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EXISTENCE OF SOLUTIONS FOR INTERVAL-VALUED CAPUTO FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS WITH IMPULSES AND LENGTH CONSTRAINTS

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Abstract. This paper investigates the existence of solutions for a class of interval-valued Caputo fractional integro-differential equations with impulses and length constraints. Within the short-memory fractional derivative framework, a hybrid impulse-integral regulatory mechanism is introduced, which integrates continuous integral perturbations with discrete impulses to smooth system dynamics and enhance stability. By applying Schauder's fixed point theorem, the existence of both forward and backward solutions for impulsive interval-valued fractional integro-differential equations is established. These results are further extended to systems with bounded length, yielding existence theorems under length-constrained conditions. We provide numerical examples to demonstrate the theoretical results and the method's applicability to systems with memory effects and uncertainties.

1. Introduction

Impulsive differential equations are fundamental tools for modeling dynamic systems with abrupt state changes, such as epidemic models and optimal control problems [21, 33], population ecology [3], and mechanical or engineering processes [14]. Fractional differential equations, particularly those of Caputo type, have attracted considerable attention because of their ability to capture memory and hereditary effects, which makes them well suited for describing systems with long-term dependencies. Incorporating impulses into fractional differential equations further broadens their applicability, allowing accurate modeling of rapid transitions in complex systems that exhibit memory effects.

Recent years have witnessed significant progress in the theory of fuzzy fractional differential equations (FFDEs). Agarwal et al. [1, 2] pioneered this direction by introducing Riemann-Liouville differentiability under Hukuhara differentiability and establishing existence results under compactness conditions. Later, Arshad and Lupulescu [6] proved existence and uniqueness theorems for FFDEs based on

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Hukuhara-type fractional Riemann-Liouville derivatives. Mazandarani et al. [17] investigated fuzzy fractional initial value problems with Caputo-type fuzzy fractional derivatives using a modified Euler method, and the same authors [18] established existence and uniqueness results for type-2 fuzzy Caputo fractional derivatives.

The close connection between fuzzy and interval-valued analysis [19] has also inspired parallel developments in interval-valued differential equations. Important contributions include linear interval-valued systems [11], second-order equations under generalized Hukuhara differentiability [23], and delay systems analyzed via fixed point techniques [9]. Further progress was achieved in [20], which examined forward and backward solutions under generalized Hukuhara differentiability and proved the existence of (I)-gH-differentiable and (II)-gH-differentiable solutions with monotonic and non-monotonic length, respectively. The concept of stopping time for interval-valued solutions was introduced in [30]. More recently, [29] established existence results for first-order interval-valued differential equations with length constraints, and [32] proposed a new framework for modeling interval-valued dynamical systems under bounded-length conditions.

Despite these developments, some critical challenges remain. First, the existence of solutions for interval-valued Caputo fractional integro-differential equations subject to length constraints has not yet been investigated. Second, classical impulsive approaches rely solely on discrete jumps, which may destabilize the system and fail to describe the continuous influence of historical states. Moreover, as pointed out in recent studies, the use of the classical long-memory Caputo derivative in impulsive systems leads to ill-posedness because the integral memory is disrupted at each impulse instant. To overcome this limitation, we adopt the short-memory Caputo fractional derivative, which resets the memory at each impulse time and thus ensures that the impulsive problem is mathematically well defined.

Motivated by these considerations, this paper investigates the existence of solutions for a class of interval-valued Caputo fractional integro-differential equations with impulses and length constraints under the short-memory framework. Specifically, we focus on the following systems:

$$\begin{cases} ({}^{C}D^{\alpha}_{t_0^+}x)(t) = f(t,x(t)) + \int_{t_0}^{t} g(\tau,x(\tau))\,d\tau, \\ x(t_0) = x_0, \end{cases} \tag{1.1}$$

and

$$\begin{cases} ({}^{C}D^{\alpha}_{t_0^-}x)(t) = f(t,x(t)) + \int_{t}^{t_0} g(\tau,x(\tau))\,d\tau, \\ x(t_0) = x_0, \end{cases} \tag{1.2}$$

where $\alpha \in (0,1]$, $({}^{C}D^{\alpha}_{t_0^+}x)(t)$ and $({}^{C}D^{\alpha}_{t_0^-}x)(t)$ denote the short-memory Caputo fractional derivatives in Definition 2.6, $x_0 \in K_C$, and $f, g \in C(\mathbb{R}\times K_C, K_C)$, with K_C denoting the set of all nonempty compact convex subsets of $\mathbb{R}$. In brief, this paper aims to (i) define the forward solution to (3.1) and the backward solution to (3.9); (ii) establish the existence of forward and backward solutions to these impulsive systems by applying Schauder's fixed point theorem; and (iii) examining the existence of forward solution to (1.1) and backward solution to (1.2) under the length constraint $\omega(x(t)) = x_+(t) - x_-(t) \le M$, where M is a fixed positive real number.

The remainder of the paper is organized as follows. Section 2 presents the necessary preliminaries. Section 3 establishes the existence of forward and backward

solutions for impulsive interval-valued Caputo fractional integro-differential equations under the short-memory framework. Section 4 examines a class of impulsive interval-valued Caputo fractional differential equations under length constraints and proves the corresponding existence results.

2. Preliminaries

In this section, we introduce some basic notations and results concerning interval-valued functions. Let K_C be the set of all nonempty compact convex subsets of $\mathbb{R}$. Let $u = [u_-, u_+], v = [v_-, v_+]$ belong to K_C, then $u + v = \{a + b \mid a \in u, b \in v\}$ and $ku = \{ka \mid a \in u\}, k \in \mathbb{R}$. Moreover, $u - v = u + (-1)v = \{a - b \mid a \in u, b \in v\}$.

Let $H : K_C \times K_C \mapsto [0, +\infty), H(u, v) = \max\{|u_- - v_-|, |u_+ - v_+|\}$. It is well known that H is Pompeiu Hausdorff metric on K_C and that (K_C, H) is a complete and locally compact metric space.

Lemma 2.1. *Let $u, v, w, z \in K_C$. The following properties hold:*
(i) $H(\lambda u, \lambda v) = |\lambda| \cdot H(u, v), \lambda \in \mathbb{R}$.
(ii) $H(u, v) \leq H(u, w) + H(w, v)$.
(iii) If $v \subseteq u$, then $H(v, \{0\}) \leq H(u, \{0\})$.
(iv) $H(u + v, w + z) \leq H(u, w) + H(v, z)$, especially, $H(u + v, \{0\}) \leq H(u, \{0\}) + H(v, \{0\})$.
(v) $\omega(u + v) = \omega(u - v) = \omega(u) + \omega(v)$.
(vi) $\omega(\alpha u) = |\alpha| \cdot \omega(u), \lambda \in \mathbb{R}$, where $\omega(u) = u_+ - u_-$.

Proof. Properties (i)-(ii) are proved in [20], (iii) in [29], and (v)-(vi) in [31]. We only prove (iv).
For $u, v, w, z \in K_C$,

$$\begin{aligned} H(u + v, w + z) &= max\{|u_- + v_- - w_- - z_-|, |u_+ + v_+ - w_+ - z_+|\} \\ &\leq max\{|u_- - w_-| + |v_- - z_-|, |u_+ - w_+| + |v_+ - z_+|\} \\ &\leq max\{|u_- - w_-|, |u_+ - w_+|\} + max\{|v_- - z_-|, |v_+ - z_+|\} \\ &= H(u, w) + H(v, z). \end{aligned}$$

□

Lemma 2.2. [7] *Suppose that $f : J \subseteq \mathbb{R} \mapsto K_C$ is integrable, then $\omega(\int_J f(t)dt) = \int_J \omega(f(t))dt$.*

Definition 2.1. [13] Let $u, v \in K_C$, Hukuhara difference of u and v is defined as $u \ominus v = [u_- - v_-, u_+ - v_+]$.

Definition 2.2. [20] The generalized Hukuhara difference (gH-difference, for short) is defined as follows:

$$u \ominus_g v = \begin{cases} u \ominus v, & \text{if u} \ominus v \text{ exists,} \\ (-1)v \ominus u, & \text{if v} \ominus u \text{ exists.} \end{cases}$$

In addition, the gH-derivative of a function $f : J \subseteq \mathbb{R} \mapsto K_C$ at $t_0 \in J$ is defined as

$$f'(t_0) = \lim_{h\to 0} \frac{f(t_0+h) \ominus_g f(t_0)}{h}.$$

An interval-valued function $f : [a, b] \mapsto K_C$ is called ω-increasing (respectively, ω-decreasing) on $[a, b]$ if the real function $t \mapsto \omega(x(t))$ is nondecreasing (respectively, nonincreasing) on $[a, b]$. If f is either ω-increasing or ω-decreasing on $[a, b]$, then we say that f is ω-monotone on $[a, b]$.

Definition 2.3. [20] Let $f : J \subseteq \mathbb{R} \mapsto K_C$ and $f(t) = [f_-(t), f_+(t)]$.

(i) f is said to be (I)-gH-differentiable at $t_0 \in J$, if $f_-(t)$ and $f_+(t)$ are both differentiable at t_0 and $f'(t_0) = [f_-(t_0), f_+(t_0)]$.
(ii) f is said to be (II)-gH-differentiable at $t_0 \in J$, if $f_-(t)$ and $f_+(t)$ are both differentiable at t_0 and $f'(t_0) = [f_+(t_0), f_-(t_0)]$.

Denote by $C([a, b], K_C)$ the set of all continuous interval-valued functions and $L^1([a, b], K_C)$ the set of all Lebesgue integrable interval-valued functions on $[a, b]$.

An interval-valued function $f : [a, b] \mapsto K_C$ is said to be absolutely continuous, if for each $\epsilon > 0$, there exists $\delta > 0$ such that, for each family $\{(s_k, t_k); k = 1, 2, \cdots, n\}$ of disjoint open interval in $[a, b]$ with $\sum_{k=0}^{n} (t_k - s_k) < \delta$, we have $\sum_{k=0}^{n} H(f(t_k), f(s_k)) < \epsilon$.

Denote by $AC([a, b], K_C)$ the set of all absolutely continuous interval-valued functions on $[a, b]$.

Definition 2.4. [1] Let $f \in L^1([a, b], K_C)$, the interval-valued Riemann-Liouville integral of order $\alpha > 0$ of the interval-valued function f is defined as follows:

$$(I^{\alpha}_{a^+} f)(t) = \tfrac{1}{\Gamma(\alpha)} \textstyle\int_a^t (t-s)^{\alpha-1} f(s)\, ds, t \in (a, b],$$
$$(I^{\alpha}_{b^-} f)(t) = \tfrac{1}{\Gamma(\alpha)} \textstyle\int_t^b (s-t)^{\alpha-1} f(s)\, ds, t \in [a, b).$$

Definition 2.5. If $f \in AC([a, b], K_C)$ is an interval-valued function and $\alpha \in (0, 1]$, then the gH-fractional Caputo derivative of the interval-valued function $f([gH]^C_\alpha$-differentiable for short) is defined as follows:

$$({}^C D^{\alpha}_{a^+} f)(t) = (I^{1-\alpha}_{a^+} f')(t) = \tfrac{1}{\Gamma(1-\alpha)} \textstyle\int_a^t (t-s)^{-\alpha} f'(s)\, ds, t \in (a, b],$$
$$({}^C D^{\alpha}_{b^-} f)(t) = -(I^{1-\alpha}_{b^-} f')(t) = -\tfrac{1}{\Gamma(1-\alpha)} \textstyle\int_t^b (s-t)^{-\alpha} f'(s)\, ds, t \in [a, b).$$

Following Definition 2.5, we now introduce its short-memory counterpart, which modifies the lower limit of integration to represent a finite memory length.

Definition 2.6. Let $f \in AC([a, b], K_C)$ be an interval-valued function and $\alpha \in (0, 1)$. For any $t \in (t_k, t_{k+1}]$, the short-memory Caputo fractional derivatives of f are defined by

$$({}^C D^{\alpha}_{t_k^+} f)(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_k}^{t} (t-s)^{-\alpha} f'(s)\, ds, t \in (a, b], \tag{2.1}$$

$$({}^C D^{\alpha}_{t_k^-} f)(t) = -\frac{1}{\Gamma(1-\alpha)} \int_{t}^{t_k} (s-t)^{-\alpha} f'(s)\, ds, t \in (a, b], \tag{2.2}$$

where t_k denotes the most recent impulse instant. When $t_k = a$ (or $t_k = b$), these definitions coincide with the classical Caputo fractional derivative.

In the following analysis, all fractional derivatives are understood in the short-memory sense, that is, the memory is locally reset at each impulse time. This formulation ensures that the impulsive fractional system remains mathematically well-posed while preserving the essential memory characteristics of fractional dynamics.

Definition 2.7. Let $f \in AC(J, K_C)$ and $f(t) = [f_-(t), f_+(t)]$.
(i) f is said to be $[(I) - gH]^C_\alpha$-differentiable at $t_0 \in J$, if $f_-(t)$ and $f_+(t)$ are both Caputo fractional differentiable at t_0 and

$$({}^C D^\alpha_{a^+} f)(t_0) = [({}^C D^\alpha_{a^+} f_-)(t_0), ({}^C D^\alpha_{a^+} f_+)(t_0)],$$
$$({}^C D^\alpha_{b^-} f)(t_0) = [({}^C D^\alpha_{b^-} f_+)(t_0), ({}^C D^\alpha_{b^-} f_-)(t_0)].$$

(ii) f is said to be $[(II) - gH]^C_\alpha$-differentiable at $t_0 \in J$, if $f_-(t)$ and $f_+(t)$ are both Caputo fractional differentiable at t_0 and

$$({}^C D^\alpha_{a^+} f)(t_0) = [({}^C D^\alpha_{a^+} f_+)(t_0), ({}^C D^\alpha_{a^+} f_-)(t_0)],$$
$$({}^C D^\alpha_{b^-} f)(t_0) = [({}^C D^\alpha_{b^-} f_-)(t_0), ({}^C D^\alpha_{b^-} f_+)(t_0)].$$

Definition 2.8. [32] Let $f : J \subseteq \mathbb{R} \mapsto K_C$ and $f(t) = [f_-(t), f_+(t)]$, the integral of f over J is defined as

$$\int_J f(t)dt = [\int_J f_-(t)dt, \int_J f_+(t)dt].$$

If $\int_J f(t)dt$ exists, $J_1 \cap J_2 = \emptyset$ and $J_1 \cup J_2 = J$, then, by the additivity property of real-valued integrals, one has

$$\int_J f(t)dt = \int_{J_1} f(t)dt + \int_{J_2} f(t)dt.$$

Lemma 2.3. [31] *Suppose that $J \in \mathbb{R}$ is a bounded closed interval and $f, g : J \subseteq \mathbb{R} \mapsto K_C$ are continuous. Then*

$$H(\int_J f(t)dt, \int_J g(t)dt) \leq \int_J H(f(t), g(t))dt.$$

Let $x_0 \in K_C$ and suppose $f, g \in C(\mathbb{R} \times K_C, K_C)$ with $\alpha \in (0, 1]$, we consider initial value problem of interval-valued Caputo fractional integro-differential equation:

$$\begin{cases} ({}^C D^\alpha_{t_0^+} x)(t) = f(t, x(t)) + \int_{t_0}^t g(\tau, x(\tau))\, d\tau, \\ x(t_0) = x_0, x_0 \in K_C; \end{cases} \tag{2.3}$$

and

$$\begin{cases} ({}^C D^\alpha_{t_0^-} x)(t) = f(t, x(t)) + \int_t^{t_0} g(\tau, x(\tau))\, d\tau, \\ x(t_0) = x_0, x_0 \in K_C. \end{cases} \tag{2.4}$$

Definition 2.9. A ω-monotone interval-valued function $x \in C([t_0, b], K_C)$ is said to be a forward solution to (2.3), if and only if

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_{t_0}^t (t-s)^{\alpha-1} \left[f\left(s, x(s)\right) + \int_{t_0}^s g\left(\tau, x(\tau)\right)\, d\tau \right] ds$$

holds for $t \in [t_0, b]$ and the interval-valued function $t \mapsto I^{\alpha}_{t_0^+} f(t)$ is a ω-increasing on $[t_0, b]$.

Similarly, a ω-monotone interval-valued function $x \in C([a, t_0], K_C)$ is said to be a backward solution to (2.4), if and only if

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_t^{t_0} (s-t)^{\alpha-1} \left[f(s, x(s)) + \int_s^{t_0} g(\tau, x(\tau))\, d\tau \right] ds$$

for $t \in [a, t_0]$ and the interval-valued function $t \mapsto I^{\alpha}_{t_0^-} f(t)$ is a ω-decreasing on $[t_0, b]$.

Remark. The requirement that the interval-valued function $t \mapsto I^{\alpha}_{t_0^+} f(t)$ be ω-increasing (or $t \mapsto I^{\alpha}_{t_0^-} f(t)$ be ω-decreasing) ensures the equivalence between the integral representation and the original interval-valued Caputo fractional integro-differential equation (2.3) (or (2.4)). Without such monotonicity constraints on the length of the solution, the integral form may not validly solve the equation. For more details, see [26].

Based on Definition 2.9, if x is a forward solution to (2.3) on $[t_0, b]$, then x is $[(I) - gH]^C_{\alpha}$-differentiable and $\omega(x(t))$ is increasing on $[t_0, b]$. Moreover, x satisfies

$$\begin{cases} ({}^C D^{\alpha}_{t_0^+} x_-)(t) = f_-(t, x(t)) + \int_{t_0}^t g_-(\tau, x(\tau))\, d\tau, \\ ({}^C D^{\alpha}_{t_0^+} x_+)(t) = f_+(t, x(t)) + \int_{t_0}^t g_+(\tau, x(\tau))\, d\tau, \\ x_-(t_0) = [x_0]_-, x_+(t_0) = [x_0]_+. \end{cases} \tag{2.5}$$

If x is a backward solution to (2.4), then x is $[(II) - gH]^C_{\alpha}$-differentiable, $\omega(x(t))$ is decreasing on $[a, t_0]$ and

$$\begin{cases} ({}^C D^{\alpha}_{t_0^-} x_-)(t) = f_-(t, x(t)) + \int_t^{t_0} g_-(\tau, x(\tau))\, d\tau, \\ ({}^C D^{\alpha}_{t_0^-} x_+)(t) = f_+(t, x(t)) + \int_t^{t_0} g_+(\tau, x(\tau))\, d\tau, \\ x_-(t_0) = [x_0]_-, x_+(t_0) = [x_0]_+. \end{cases} \tag{2.6}$$

Let $J = [a, b], a = t_0 < t_1 < \cdots < t_n < t_{n+1} = b$. Define $PC(J, K_C) = \{x : J \mapsto K_C \mid x$ is continuous on $J \backslash \{t_1, t_2, \cdots, t_n\}; x\left(t_k^+\right)$ and $x\left(t_k^-\right)$ exist and $x\left(t_k^+\right) = x\left(t_k\right)$ for $k = 1, 2, \cdots, n\}$. For $x_1, x_2 \in PC(J, K_C)$, define the metric $d\left(x_1, x_2\right) = \sup_{t \in J} H\left(x_1(t), x_2(t)\right)$. We can prove that the following lemma holds.

Lemma 2.4. *$(PC(J, K_C), d)$ is a complete metric space.*

Proof. Let $J_k = [t_{k-1}, t_k)$ for $k = 1, 2, \ldots, n+1$, and assume that $\{x_n\}$ is a Cauchy sequence in $PC(J, K_C)$, then $d(x_n, x_m) = \sup_{t \in J} H(x_n(t), x_m(t)) \to 0$ as $m, n \to \infty$. Hence, for each $t \in J$, $H(x_n(t), x_m(t)) \to 0$, which implies $\{x_n(t)\}$ is a Cauchy sequence in K_C for each fixed $t \in J$. Since K_C is complete, it follows that each sequence $\{x_n(t)\}$ converges to some $x(t) \in K_C$. And then, $\forall t \in J$, $H(x_n(t), x(t)) \to 0$ as $n \to \infty$, that is $\{x_n(t)\}$ converges uniformly to $x(t)$. As a result, $x(t)$ is also continuous on each J_k. Furthermore, the left and right limits of $x(t)$ at each t_k exist and $x(t_k^+) = x(t_k)$ for $k = 1, 2, \ldots, n$.

Therefore, $x \in PC(J, K_C)$ and we can conclude that $(PC(J, K_C), d)$ is a complete metric space. □

3. Interval-valued Caputo fractional integro-differential equation with impulses

In this section, we consider impulsive interval-valued Caputo fractional integro-differential equations. It was pointed out in Remark 1.1 of [27] that impulsive fractional differential equations are not well-defined when classical long-memory Caputo derivatives are used. Therefore, in this section we adopt the short-memory Caputo derivative, where on each subinterval $(t_k, t_{k+1}]$ the lower limit of the fractional integral is taken at t_k. This ensures that the impulsive problem is well-defined and its integral representation is consistent with the differential formulation.

$$\begin{cases} ({}^{C}D^{\alpha}_{t_k^+}x)(t) = f(t,x(t)) + \int_{t_k}^{t} g(\tau,x(\tau))\,d\tau,\ t \in [0,T]\backslash\{t_1,\cdots,t_n\}, k = 1,\cdots,n-1, \\ x(0) = x_0, \\ x(t_k^+) = I_k(x(t_k)), k = 1,\cdots,n. \end{cases} \tag{3.1}$$

where $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = T$, $I_k : K_C \mapsto K_C$ is continuous. Let $I_0 : K_C \mapsto K_C$ be the identity mapping, i.e., $I_0(u) = u$ for every $u \in K_C$.

Definition 3.1. $x \in PC([0,T], K_C)$ is said to be a forward solution to (3.1), if x satisfies

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[f(s,x(s)) + \int_0^s g(\tau,x(\tau))\,d\tau \right] ds$$

on (t_0, t_1), and

$$x(t) = I_k(x(t_k)) + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t-s)^{\alpha-1} \left[f(s,x(s)) + \int_{t_k}^s g(\tau,x(\tau))\,d\tau \right] ds$$

on each $(t_k, t_{k+1}), k = 1, 2, \ldots, n$.

For each $x \in PC([0,T], K_C)$, we define the operator $A : PC([0,T], K_C) \mapsto PC([0,T], K_C)$, denoted by $(Ax)(t)$ as follows:

$$\begin{cases} x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[f(s,x(s)) + \int_0^s g(\tau,x(\tau))\,d\tau \right] ds, & t \in [t_0, t_1], \\ I_k(x(t_k)) + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t-s)^{\alpha-1} \left[f(s,x(s)) + \int_{t_k}^s g(\tau,x(\tau))\,d\tau \right] ds, & t \in (t_k, t_{k+1}], \end{cases}$$

where $k = 1, 2, \ldots, n$. Then, we have the following theorem.

Theorem 3.1. *Suppose that $I_k(u) \subseteq u$ for every $u \in K_C$ and $k = 1, 2, \ldots, n$. If there exist $\sigma \in (0,1)$ and a constant $N > 0$ such that*

$$H(f(t,u), \{0\}) \le \sigma H(u, \{0\}), \quad H(g(t,u), \{0\}) \le N \tag{3.2}$$

holds for every $u \in K_C$, where $\frac{1}{\Gamma(\alpha+1)} \sum_{i=0}^{k} (t_{i+1} - t_i)^{\alpha} < \frac{1}{\sigma}$, then the problem (3.1) has at least one forward solution.

Proof. We shall use Schauder's fixed point theorem to prove that the problem (3.1) has at least one forward solution defined on $[0,T]$. Let $r>0$ and define $E_r=\{x\in PC([0,T],K_C)|d(x,\{0\})\le r\}$.

First of all, we prove $A(E_r)\subseteq E_r$. $\forall x\in E_r$, by Lemma 2.1(i), (iv) and Lemma 2.3, for each $t\in[0,t_1]$, we have

$$\begin{aligned}
&H((Ax)(t),\{0\})\\
=&H\left(x_0+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\left[f(s,x(s))+\int_0^s g(\tau,x(\tau))\,d\tau\right]ds,\{0\}\right)\\
\le&H(x_0,\{0\})+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\left[H\left(f(s,x(s))+\int_0^s g(\tau,x(\tau))\,d\tau,\{0\}\right)\right]ds\\
\le&H(x_0,\{0\})+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}(\sigma H(x(s),\{0\}))\,ds+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}(Ns)\,ds\\
\le&H(x_0,\{0\})+\frac{\sigma\cdot r+NT}{\Gamma(\alpha+1)}(t_1-t_0)^{\alpha}.
\end{aligned}\tag{3.3}$$

Let $I_0(x(t_0))=x_0$. For $k=1,2,\cdots,n$, Lemma 2.1(iii) and $I_k(u)\subseteq u$ provide that $H(I_k(x(t_k)),\{0\})\le H(x(t_k),\{0\})$. Let $t\in(t_k,t_{k+1}]$, by Lemma 2.1(i), (iv) and Lemma 2.3, we have

$$\begin{aligned}
&H((Ax)(t),\{0\})\\
=&H(I_k(x(t_k))+\frac{1}{\Gamma(\alpha)}\int_{t_k}^t(t-s)^{\alpha-1}\left[f(s,x(s))+\int_{t_k}^s g(\tau,x(\tau))\,d\tau\right]ds,\{0\})\\
\le&H(I_k(x(t_k)),\{0\})+\frac{1}{\Gamma(\alpha)}\int_{t_k}^t(t-s)^{\alpha-1}\left[H\left(f(s,x(s))+\int_{t_k}^s g(\tau,x(\tau))\,d\tau,\{0\}\right)\right]ds\\
\le&H(x(t_k)),\{0\})+\frac{1}{\Gamma(\alpha)}\int_{t_k}^t(t-s)^{\alpha-1}\left[H\left(f(s,x(s))+\int_{t_k}^s g(\tau,x(\tau))\,d\tau,\{0\}\right)\right]ds\\
\le&H(I_{k-1}(x(t_{k-1}))),\{0\})\\
&+\frac{1}{\Gamma(\alpha)}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}\left[H\left(f(s,x(s))+\int_{t_{k-1}}^s g(\tau,x(\tau))\,d\tau,\{0\}\right)\right]ds\\
&+\frac{1}{\Gamma(\alpha)}\int_{t_k}^t(t-s)^{\alpha-1}\left[H\left(f(s,x(s))+\int_{t_k}^s g(\tau,x(\tau))\,d\tau,\{0\}\right)\right]ds.
\end{aligned}\tag{3.4}$$

Repeat the above steps, we get

$$\begin{aligned}
&H((Ax)(t),\{0\})\\
\le&H(x_0,\{0\})\\
&+\frac{1}{\Gamma(\alpha)}\left(\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}}(t_{i+1}-s)^{\alpha-1}\left[H\left(f(s,x(s))+\int_{t_i}^s g(\tau,x(\tau))\,d\tau,\{0\}\right)\right]\right)ds\\
&+\frac{1}{\Gamma(\alpha)}\int_{t_k}^t(t-s)^{\alpha-1}\left[H\left(f(s,x(s))+\int_{t_k}^s g(\tau,x(\tau))\,d\tau,\{0\}\right)\right]ds\\
\le&H(x_0,\{0\})+\frac{\sigma\cdot r+NT}{\Gamma(\alpha+1)}\sum_{i=0}^{k}(t_{i+1}-t_i)^{\alpha}.
\end{aligned}\tag{3.5}$$

Thus, for each $t \in [0,T]$,

$$H((Ax)(t),\{0\}) \le H(x_0,\{0\}) + \frac{\sigma\cdot r + NT}{\Gamma(\alpha+1)}\sum_{i=0}^{k}(t_{i+1}-t_i)^{\alpha}. \tag{3.6}$$

Let r be a bigger positive number if it is necessary, then we have that $d(Ax,\{0\}) \le r$. That is $A(E_r) \subseteq E_r$.

Next, we prove the operator A is continuous. Indeed, let $\{x_n\}_{n\ge1}$ be a sequence such that $x_n \to x$ in $PC([0,T],K_C)$. For each $t \in [t_k,t_{k+1}], k = 0,1,2,\ldots,n$, one has

$$\begin{aligned}
&H((Ax_n)(t),(Ax)(t))\\
\le&\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}\left[H\left(f(s,x_n(s))+\int_{t_k}^{s}g(\tau,x_n(\tau))\,d\tau, f(s,x(s))+\int_{t_k}^{s}g(\tau,x(\tau))\,d\tau\right)\right]ds\\
\le&\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}\left[H\left(f(s,x_n(s)),f(s,x(s))\right)+\int_{t_k}^{s}H\left(g(\tau,x_n(\tau)),g(\tau,x(\tau))\right)\,d\tau\right]ds\\
\le&\sup_{s\in[t_k,t_{k+1}]}H(f(s,x_n(s)),f(s,x(s)))\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}\,ds\\
&+T\cdot\sup_{\tau\in[t_k,t_{k+1}]}H(g(\tau,x_n(\tau)),g(\tau,x(\tau)))\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}\,ds\\
\le&\left[\sup_{s\in[0,T]}H(f(s,x_n(s)),f(s,x(s)))+T\cdot\sup_{\tau\in[0,T]}H(g(\tau,x_n(\tau)),g(\tau,x(\tau)))\right]\\
&\times\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}\,ds.
\end{aligned}$$

by the Lebesgue dominated convergence theorem it yields that $H((Ax_n)(t),(Ax)(t)) \to 0$ as $n\to\infty$. And then we can get $\sup\limits_{t\in[0,T]}H((Ax_n)(t),(Ax)(t)) \to 0$ as $n\to\infty$.

On the other hand, we prove $A(E_r)$ is equicontinuous. For each $x \in E_r$, if $t_k \le \tilde{t} \le \bar{t} \le t_{k+1}, k = 0,1,2,\ldots,n$, then

$$\begin{aligned}
&H((Ax)(\tilde{t}),(Ax)(\bar{t}))\\
=&\max\{|(A_-x)(\tilde{t})-(A_-x)(\bar{t})|,|(A_+x)(\tilde{t})-(A_+x)(\bar{t})|\}\\
\le&\frac{1}{\Gamma(\alpha)}\int_{t_k}^{\tilde{t}}[(\tilde{t}-s)^{\alpha-1}-(\bar{t}-s)^{\alpha-1}]\left[H(f(s,x(s))+\int_{t_k}^{s}g(\tau,x(\tau))\,d\tau,\{0\})\right]ds\\
&+\frac{1}{\Gamma(\alpha)}\int_{\tilde{t}}^{\bar{t}}(\bar{t}-s)^{\alpha-1}\left[H(f(s,x(s))+\int_{t_k}^{s}g(\tau,x(\tau))\,d\tau,\{0\})\right]ds\\
\le&\frac{\sigma r+NT}{\Gamma(\alpha)}\left[\int_{t_k}^{\tilde{t}}(\tilde{t}-s)^{\alpha-1}-(\bar{t}-s)^{\alpha-1}\,ds+\int_{\tilde{t}}^{\bar{t}}(\bar{t}-s)^{\alpha-1}\,ds\right]\\
\le&\frac{2(\sigma r+NT)}{\Gamma(\alpha+1)}|\bar{t}-\tilde{t}|.
\end{aligned} \tag{3.7}$$

If $t_l \le \tilde{t} \le t_{l+1}, t_m \le \bar{t} \le t_{m+1}$, $0 \le l \le m \le n$, by Lemma 2.1(ii), we have

$$\begin{aligned}
&H((Ax)(\tilde{t}),(Ax)(\bar{t}))\\
\le&H((Ax)(\tilde{t}),(Ax)(t_{l+1}))+H((Ax)(t_{l+1}),(Ax)(t_{l+2}))+\ldots+H((Ax)(t_m),(Ax)(\bar{t})).
\end{aligned}$$

By (3.7), the right hand side of the above inequality tends to zero as $\bar{t} \to \tilde{t}$.

By Arzela-Ascoli theorem, we know that $A_-(E_r)$ and $A_+(E_r)$ are both uniformly bounded and are equicontinuous, thus $A_-(E_r)$ and $A_+(E_r)$ are sequentially compact, which provide that $A(E_r)$ is also sequentially compact.

Let $C^*[0,T] = \{x : [0,T] \to \mathbb{R} \mid x \text{ is continuous on } [0,T] \backslash \{t_1, t_2, \dots t_n\}; x\left(t_k^+\right) \text{and } x\left(t_k^-\right) \text{exist and } x\left(t_k^+\right) = x\left(t_k\right) \text{ for } k = 1, 2, \dots, n\}$, then $C^*[0,T] \times C^*[0,T]$ with norm $\|(x,y)\| = \left\{\max_{t\in[0,T]} |x(t)|, \max_{t\in[0,T]} |y(t)|\right\}$ is a Banach space. Now we consider E_r as a subset of $C^*[0,T] \times C^*[0,T]$, which is nonempty bounded and convex. In addition, Lemma 2.4 ensures that E_r is closed. By Schauder's fixed point theorem, A has at least one fixed point in E_r, which is also a forward solution to problem (3.1) □

Example 3.1. Consider the impulsive interval-valued Caputo fractional integro-differential equation.

$$\begin{cases} ({}^C D^{0.5}_{t_k^+} x)(t) = 0.3x(t) + \int_{t_k}^t 0.5\, d\tau, t \in [0,1] \backslash \{0.5\}, \quad k = 0, 1, \\ x(0) = [1,2], \\ x(t^+) = \left[x_-(t), x_+(t) - \frac{x_+(t) - x_-(t)}{4}\right], t = 0.5. \end{cases} \tag{3.8}$$

Firstly, we can get $I_k(u) = \left[u_-, u_+ - \frac{u_+ - u_-}{4}\right] \subseteq u$. In addition, $H(0.3u, \{0\}) = 0.3H(u, \{0\})$, $H(\{0.5\}, \{0\}) \leq N = 0.5$ and $\frac{1}{\Gamma(\alpha+1)} \sum_{i=0}^{1} (t_{i+1} - t_i)^\alpha = 2 < \frac{1}{\sigma}$. Therefore, by Theorem 3.1, problem (3.8) has at least one forward solution. A forward solution $x(t)$ of (3.8) can be obtained as follows:

$$[1,2] \cdot E_{0.5,1}\left(0.3t^{0.5}\right) + \int_0^t (t-\tau)^{-0.5} E_{0.5,0.5}\left(0.3(t-\tau)^{0.5}\right)(0.5\tau)\, d\tau,$$

for $t \in [0, 0.5]$ and

$$[1.234, 2.359] \cdot E_{0.5,1}\left(0.3(t-0.5)^{0.5}\right) + \int_{0.5}^t (t-\tau)^{-0.5} E_{0.5,0.5}\left(0.3(t-\tau)^{0.5}\right)(0.5\tau)\, d\tau,$$

for $t \in (0.5, 1]$.

Similarly, we consider impulsive interval-valued Caputo fractional integro-differential equation:

$$\begin{cases} ({}^C D^{\alpha}_{t_k -} x)(t) = f(t, x(t)) + \int_{t_k}^0 g(\tau, x(\tau))\, d\tau, \quad t \in [-T, 0] \backslash \{t_1, \dots, t_n\}, k = 1, \dots, n-1, \\ x(0) = x_0, \\ x(t_k^-) = I_k(x(t_k)), k = 1, \cdots n. \end{cases} \tag{3.9}$$

where $-T = t_{n+1} < t_n < \cdots < t_1 < t_0 = 0$, I_k is defined as that in (3.1). We define the backward solution as follows.

Definition 3.2. $x \in PC([-T, 0], K_C)$ is said to be a backward solution to (3.9), if x satisfies

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_t^0 (s-t)^{\alpha-1} \left[f(s, x(s)) + \int_s^0 g(\tau, x(\tau))\, d\tau\right] ds$$

on (t_1, t_0), and

$$x(t) = I_k(x(t_k)) + \frac{1}{\Gamma(\alpha)} \int_t^{t_k} (s-t)^{\alpha-1} \left[f(s, x(s)) + \int_s^{t_k} g(\tau, x(\tau))\, d\tau \right] ds$$

on each $(t_{k+1}, t_k), k = 1, 2, \ldots, n$.

For each $x \in PC([-T, 0], K_C)$, we define a operator $B : PC([-T, 0], K_C) \mapsto PC([-T, 0], K_C)$ as follows:

$$(Bx)(t) = \begin{cases} x_0 + \frac{1}{\Gamma(\alpha)} \int_t^0 (s-t)^{\alpha-1} \left[f(s, x(s)) + \int_s^0 g(\tau, x(\tau))\, d\tau \right] ds, & t \in [t_1, 0], \\ I_k(x(t_k)) + \frac{1}{\Gamma(\alpha)} \int_t^{t_k} (s-t)^{\alpha-1} \left[f(s, x(s)) + \int_s^{t_k} g(\tau, x(\tau))\, d\tau \right] ds, & t \in [t_{k+1}, t_k), \end{cases}$$

where $k = 1, 2, \ldots, n$. For the existence of backward solution to (3.9), we have the following theorem.

Theorem 3.2. *Suppose that $I_k(u) \subseteq u$ for every $u \in K_C$ and $k = 1, 2, \ldots, n$. If there exist $\sigma \in (0, 1)$ and a constant $N > 0$ such that (3.2) holds for every $u \in K_C$, where $\frac{1}{\Gamma(\alpha+1)} \sum_{i=0}^{k} (t_i - t_{i+1})^\alpha < \frac{1}{\sigma}$, then the problem (3.9) has at least one backward solution.*

Proof. Let $r > 0$, $E_r = \{x \in PC([-T, 0], K_C) | d(x, \{0\}) \leq r\}$. By using the same method in the proof of Theorem 3.1, we can prove that B has at least one fixed point in E_r, so problem (3.9) has a backward solution. □

Example 3.2. Consider the impulsive interval-valued Caputo fractional integro-differential equation,

$$\begin{cases} ({}^C D^{0.5}_{t_k^-} x)(t) = 0.3x(t) + \int_t^{t_k} 0.5\, d\tau, t \in [-1, 0] \backslash \{-0.5\}, k = 0, 1, \\ x(0) = [1, 2], \\ x(t^+) = \left[x_+(t) - \frac{x_+(t) - x_-(t)}{4}, x_+(t) \right], t = -0.5. \end{cases} \tag{3.10}$$

We can get $I_k(u) = \left[u_+ - \frac{u_+ - u_-}{4}, u_+ \right] \subseteq u$, $\frac{1}{\Gamma(\alpha+1)} \sum_{i=0}^{1} (t_i - t_{i+1})^\alpha = 2 < \frac{1}{\sigma}$ and (3.2) holds. Therefore, by Theorem 3.2, problem (3.10) has at least one backward solution. A backward solution $x(t)$ to (3.10) can be constructed as follows:

$$[1, 2] \cdot E_{0.5,1} \left(0.3(-t)^{0.5} \right) + \int_t^0 (\tau - t)^{-0.5} E_{0.5,0.5} \left(0.3(\tau - t)^{0.5} \right) (0.5\tau)\, d\tau,$$

for $t \in [-0.5, 0]$ and

$$[1.234, 2.359] \cdot E_{0.5,1} \left(0.3(-t + 0.5)^{0.5} \right) + \int_t^{-0.5} (\tau - t)^{-0.5} E_{0.5,0.5} \left(0.3(\tau - t)^{0.5} \right) (0.5\tau)\, d\tau,$$

for $t \in [-1, -0.5)$.

Remark. Based on the proof of Theorem 3.1, it can be observed that the condition $I_k(u) \subseteq u$ is not essential. If there exists a constant $\delta > 0$ such that $H(I_k(u), u) \leq \delta$ for all $u \in K_C$ and $k = 1, 2, \ldots, n$, then inequalities (3.3)-(3.6) remain valid. The following results are therefore stated without proof.

Theorem 3.3. *Suppose that there exists $\delta > 0$ such that $H(I_k(u), u) \leq \delta$ for every $u \in K_C, k = 1, 2, \ldots, n$*
(i) If there exist $\sigma \in (0, 1)$ and a constant $N > 0$ such that (3.2) holds for every $u \in K_C$, where $\frac{1}{\Gamma(\alpha+1)} \sum_{i=0}^{k} (t_{i+1} - t_i)^\alpha < \frac{1}{\sigma}$, then the problem (3.1) has at least one forward solution.
(ii) If there exist $\sigma \in (0, 1)$ and a constant $N > 0$ such that (3.2) holds for every $u \in K_C$, where $\frac{1}{\Gamma(\alpha+1)} \sum_{i=0}^{k} (t_i - t_{i+1})^\alpha < \frac{1}{\sigma}$, then the problem (3.9) has at least one backward solution.

4. Interval-valued Caputo fractional integro-differential equation under length constraints

In this section, we investigate the existence of forward and backward solutions for interval-valued Caputo fractional integro-differential equations under length constraints. The short-memory setting established in Section 2 naturally extends the impulsive framework of Section 3 to systems with bounded length, allowing the analysis of both forward and backward dynamics within a unified formulation.

Throughout this section, all fractional derivatives are considered in the short-memory sense, that is, the memory is locally reset at each impulse instant. When no impulses occur on $[t_0, T]$, the short-memory Caputo derivative coincides exactly with the classical Caputo derivative (with lower limit t_0); when impulses appear at times t_k, the lower limit of the fractional integral is shifted to t_k. This unified notation guarantees the well-posedness of the model and makes the transition between impulsive and non-impulsive cases mathematically consistent. We begin with the following equations:

$$\begin{cases} ({}^C D^{\alpha}_{t_0^+} x)(t) = f(t, x(t)) + \int_{t_0}^{t} g(\tau, x(\tau))\, d\tau, \\ x(t_0) = x_0, \end{cases} \tag{4.1}$$

and

$$\begin{cases} ({}^C D^{\alpha}_{t_0^-} x)(t) = f(t, x(t)) + \int_{t}^{t_0} g(\tau, x(\tau))\, d\tau, \\ x(t_0) = x_0, \end{cases} \tag{4.2}$$

where $x_0 \in K_C$, $f, g : \mathbb{R} \times K_C \to K_C$ are both continuous. Moreover,
(H1) There exist $L > 0$ and $C > 0$ such that $\omega(f(t, u)) \leq L \cdot \omega(u)$, $\omega(g(t, u)) \leq C \cdot \omega(u)$ for every $u \in K_C$.

Lemma 4.1. *Suppose that (H1) is satisfied, $0 < \omega(x_0) < M$, and $t \in [-T, T]$, where $0 \leq T < +\infty$.*
(i) The forward solution to problem (4.1) satisfies $\omega(x(t)) \leq M$ for $t \in [t_0, \alpha^+_{t_0})$, where

$$\alpha^+_{t_0} = \begin{cases} P^{-1}_{t_0}\left(\tilde{M}_{x_0}\right) & \text{for } \int_{t_0}^{T} (T-s)^{\alpha-1}\, ds > \tilde{M}_{x_0}, \\ T & \text{for } \int_{t_0}^{T} (T-s)^{\alpha-1}\, ds \leq \tilde{M}_{x_0}, \end{cases}$$

$\tilde{M}_{x_0} = \frac{\Gamma(\alpha)}{L+CT} \ln \frac{M}{len(x_0)}$, $P^{-1}_{t_0}$ is the inverse function of $P_{t_0} : [t_0, T] \mapsto [0, +\infty)$, given by $P_{t_0}(t) = \int_{t_0}^{t} (t-s)^{\alpha-1} ds$.
(ii) The backward solution to problem (4.2) satisfies $\omega(x(t)) \leq M$ for $t \in (\alpha^-_{t_0}, t_0]$,

where

$$\alpha_{t_0}^- = \begin{cases} Q_{t_0}^{-1}\left(\tilde{M}_{x_0}\right) & \text{for } \int_{-T}^{t_0}(s+T)^{\alpha-1}ds > \tilde{M}_{x_0}, \\ -T & \text{for } \int_{-T}^{t_0}(s+T)^{\alpha-1}ds \le \tilde{M}_{x_0}, \end{cases}$$

$Q_{t_0}^{-1}$ *is the inverse function of* $Q_{t_0} : [-T, t_0] \mapsto [0, +\infty)$*, given by*

$$Q_{t_0}(t) = \int_t^{t_0} (s-t)^{\alpha-1}\, ds.$$

Proof. To prove (i) holds, we suppose that x is a forward solution to (4.1) on $[t_0, T]$, by Lemma 2.1(v) and Lemma 2.2 together with assumption (H1), we have

$$\begin{aligned} \omega(x(t)) &= \omega(x_0) + \frac{1}{\Gamma(\alpha)}\omega\left(\int_{t_0}^t (t-s)^{\alpha-1}\left[f(s,x(s)) + \int_{t_0}^s g(\tau, x(\tau))\, d\tau\right] ds\right) \\ &= \omega(x_0) + \frac{1}{\Gamma(\alpha)}\int_{t_0}^t (t-s)^{\alpha-1}\left[\omega(f(s,x(s)) + \int_{t_0}^s g(\tau, x(\tau))\, d\tau\right] ds \\ &\le \omega(x_0) + \frac{1}{\Gamma(\alpha)}\int_{t_0}^t (t-s)^{\alpha-1}\left[L\cdot\omega(x(s)) + \int_{t_0}^s C\cdot\omega(x(\tau))\, d\tau\right] ds. \end{aligned} \tag{4.3}$$

Let $z(t) = \sup_{0\le u\le t} \omega(x(u))$, then

$$\begin{aligned} z(t) &\le \omega(x_0) + \frac{1}{\Gamma(\alpha)}\int_{t_0}^t (t-s)^{\alpha-1}\left[L\cdot z(t) + \int_{t_0}^s C\cdot z(t)\, d\tau\right] ds \\ &\le \omega(x_0) + \frac{1}{\Gamma(\alpha)}\int_{t_0}^t (t-s)^{\alpha-1}(L+CT)\cdot z(t)\, ds. \end{aligned} \tag{4.4}$$

By the Gronwall-Bellman inequality, it follows that

$$\omega(x(t)) \le z(t) \le \omega(x_0)\cdot e^{\frac{L+CT}{\Gamma(\alpha)}\int_{t_0}^t (t-s)^{\alpha-1}\, ds}.$$

From the definition of $\alpha_{t_0}^+$, we have

$$\int_{t_0}^{\alpha_{t_0}^+} (\alpha_{t_0}^+ - t)^{\alpha-1}\, ds \le \tilde{M}_{x_0} = \frac{\Gamma(\alpha)}{L+CT}\ln\frac{M}{\omega(x_0)}.$$

which implies

$$\omega(x_0)\cdot e^{\frac{L+CT}{\Gamma(\alpha)}\int_{t_0}^{\alpha_{t_0}^+} (\alpha_{t_0}^+ - s)^{\alpha-1}\, ds} \le M.$$

Therefore,

$$\omega(x(t)) \le \omega(x_0)\cdot e^{\frac{L+CT}{\Gamma(\alpha)}\int_{t_0}^t (t-s)^{\alpha-1}\, ds} \le \omega(x_0)\cdot e^{\frac{L+CT}{\Gamma(\alpha)}\int_{t_0}^{\alpha_{t_0}^+} (\alpha_{t_0}^+ - s)^{\alpha-1}\, ds} \le M.$$

holds for every $t \in [t_0, \alpha_{t_0})$

The proof of (ii) follows similarly. □

Now, we consider problem (4.1), $0 < \omega(x_0) < M$. Moreover, we introduce the following hypotheses.

(H2)$\int_0^T (T-s)^{\alpha-1}\, ds > \tilde{M}_{x_0}$.

(H2')$\int_{-T}^0 (s+T)^{\alpha-1}\, ds > \tilde{M}_{x_0}$.

(H3)$H(f(t,u),\{0\}) \le \sigma H(u,\{0\}), H(g(t,u),\{0\}) \le N$, for all $u \in K_C$.

4.1. **Existence of forward solution on** $[0,T]$**.** By Lemma 4.1(i), if $\int_{t_0}^{T}(T-s)^{\alpha-1}\,ds \le \tilde{M}_{x_0}$ and (H1) hold, then the forward solution to problem (4.1), if exists, satisfies $\omega(x(t)) < M$ and is $[gH]_{\alpha}^{C}$-differentiable on $[0,T]$.

If (H1) and (H2) hold, then the forward solution to problem (4.1) on $[0,T]$, if it exists, may exhibit one or more impulses to ensure that the length constraint is maintained. Let the impulsive condition be defined as

$$x(t_k^+) = I(x(t_k)), k = 1,2,\ldots,n, \tag{4.5}$$

where $I : K_C \mapsto K_C$ is a continuous mapping satisfying $\omega(I(u)) < M$ for every $u \in K_C$, $t_0 = 0, t_k = P_{t_{k-1}}^{-1}(\tilde{M}_{I(x(t_{k-1}))})$ for $1 \le k \le n$ and $t_{n+1} = T$. For the value of n, we have the following lemma.

Lemma 4.2. *Suppose that (H1) and (H2) are satisfied and that* $\omega(I(u)) = \omega(x_0)$ *holds for every* $u \in K_C$*. Then* $n = \left[\frac{\sum_{k=1}^{n+1}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}\,ds}{\tilde{M}_{x_0}}\right]$ *and* $t_k = P_{t_{k-1}}^{-1}(\tilde{M}_{x_0})$ *for* $1 \le k \le n$.

Proof. By $\omega(I(u)) = \omega(x_0)$, we have that $\omega(I(x(t_{k-1}))) = \omega(x_0)$ for each $1 \le k \le n$. Thus,

$$\tilde{M}_{I(x(t_{k-1}))} = \frac{\Gamma(\alpha)}{L+CT}\ln\frac{M}{\omega(I(x(t_{k-1})))} = \frac{\Gamma(\alpha)}{L+CT}\ln\frac{M}{\omega(x_0)} = \tilde{M}_{x_0}, 1 \le k \le n.$$

And by Lemma 4.1, we obtain $\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}\,ds = \tilde{M}_{x_0}$ for $k = 1,2,\ldots,n$ and $\int_{t_n}^{T}(T-s)^{\alpha-1}\,ds \le \tilde{M}_{x_0}$. Therefore, $t_k = P_{t_{k-1}}^{-1}(\tilde{M}_{x_0})$ for $k = 1,2,\ldots,n$, where $n = \left[\frac{\sum_{k=1}^{n+1}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}\,ds}{\tilde{M}_{x_0}}\right]$. □

Under the assumptions of Lemma 4.2, the number of impulses n is fixed for a given M. Thus, the following theorem establishes the existence of a forward solution to problem (4.1).

Theorem 4.3. *If (H1) and (H3) are both satisfied, then problem* (4.1) *has at least one forward solution satisfying* $\omega(x(t)) \le M$ *on* $[0,T]$.

Proof. If $\int_{t_0}^{T}(T-s)^{\alpha-1}\,ds \le \tilde{M}_{x_0}$ holds, Theorem 3.1 guarantees that problem (3.1) with $n = 0$ has at least one forward solution on $[0,T]$, thus problem (4.1) has a forward solution.

Suppose $\int_{t_0}^{T}(T-s)^{\alpha-1}\,ds > \tilde{M}_{x_0}$ holds. In this case, we let $I : K_C \to K_C$ be a continuous mapping satisfying $I(u) \subseteq u$ and $\omega(I(u)) = \omega(x_0)$ for every $u \in K_C$. Lemma 4.2 guarantees the value of n, then by Theorem 3.1, problem (4.1)-(4.5) has at least one forward solution. Finally, the definition of t_k and Lemma 4.1 imply that the forward solution satisfies $\omega(x(x)) \le M$ on $[0,T]$. □

Example 4.1. we consider the following equation

$$\begin{cases} ({}^{C}D_{0^+}^{0.5}x)(t) = \frac{1}{5}\cdot x(t), t \in [0, 2\pi], \\ x(0) = [1, 4]. \end{cases} \tag{4.6}$$

Firstly, $\omega(f(t, x(t))) = |\frac{1}{5}| \cdot \omega(x(t)) \leq \frac{1}{5} \cdot \omega(x(t))$ and $g(t, x(t)) = 0$, thus (H1) holds. Besides,
$H(f(t, x(t)), \{0\}) \leq \frac{1}{5} \cdot H(x(t), \{0\})$, $H(g(t, x(t)), \{0\}) \leq N = 0$, which imply (H3) holds. By Theorem 4.3, problem (4.6) has at least one forward solution satisfying $\omega(x(t)) \leq M$ on $[0, 2\pi]$.

To get more details, we calculate that $\int_0^{2\pi}(2\pi - s)^{-0.5}\, ds = 2\sqrt{2\pi}$, if $M \geq 3e^{\frac{2\sqrt{2\pi}}{5}}$, then $\int_0^{2\pi}(2\pi - s)^{-0.5}\, ds \leq \tilde{M}_{x_0}$. Based on the proof of Theorem 4.3, problem (4.6) has at least one forward solution satisfying $\omega(x(t)) \leq M$ on $[0, 2\pi]$ without impulse.

If M=4, $\tilde{M}_{x_0} = \frac{\Gamma(\alpha)}{L+CT} \ln \frac{M}{\omega(x_0)} = 5\sqrt{\pi} ln\left(\frac{4}{3}\right) \leq 2\sqrt{2\pi}$, so (H2) holds. By Theorem 4.3, problem (4.6) has at least one forward solution satisfying $\omega(x(t)) \leq M$ on $[0, 2\pi]$ with three impulses, $t_1 = \left(\frac{5\sqrt{\pi} ln(\frac{4}{3})}{2}\right)^2 \approx 1.625$,
$t_2 \approx 3.250$ and $t_3 \approx 4.875$, where we set $I(u) = [u_-, u_- + \omega(x_0)]$.

4.2. **Existence of forward solution on $[-T, 0]$.** We consider the backward solution of problem (4.2), if $\int_{-T}^{t_0}(s + T)^{\alpha-1}\, ds \leq \tilde{M}_{x_0}$ and (H1) hold, then Lemma 4.1(ii) guarantees that the backward solution to problem (4.2), if exists, satisfies $\omega(x(t)) < M$ and is $[gH]_\alpha^C$-differentiable on $[-T, 0]$.

If (H1) and (H2') are both satisfied, Let $I : K_C \to K_C$ is a continuous mapping satisfying $\omega(I(u)) < M$ for every $u \in K_C$, and let $t_0 = 0, t_k = Q_{t_{k-1}}^{-1}(\tilde{M}_{I(x(t_{k-1}))})$ for $1 \leq k \leq m$ and $t_{m+1} = -T$. Then Lemma 4.1(ii) guarantees that the backward solution to problem (4.2) with the impulsive condition

$$x(t_k^-) = I(x(t_k)), k = 1, 2, \ldots, m, \tag{4.7}$$

if it exists, satisfies $\omega(x(t)) < M$ on each (t_k, t_{k-1}) for $k = 1, 2, \ldots, m + 1$. Analogous to Lemma 4.2, we present the following result on the value of m without proof.

Lemma 4.4. *Suppose that (H1) and (H2) are satisfied, and that $\omega(I(u)) = \omega(x_0)$ for every $u \in K_C$. Then $t_k = Q_{t_{k-1}}^{-1}(\tilde{M}_{x_0})$ for $1 \leq k \leq m$ and $m = \left[\frac{\sum_{k=1}^{n+1}\int_{t_k}^{t_{k-1}}(s-t_k)^{\alpha-1}\, ds}{\tilde{M}_{x_0}}\right]$.*

Using the same reasoning as in the proof of Theorem 4.3, we can establish the following result.

Theorem 4.5. *If (H1) and (H3) are both satisfied, then problem* (4.2) *has at least one backward solution satisfying $\omega(x(t)) \leq M$ on $[-T, 0]$.*

Example 4.2. Consider the following example

$$\begin{cases} ({}^{C}D_{0^-}^{0.5}x)(t) = \frac{1}{5}\cdot x(t), t \in [-2\pi, 0], \\ x(0) = [1, 4]. \end{cases} \tag{4.8}$$

From Example 4.1, conditions (H1) and (H3) are satisfied. By Theorem 4.5, problem (4.8) has at least one backward solution satisfying $\omega(x(t)) \leq M$ on $[-2\pi, 0]$.

If $M \geq 3e^{\frac{2\sqrt{2\pi}}{5}}$, then $\int_{-2\pi}^{0}(s+2\pi)^{-0.5}\,ds \leq \tilde{M}_{x_0}$. Hence by Theorem 4.5, problem (4.8) has at least one backward solution satisfying $\omega(x(t)) \leq M$ on $[-2\pi, 0]$ without impulse.

If M=4, then $\tilde{M}_{x_0} = \frac{\Gamma(\alpha)}{L+CT}\ln\frac{M}{\omega(x_0)} = 5\sqrt{\pi}ln\left(\frac{4}{3}\right) \leq 2\sqrt{2\pi}$, and $\int_{-2\pi}^{0}(s+2\pi)^{-0.5}\,ds = 2\sqrt{2\pi}$, so $(H2')$ holds. By Theorem 4.5, problem (4.8) has at least one forward solution satisfying $\omega(x(t)) \leq M$ on $[-2\pi, 0]$ with three impulses: $t_1 = -\left(\frac{5\sqrt{\pi}ln(\frac{4}{3})}{2}\right)^2 \approx -1.625, t_2 \approx -3.250$ and $t_3 \approx -4.875$, where we set $I(u) = [u_-, u_- + \omega(x_0)]$.

5. Conclusions and future works

This paper establishes the existence of solutions for a class of impulsive interval-valued Caputo fractional integro-differential equations under length constraints within the short-memory fractional derivative framework. By adopting the short-memory Caputo derivative, the system memory is locally reset at each impulse instant, which effectively resolves the ill-posedness caused by the disruption of long-memory kernels in classical impulsive formulations. This modification ensures the mathematical well-posedness of the model while retaining the essential memory characteristics of fractional dynamics.

A hybrid impulse-integral regulatory mechanism was developed to combine discrete impulses with continuous integral perturbations, thereby smoothing the system evolution and enhancing stability. Using Schauder's fixed point theorem, the existence of both forward and backward solutions was rigorously established. Furthermore, by incorporating bounded length constraints, we provided existence theorems that guarantee the admissibility of solutions within a physically meaningful range. Illustrative examples validated the theoretical results and demonstrated the applicability of the proposed approach to systems exhibiting both uncertainty and limited memory effects.

Future research can extend the present framework in several directions. First, generalizing the length constraints to time-dependent or state-dependent forms, such as $\omega(x(t)) \leq M(t)$ or $\omega(x(t)) \leq \psi(x(t))$, would enhance the modeling flexibility for adaptive systems. Second, analyzing the stability of solutions, including Ulam-Hyers and Lyapunov-type stability, would provide deeper insights into the robustness of hybrid fractional systems. Finally, incorporating stochastic or fuzzy fractional operators into the short-memory framework could further broaden its applications in fields such as finance, biology, and control engineering.

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