int_0^x K_1(x,y,z)dy,$$

$$(Bz)(x) = \frac{1}{2}\int_0^x K_2(x,y,z)dy,$$

$$(Sz)(x) = \int_0^x K_3(x,y,z)dy,$$

$$(Tz)(x) = \frac{1}{2}\int_0^x K_4(x,y,z)dy.$$

Theorem 4.1. Let $x, y \in [0,1]$, $z \in Y$, $K_1, K_2, K_3, K_4 : [0,1] \times [0,1] \times Y \to \mathbb{C}$ are continuous kernals. Suppose that

$$|K_1(x,y,v) - \frac{1}{2}K_2(x,y,w)| \leq hW(v,w), \tag{4.5}$$

where, $h \in (0,1)$ and

$$\begin{aligned} &W(v,w) \\ \in\ &\left\{ \sup_{m\in[0,1]} |Sv(m) - Tw(m)|, \sup_{m\in[0,1]} |Av(m) - Sv(m)|, \sup_{m\in[0,1]} |Bw(m) - Tw(m)|, \right. \\ &\left. \frac{\sup_{m\in[0,1]} |Av(m) - Tw(m)| + \sup_{m\in[0,1]} |Bw(m) - Sv(m)|}{2} \right\}. \end{aligned}$$

If $A(Y) \subseteq T(Y)$, $B(Y) \subseteq S(Y)$, one of $A(Y)$, $B(Y)$, $S(Y)$ or $T(Y)$ is a complete subspaces of Y and $\{A, S\}$ and $\{B, T\}$ are weakly compatible, then the integral equations (4.1), (4.2), (4.3) and (4.4) have a unique common solution.

Proof: The integral equations (4.1), (4.2), (4.3) and (4.4) can be written as $Az = z$, $Br = r$, $Sv = v$ and $Tw = w$ respectively, so the common fixed point of A, B, S and T will be the common solution of (4.1), (4.2), (4.3) and (4.4). Now, we prove the conditions of Theorem 3.2 are satisfied to ensure the existance of common fixed point.

$$\begin{aligned} |Av(x) - Bw(x)| &= |\int_0^x K_1(x,y,v)dy - \frac{1}{2}\int_0^x K_2(x,y,w)dy| \\ &= |\int_0^x (K_1(x,y,v) - \frac{1}{2}K_2(x,y,w))dy| \\ &\leq \int_0^x hW(v,w)dy \\ &= hW(v,w)\,|y|_0^x = hW(v,w)x \\ \sup_{x\in[0,1]} |Av(x) - Bw(x)| &\leq hW(v,w). \end{aligned}$$

This implies

$$\begin{aligned} i \sup_{x\in[0,1]} |Av(x) - Bw(x)| &\preceq ihW(v,w) \\ d(Av,Bw) &\preceq ihW(v,w). \end{aligned}$$

Hence, we have

$$d(Av,Bw) \preceq h\,(ABST)_{(v,w)}$$

$$(ABST)_{(v,w)} \in \left\{ d\,(Sv,Tw)\,, d\,(Av,Sv)\,, d\,(Bw,Tw)\,, \frac{d\,(Av,Tw) + d\,(Bw,Sv)}{2} \right\},$$

which is contractive condition of Theorem 3.2. As $A\,(Y) \subseteq T\,(Y)\,,\ B\,(Y) \subseteq S\,(Y)$, one of $A\,(Y)\,,\ B\,(Y)\,, S\,(Y)$ or $T\,(Y)$ is a complete subspaces of Y and $\{A,S\}$ and $\{B,T\}$ are weakly compatible. So all conditions of Theorem 3.2 are satisfied. Therefore, the mappings A, B, S and T have a unique common fixed point, which is a unique common solution of the integral equations (4.1), (4.2), (4.3) and (4.4).

5. Conclusion

In this article, some common fixed point theorems for four mappings under the framework based on $ATBS$-type iterative sequence in rectangular complex-valued metric spaces have been established. The main results discuss not only the existence but also the uniqueness of common fixed point for mappings which satisfy some contractive conditions. Some detailed examples that verify the validity of results and illustrate how the conditions are satisfied in practical scenario also included. Furthermore, the results of this articles contribute to the ongoing development of fixed point theory in complex-valued metric spaces and give a foundation for future applications in areas such as integral equations and metric-type structures.

Acknowledgments. The authors would like to thank the anonymous referees for their comments that helped us improve this article.

References

[1] M. A. Ahmed, *Common fixed point theorems for weakly compatible mappings*, Rocky Mountain J. Math. **33** 4 (2003) 1189–1203.
[2] M. Arshad, S. Alshoraify, A. Shoaib, E. Ameer, *Common fixed points for generalized $\alpha - \psi$-Meir–Keeler–Khan mappings in metric spaces,* J. Anal. **29** **1** (2021) 1–14.
[3] M. S. Aslam, M. S. R. Chowdhury, L. Guran, I. Manzoor, T. Abdeljawad, D. Santina, N. Mlaiki, *Complex-valued double controlled metric like spaces with applications to fixed point theorems and Fredholm type integral equations*, AIMS Math. **8** (2023) 4944–4963.
[4] A. Azam, B. Fisher, M. Khan, *Common fixed point theorems in complex valued metric spaces*, Numer. Funct. Anal. Optim. **32** **3** (2011) 243–253.
[5] A. Azam, J. Ahmad, P. Kumam, *Common fixed point theorems for multi-valued mappings in complex-valued metric spaces*, J. Inequal. Appl., **2013**, (2013) 578.
[6] F. M. Azmi, N. Mlaiki, S. Haque, W. Shatanawi, *Complex valued controlled rectangular metric type spaces and application to linear systems*, AIMS Math. **8** **7** (2023) 16584–16598.
[7] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math. **3** (1922) 133–181.
[8] D. Barman, K. Sarkar, K. Tiwary, *Fixed point theorems in complex valued b-metric spaces*, Math. Moravica **27** **1** (2023) 85–96.
[9] K. Berrah, A. Aliouche, T. Oussaeif, *Applications and theorem on common fixed point in complex valued b-metric spaces*, AIMS Math. **4** **3** (2019) 1019–1033.
[10] K. Berrah, T. E. Oussaeif, A. Aliouche, *Common fixed point theorems of Meir–Keeler contraction type in complex-valued metric spaces and an application to dynamic programming*, J. Interdiscip. Math. **25** **2** (2022) 445–461.

[11] S. Bhatt, S. Chaukiyal, R. C. Dimri, *A common fixed point theorem for weakly compatible maps in complex valued metric spaces*, Int. J. Math. Sci. Appl. **1** **3** (2011) 1385–1389.
[12] S. Chourasiya, H. Khare, K. Shrivastava, *Common fixed point theorems in complex valued b-metric spaces and using weak compatible mappings*, Reliab. Theory Appl. **20** **2** (2025) 271–281.
[13] O. Ege, I. Karaca, *Common fixed point results on complex valued G_b-metric spaces*, Thai J. Math. **16** **3** (2018) 775–787.
[14] I. Iqbal, N. Hussain, M. A. Kutbi, *Existence of the solution to variational inequality, optimization problem, and elliptic boundary value problem through revisited best proximity point results,* J. Comput. Appl. Math. **375** (2020) 112804.
[15] G. Jungck, *Compatible mappings and common fixed points*, Int. J. Math. Math. Sci. **9** (1986) 771–779.
[16] A. Hosseini, M. M. Karizaki, *On the complex valued metric-like spaces*, Filomat **37** **15** (2023) 4903–4917.
[17] S. U. Khan, M. Arshad, H. K. Nashine, M. Nazam, *Some common fixed points of generalized contractive mappings on complex valued metric spaces.* J. Ana. Num. Theor, **5** **1** (2017) 73-80.
[18] D. Kumar, S. Chandok, *Existence of fixed point for a pair of mappings in complex valued metric spaces and application to integral equations*, Nonlinear Stud. **30** **1** (2023).
[19] M. Mehmood, A. Shoaib, N. Mlaiki, *Fixed point results on triple controlled quasi rectangular metric like spaces,* AIMS Math. **8** **5** (2023) 10049–10066.
[20] F. Mehmood, R. Ali, N. Hussain, *Contractions in fuzzy rectangular b-metric spaces with application, J. Intell.* Fuzzy Syst. **37** **1** (2019) 1275–1285.
[21] A. Moussaoui, *b-metric like spaces: a survey of concepts and applications*, Vojnoteh. Glasn. **73** **3** (2025) 1068–1097.
[22] A. Moussaoui, S. Radenović, S. Melliani, *A generalization of parametric metric spaces via β-actions*, Filomat **38** **15** (2024) 5475–5485.
[23] M. Nazam, M. Arshad, M. Postolache, *Coincidence and common fixed point theorems for four mappings satisfying (α_s, F)-contraction*, Nonlinear Analysis: Modelling and Control, **23** **5** (2018) 664-690.
[24] R. E. Rhoades, *Some theorems on weakly contractive maps*, Nonlinear Anal. **47** (2001) 2483–2693.
[25] F. Rouzkard, M. Imdad, *Some common fixed point theorems on complex valued metric spaces*, Comput. Math. Appl. **64** (2012) 1866–1874.
[26] A. Shahzad, A. Shoaib, N. Mlaiki, S. S. Aiadi, *Results for fuzzy mappings and stability of fuzzy sets with applications*, Fractal Fract. **6** **10** (2022) 556.
[27] A. M. Saddeek, G. M. Ahmed, *Common fixed point results for quadruple maps with weak compatibility in complex b-metric spaces with a practical application*, Math. Anal. Contemp. Appl. **6** **3** (2024) 47–63.
[28] A. E. Shammaky, J. Ahmad, *Generalization of Fixed-Point Results in Complex-Valued Bipolar Metric Space with Applications*, Axioms, **13** **8** (2024) 550.
[29] A. Shoaib, K. Khaliq, *Fixed-point results for generalized contraction in K-sequentially complete ordered dislocated fuzzy quasimetric spaces*, Fixed Point Theory Algorithms Sci. Eng. **2022** (2022) 27.
[30] A. Shoaib, P. Kumam, K. Sitthithakerngkiet, *Interpolative Hardy–Rogers type contraction on a closed ball in ordered dislocated metric spaces and some results*, AIMS Math. **7** **8** (2022) 13821–13831.
[31] D. Singh, O. P. Chauhan, N. Singh, V. Joshi, *Complex valued rectangular metric spaces and common fixed point theorems,* Bull. Math. Anal. Appl. **7** **2** (2015) 1–13.
[32] S. L. Singh, *On common fixed point of commuting mappings*, Math. Sem. Notes Kobe Univ. **5** (1977) 131–134.
[33] W. Sintunavarat, M. B. Zada, M. Sarwar, *Common solution of Urysohn integral equations with the help of common fixed point results in complex valued metric spaces*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. **111** (2017) 531–545.

Faisal Mahmood
Department of Mathematics and Statistics, Riphah International University, Islamabad - 44000, Pakistan
E-mail address: mahmoodfaisal143@gmail.com

Mumtaz Ali
Federal Governmentd College for boys Mardan Cantonment, Pakistan
E-mail address: mumtaz6767@yahoo.com

Abdullah Shoaib
Department of Mathematics and Statistics, Riphah International University, Islamabad - 44000, Pakistan
E-mail address: abdullahshoaib15@yahoo.com