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ABSOLUTE GENERALIZED MEANS, AND THEIR ASSOCIATED SERIES SPACE DERIVED FROM MADDOX'S SPACE $\ell(p)$

CANAN HAZAR GÜLEÇ

Abstract. Mursaleen and Noman [22, 23] developed a comprehensive framework for sequence spaces that generalized many well-known sequence spaces. Motivated by their work, in the present study, we introduce a new paranormed series space $\left|\bar{A}_{(r,s,t,u)}\right|(p)$ over the Maddox's space $\ell(p)$ using absolute generalized means method, where $p=(p_k)$ is a bounded sequence of positive real numbers. Also, we examine various topological properties of this space and determine its α-, β-, and γ- duals. Furthermore, we characterize the matrix classes $(\left|\bar{A}_{(r,s,t,u)}\right|(p),\lambda)$ and $(\lambda,\left|\bar{A}_{(r,s,t,u)}\right|(p))$ for a given sequence space λ.

1. Introduction

A prominent research direction in the theory of sequence spaces involves generating new spaces through the matrix domains of triangle matrices. This approach has been extensively explored using classical means such as the weighted, Cesàro, Jordan totient, and Nörlund means [8, 9, 10, 11, 12, 13, 14, 28, 29, 30]. The reader can refer to the articles [1, 2, 3, 4, 6, 26, 32] in order to the reader is aware of some new results on some new paranormed spaces obtained certain triangles that isomorphic to the Maddox's spaces. The standard analysis of such spaces focuses on three pillars: the investigation of topological properties, the characterization of matrix transformations, and the examination of inclusion relations. A significant contribution to this field was made by Mursaleen and Noman [22, 23], who developed a comprehensive study for generalized sequence spaces that successfully generalized many well-known sequence spaces.

The study of sequence spaces forms a cornerstone of functional and modern analysis, with critical applications spanning summability theory, operator theory, and approximation theory. The pursuit of greater generality and structural flexibility within this field naturally leads to the framework of paranormed sequence spaces. Unlike their normed counterparts, paranormed spaces have more general

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properties as a generalization of the normed spaces. In addition, in order to expand these studies, some new paranormed spaces have been defined and some of their properties have been investigated [15, 16, 17, 34, 35, 36].

Motivated by these developments, this study aims to further enrich the field by introducing a novel paranormed series space, denoted $\left|\bar{A}_{(r,s,t,u)}\right|(p)$. This space is constructed by applying absolute generalized means to the paranormed space $\ell(p)$, where $p=(p_k)$ is a bounded sequence of positive real numbers.

Any vector subspace of ω, the space of all complex sequences, is called a sequence space. Let ℓ_∞, c and c_0 denote the sets of all bounded, convergent and null sequences, respectively. The space ℓ_p consists of all absolutely p-summable sequences for $1 \leq p < \infty$. Additionally, bs and cs denote the spaces of all bounded and convergent series, respectively.

A linear topological space X over the real field $\mathbb{R}$ is said to be a paranormed space if there is a subadditive function $g : X \to \mathbb{R}$ such that $g(\theta)=0$, $g(x)=g(-x)$ and scalar multiplication is continuous, i.e.,

$$|\alpha_n - \alpha| \to 0 \text{ and } g(x_n - x) \to 0 \text{ imply } g(\alpha_n x_n - \alpha x) \to 0$$

for all $\alpha^{'}$s in $\mathbb{R}$ and all $x^{'}$s in X, where θ is the zero vector in the linear space X.

Throughout this paper, (p_k) denotes a bounded sequence of strictly positive real numbers such that $H=\sup_k p_k$ and $M=\max\{1,H\}$. The linear space $\ell(p)$ was defined by Maddox [19, 20] (see also Nakano [25] and Simons [31]) as follows:

$$\ell(p)=\left\{x=(x_k)\in\omega : \sum_k |x_k|^{p_k} < \infty\right\}, (0<p_k\leq H<\infty)$$

which is a complete space under the paranorm given by

$$g(x)=\left(\sum_k |x_k|^{p_k}\right)^{1/M}.$$

Furthermore, we assume throughout that $p_k^{-1}+(q_k)^{-1}=1$ provided $0<\inf p_k \leq H<\infty$, with $1/q_k=0$ when $p_k=1$, and $\mathcal{F}$ will denote the collection of all finite subsets of $\mathbb{N}$, where $\mathbb{N}=\{0,1,2,...\}$.

Let X and Y be two sequence spaces and $A=(a_{nk})$ be an infinite matrix of real numbers. Then, A defines a matrix transformation from X into Y, if for every sequence $x=(x_k)\in X$, the sequence $Ax=(A_n(x))$, the A-transform of x, is in Y, where

$$A_n(x)=\sum_{k=0}^{\infty} a_{nk}x_k$$

provided that the series is convergent for each $n\in\mathbb{N}$. The class of all such matrices is denoted by (X,Y). Also, we write A_n for the sequence in the n-th row of A, that is, $A_n=(a_{nk})_{k=1}^{\infty}$ for every $n\in\mathbb{N}$. Central to the study of these transformations are the α-, β- and γ- duals of a sequence space X. These are defined, respectively, as $X^\alpha=S(X,\ell_1)$, $X^\beta=S(X,cs)$ and $X^\gamma=S(X,bs)$, where

$$S(X,Y)=\{u=(u_k)\in\omega : xu=(x_ku_k)\in Y \text{ for all } x\in X\}.$$

The matrix domain of an infinite matrix A in a sequence space X is defined by

$$X_A=\{x\in\omega : Ax\in X\} \tag{1.1}$$

which is also a sequence space.

An infinite matrix $A = (a_{nv})$ is called a triangle if $a_{nn} \neq 0$ and $a_{nv} = 0$ for all $n, v \in \mathbb{N}$ with $v > n$ [33].

To provide the necessary foundation for our investigation, we begin by recalling the concept of generalized means.

For any sequences $s, t \in \omega$, their convolution $s * t$ is a sequence defined by

$$(s * t)_n = \sum_{k=0}^{n} s_{n-k} t_k; \ \ (n \in \mathbb{N}).$$

Let Λ and Λ_0 be the sets of all sequences with non-zero terms and non-zero first terms, respectively, that is,

$$\Lambda = \{a = (a_k) \in \omega : a_k \neq 0 \text{ for all } k \in \mathbb{N}\}$$

and

$$\Lambda_0 = \{a = (a_k) \in \omega : a_0 \neq 0 \ \}.$$

Throughout this work, we assume that $t = (t_n), r = (r_n) \in \Lambda$, $s = (s_n) \in \Lambda_0$ and $u = (u_n)$ denotes a sequence of positive terms. Also, a negative subscript denotes zero.

The infinite matrix $\bar{A}(r, s, t)$ of generalized means has been defined by

$$\left(\bar{A}(r, s, t)\right)_{nk} = \begin{cases} \dfrac{s_{n-k} t_k}{r_n}, \ 0 \leq k \leq n, \\ 0, \ k > n \end{cases}$$

for all $n, k \in \mathbb{N}$ [22, 23].

Also, the matrix $\bar{A}(r, s, t)$ has a unique inverse matrix $B = (b_{nk})_{n,k=0}^{\infty}$ that is defined by

$$b_{nk} = \begin{cases} (-1)^{n-k} D_{n-k}^{(s)} \dfrac{r_k}{t_n}, \ 0 \leq k \leq n, \\ 0, \ k > n \end{cases}$$

where $D_0^{(s)} = 1/s_0$ and

$$D_n^{(s)} = \frac{1}{s_0^{n+1}} \begin{vmatrix} s_1 & s_0 & 0 & 0 & ... & 0 \\ s_2 & s_1 & s_0 & 0 & ... & 0 \\ s_3 & s_2 & s_1 & s_0 & ... & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ s_{n-1} & s_{n-2} & s_{n-3} & s_{n-4} & ... & s_0 \\ s_n & s_{n-1} & s_{n-2} & s_{n-3} & ... & s_1 \end{vmatrix}; (n = 1, 2, 3, ...).$$

Building on the theory of generalized means, Mursaleen and Noman [22, 23] introduced the sequence spaces

$$X(r, s, t) = \left\{ x \in \omega \ : \ \overline{x} = \left(\frac{1}{r_n} \sum_{v=0}^{n} s_{n-v} t_v x_v \right)_{n=0}^{\infty} \in X \right\}$$

for arbitrary subset X of ω. Since generalized means subsume many significant classical means, these spaces include several known sequence spaces in the special cases. After exploring the topological structure of these spaces, they also characterized the relevant compact operators and matrix mappings. The reader can refer to the recent textbooks [5, 24] for some new results on normed and paranormed sequence

spaces derived from certain triangle matrices, as well as fundamental theorems in functional analysis, summability theory, and sequence spaces.

2. The Paranormed Series Space $\left|\bar{A}_{(r,s,t,u)}\right|(p)$

The main purpose of this study is to construct a new paranormed space of infinite series $\left|\bar{A}_{(r,s,t,u)}\right|(p)$ over the paranormed space $\ell(p)$ using the absolute generalized means summability method, where $p=(p_k)$ is a bounded sequence of positive real numbers.

For a sequence $x=(x_v)\in\omega$, let $z=(z_n)$ denote the sequence of partial sums of the infinite series $\sum x_n$. Then, the generalized mean transform $\bar{A}(z)=\left(\bar{A}_n(z)\right)$ of the sequence $z=(z_n)$ is defined by the convolution:

$$\bar{A}_n(z)=\frac{1}{r_n}\sum_{k=0}^{n}s_{n-k}t_kz_k$$

where $t=(t_n),r=(r_n)\in\Lambda$ and $s=(s_n)\in\Lambda_0$.

Now, we define the series space $\left|\bar{A}_{(r,s,t,u)}\right|(p)$ as the set of all series summable by the absolute generalized means summability method $\left|\bar{A}_{(r,s,t)},u_n\right|(p)$, that is,

$$\left|\bar{A}_{(r,s,t,u)}\right|(p)=\left\{x=(x_v)\in\omega:\sum_{n=0}^{\infty}u_n^{p_n-1}\left|\Delta\bar{A}_n(z)\right|^{p_n}<\infty\right\},(0<p_n\leq H<\infty),$$

where $\Delta\bar{A}_n(z)=\bar{A}_n(z)-\bar{A}_{n-1}(z)$ and $\bar{A}_{-1}(z)=0$ for all $n\in\mathbb{N}$, using definition of Sargl [27].

Since the sequence $z=(z_n)$ is partial sums of infinite series of $\sum x_n$, we can write the generalized means $\bar{A}(z)=\left(\bar{A}_n(z)\right)$ by

$$\bar{A}_n(z)=\frac{1}{r_n}\sum_{k=0}^{n}s_{n-k}t_kz_k=\frac{1}{r_n}\sum_{j=0}^{n}x_j\sum_{k=j}^{n}s_{n-k}t_k,$$

then $\left|\bar{A}_{(r,s,t,u)}\right|(p)$ series space can be restated in terms of the sequence $x=(x_v)$ as follows:

$$\left|\bar{A}_{(r,s,t,u)}\right|(p)=\left\{x\in\omega:\sum_{n=1}^{\infty}u_n^{p_n-1}\left|\sum_{j=0}^{n}\left(\sum_{v=j}^{n}\left(\frac{s_{n-v}}{r_n}-\frac{s_{n-v-1}}{r_{n-1}}\right)t_\nu\right)x_j\right|^{p_n}<\infty\right\}.$$

If we define $\Phi(x)=(\Phi_n(x))$ by

$$\Phi_n(x)=u_n^{1/q_n}\sum_{j=0}^{n}\left(\sum_{v=j}^{n}\left(\frac{s_{n-v}}{r_n}-\frac{s_{n-v-1}}{r_{n-1}}\right)t_\nu\right)x_j,\ n\geq1\text{ and }\Phi_0(x)=u_0^{1/q_0}\frac{s_0t_0}{r_0}x_0$$

then, it can be written that the series $\sum x_n$ is summable by $\left|\bar{A}_{(r,s,t)},u_n\right|(p)$ if and only if $x=(x_v)\in\left|\bar{A}_{(r,s,t,u)}\right|(p)$, or equivalently $(\Phi_n(x))\in\ell(p)$, where $s_{-1}=0$.

If $p_n=p$ for every $n\in\mathbb{N}$, the space $\left|\bar{A}_{(r,s,t,u)}\right|(p)$ is reduced to the space $\left|\bar{A}_{(r,s,t,u)}\right|_p$ for $1\leq p<\infty$.

A powerful characterization of this space can be achieved through of matrix domains. This is facilitated by defining two crucial matrices. The first is the

matrix $\tilde{A} = \tilde{A}(r,s,t) = \left(\tilde{a}_{nj}^{(r,s,t)}\right)$ defined by

$$\tilde{a}_{nj}^{(r,s,t)} = \begin{cases} \frac{1}{r_n}\sum_{v=j}^{n} s_{n-v}t_\nu, 0 \le j \le n \\ 0, j > n, \end{cases} \tag{2.1}$$

and the second is the matrix $F(p) = (f_{nv}(p))$ defined by

$$f_{nv}(p) = \begin{cases} -u_n^{1/q_n}, \ v = n-1 \\ u_n^{1/q_n}, \qquad v = n \\ 0, \quad \text{otherwise}, \end{cases} \tag{2.2}$$

where q_n denotes the conjugate exponent satisfying $1/p_n + 1/q_n = 1$, with $1/q_n = 0$ when $p_n = 1$. Equipped with these definitions, the series space $\left|\bar{A}_{(r,s,t,u)}\right|(p)$ admits representation as the matrix domain of the composite matrix in the base space $\ell(p)$. This is expressed by the identity

$$\left|\bar{A}_{(r,s,t,u)}\right|(p) = (\ell(p))_{F(p)\tilde{A}}.$$

using (1.1).

We define an associated sequence $\varphi = (\varphi_n)$ as the image of the partial sums sequence z under the $\bar{A}$ transform, given by

$$\bar{A}_n(z) = \varphi_n = \frac{1}{r_n}\sum_{j=0}^{n} x_j \sum_{k=j}^{n} s_{n-k}t_k,$$

then the inverse transformation is calculated by $x_0 = \frac{r_0\varphi_0}{t_0 s_0}$ and

$$x_n = \frac{r_n\varphi_n}{t_n s_0} + \sum_{k=0}^{n-1}\left(\frac{(-1)^{n-k}D_{n-k}^{(s)}r_k}{t_n} - \frac{(-1)^{n-k-1}D_{n-k-1}^{(s)}r_k}{t_{n-1}}\right)\varphi_k \ , \ n \ge 1 \ . \tag{2.3}$$

From (2.3), the inverse matrix $\tilde{B} = \left(\tilde{b}_{nk}\right)_{n,k=0}^{\infty}$ is given by

$$\tilde{b}_{nk} = \begin{cases} \frac{r_0}{t_0 s_0}, & n = 0, \\ \frac{(-1)^{n-k}D_{n-k}^{(s)}r_k}{t_n} - \frac{(-1)^{n-k-1}D_{n-k-1}^{(s)}r_k}{t_{n-1}}, & 0 \le k \le n-1, \ (n \ge 1) \\ \frac{r_n}{t_n s_0}, & k = n, \ (n \ge 1) \\ 0, & k > n. \end{cases} \tag{2.4}$$

So, by means of the inverse matrix $\tilde{B} = \left(\tilde{b}_{nk}\right)_{n,k=0}^{\infty}$ we can write that

$$x_n = \sum_{k=0}^{n} \tilde{b}_{nk}\varphi_k. \tag{2.5}$$

Given this relationship, series space $\left|\bar{A}_{(r,s,t,u)}\right|(p)$ can be redefined by all sequences whose $F(p)\tilde{A}$ transform belongs to the sequence space $\ell(p)$. This means $x = (x_n) \in \left|\bar{A}_{(r,s,t,u)}\right|(p)$ if and only if $\left(F(p)\tilde{A}\right)(x) \in \ell(p)$.

We now present some significant results regarding the topological structure of the newly defined space $\left|\bar{A}_{(r,s,t,u)}\right|(p)$.

Theorem 2.1. *The series space* $\left|\bar{A}_{(r,s,t,u)}\right|(p)$ *is a complete linear metric space paranormed by* g*, defined as*

$$g(x)=\left(\sum_n\left|\left(F(p)\circ\tilde{A}\right)_n(x)\right|^{p_n}\right)^{1/M},$$

where $0<p_n\leq H<\infty$ *for all* $n\in\mathbb{N}$, $H=\sup_n p_n$ *and* $M=\max\{1,H\}$.

Proof. The linearity of the series space $\left|\bar{A}_{(r,s,t,u)}\right|(p)$ under coordinatewise addition and scalar multiplication is confirmed by the following inequalities, which hold for any $x,y\in\left|\bar{A}_{(r,s,t,u)}\right|(p)$ (see, [21], p.30):

$$\left(\sum_n\left|\left(F(p)\circ\tilde{A}\right)_n(x+y)\right|^{p_n}\right)^{1/M} \tag{2.6}$$

$$\begin{aligned}\leq\ & \left(\sum_n\left|\left(F(p)\circ\tilde{A}\right)_n(x)\right|^{p_n}\right)^{1/M}+\left(\sum_n\left|\left(F(p)\circ\tilde{A}\right)_n(y)\right|^{p_n}\right)^{1/M}\\ <\ & \infty,\end{aligned}$$

and for any $\alpha\in\mathbb{R}$ (see, [19])

$$|\alpha|^{p_k}\leq\max\left\{1,|\alpha|^M\right\}. \tag{2.7}$$

It is immediate from the definition that $g(\theta)=0$ and $g(x)=g(-x)$ for all $x\in\left|\bar{A}_{(r,s,t,u)}\right|(p)$, where θ is the zero sequence.

Furthermore, inequalities (2.6) and (2.7) directly imply the subadditivity of g, and $g(\alpha x)\leq\max\{1,|\alpha|\}g(x)$ for any $\alpha\in\mathbb{R}$.

Consider a sequence (x^n) in $\left|\bar{A}_{(r,s,t,u)}\right|(p)$ for each $n\in\mathbb{N}$ such that $g(x^n-x)\to 0$ as $n\to\infty$ and also let (α_n) be a sequence of scalars such that $\alpha_n\to\alpha$ as $n\to\infty$. The subadditivity property of g yields the inequality

$$g(x^n)\leq g(x)+g(x-x^n),$$

from which it follows that the sequence $\{g(x^n)\}$ is bounded. So, we observe

$$\begin{aligned}g\left(\alpha_k x^k-\alpha x\right) &= \left(\sum_n\left|\left(F(p)\circ\tilde{A}\right)_n\left(\alpha_k x^k-\alpha x\right)\right|^{p_n}\right)^{1/M}\\ &\leq \left\{|\alpha_k-\alpha|\,g\left(x^k\right)+|\alpha|\,g\left(x^k-x\right)\right\}\end{aligned}$$

which tends to zero as $k\to\infty$. This satisfies that the scalar multiplication is continuous. Hence g is a paranorm on the space $\left|\bar{A}_{(r,s,t,u)}\right|(p)$.

We now proceed to demonstrate that the space $\left|\bar{A}_{(r,s,t,u)}\right|(p)$ is complete. Let $\left(x^i\right)$ be an arbitrary Cauchy sequence in $\left|\bar{A}_{(r,s,t,u)}\right|(p)$, where each term is given by $x^i=\left\{x_0^{(i)},x_1^{(i)},x_2^{(i)},...\right\}$ for $i\in\mathbb{N}$. Then, for any $\varepsilon>0$, there exists a positive integer $n_0(\varepsilon)$ such that

$$g\left(x^i-x^j\right)<\varepsilon \text{ for all } i,j\geq n_0(\varepsilon).$$

From the definition of g, it follows for each fixed $n\in\mathbb{N}$ that

$$\left|\left(F(p)\circ\tilde{A}\right)_n\left(x^i\right)-\left(F(p)\circ\tilde{A}\right)_n\left(x^j\right)\right| \tag{2.8}$$

$$\leq \left(\sum_{n}\left|\left(F(p)\circ\tilde{A}\right)_{n}\left(x^{i}\right)-\left(F(p)\circ\tilde{A}\right)_{n}\left(x^{j}\right)\right|^{p_{n}}\right)^{1/M}<\varepsilon,$$

for all $i,j\geq n_{0}\left(\varepsilon\right)$. This implies that the sequence $\left\{\left(F(p)\circ\tilde{A}\right)_{n}\left(x^{0}\right),\left(F(p)\circ\tilde{A}\right)_{n}\left(x^{1}\right),\left(F(p)\circ\tilde{A}\right)_{n}\left(x^{2}\right)...\right\}$ is a Cauchy sequence in $\mathbb{R}$ for every fixed $n\in\mathbb{N}$. Since $\mathbb{R}$ is complete, it converges for every $n\in\mathbb{N}$, and we may write that

$$\lim_{i\to\infty}\left(F(p)\circ\tilde{A}\right)_{n}\left(x^{i}\right)=\left(F(p)\circ\tilde{A}\right)_{n}\left(x\right).$$

With the help of these infinitely many limits $\left(F(p)\circ\tilde{A}\right)_{0}\left(x\right),\left(F(p)\circ\tilde{A}\right)_{1}\left(x\right),\left(F(p)\circ\tilde{A}\right)_{2}\left(x\right),...$, we construct the sequence $\left\{\left(F(p)\circ\tilde{A}\right)_{0}\left(x\right),\left(F(p)\circ\tilde{A}\right)_{1}\left(x\right),\left(F(p)\circ\tilde{A}\right)_{2}\left(x\right),...\right\}$. From inequality (2.8), we can deduce that

$$\sum_{n=0}^{m}\left|\left(F(p)\circ\tilde{A}\right)_{n}\left(x^{i}\right)-\left(F(p)\circ\tilde{A}\right)_{n}\left(x^{j}\right)\right|^{p_{n}}\leq g\left(x^{i}-x^{j}\right)^{M}<\varepsilon^{M}\qquad(2.9)$$

for each $m\in\mathbb{N}$ and $i,j\geq n_{0}\left(\varepsilon\right)$. Take any $i\geq n_{0}\left(\varepsilon\right)$. If we take to limit first as $j\to\infty$ and next as $m\to\infty$ in (2.9), then, we get $g\left(x^{i}-x\right)<\varepsilon$. To establish that the limit point x belongs to the space, take $\varepsilon=1$ in (2.9) and consider any $i\geq n_{0}\left(1\right)$. Applying Minkowski's inequality for each fixed $m\in\mathbb{N}$ yields

$$\left(\sum_{n=0}^{m}\left|\left(F(p)\circ\tilde{A}\right)_{n}\left(x\right)\right|^{p_{n}}\right)^{1/M}\leq g\left(x^{i}-x\right)+g\left(x^{i}\right)\leq 1+g\left(x^{i}\right)$$

which shows that $x\in\left|\bar{A}_{(r,s,t,u)}\right|\left(p\right)$. Since $g\left(x^{i}-x\right)<\varepsilon$ for all $i\geq n_{0}\left(\varepsilon\right)$, we have that $x^{i}\to x$ as $i\to\infty$. This concludes that $\left|\bar{A}_{(r,s,t,u)}\right|\left(p\right)$ is complete.

□

Since both matrices $F(p)$ and $\tilde{A}=\tilde{A}\left(r,s,t\right)$ are triangle, the matrix $F(p)\tilde{A}=F(p)\circ\tilde{A}$ obtained by their composition is also triangle, where the matrices $F(p)$ and $\tilde{A}=\tilde{A}\left(r,s,t\right)$ are defined as in (2.2) and (2.1), respectively. Therefore, the matrix $F(p)\tilde{A}$ defines a bijective linear operator between the spaces $\left|\bar{A}_{(r,s,t,u)}\right|\left(p\right)$ and $\ell\left(p\right)$. This leads us to the conclusion that the spaces $\left|\bar{A}_{(r,s,t,u)}\right|\left(p\right)$ and $\ell\left(p\right)$ are paranorm isomorphic.

Let (X,g) be a paranormed space. A sequence (b_{k}) of the elements of X is called a basis for a sequence space X paranormed by g if and only if, for each $x\in X$, there exists a unique sequence (λ_{n}) of scalars such that

$$\lim_{n\to\infty}g\left(x-\sum_{k=0}^{n}\lambda_{k}b_{k}\right)=0\ ,$$

and in this case, we write $x=\sum_{k=0}^{\infty}\lambda_{k}b_{k}$.

Given the established linear isomorphism $\left|\bar{A}_{(r,s,t,u)}\right|\left(p\right)\cong\ell\left(p\right)$, the Schauder basis of the space $\left|\bar{A}_{(r,s,t,u)}\right|\left(p\right)$ can be obtained by taking the inverse image of the basis of $\ell\left(p\right)$. This directly leads to the following result, which is stated without proof.

Theorem 2.2. *Let* $1 < p_n \leq H < \infty$ *and* $y_j = \left(\left(F(p) \circ \tilde{A}\right) x\right)_j$, *for all* $j \in \mathbb{N}$ *and define the sequence* $v^{(j)} = \left(v_n^{(j)}\right)$ *as*

$$v_n^{(j)} = \begin{cases} u_j^{-1/q_j} \sum_{k=j}^{n} \tilde{b}_{nk}, 0 \leq j \leq n \\ 0, \qquad j > n. \end{cases}$$

The sequence $v^{(j)}$ *is a basis for the space* $\left|\bar{A}_{(r,s,t,u)}\right|(p)$ *and any* $x \in \left|\bar{A}_{(r,s,t,u)}\right|(p)$ *has a unique representation of the form*

$$x = \sum_{j=0}^{\infty} y_j v^{(j)}.$$

3. Dual spaces and some matrix transformations

This section is dedicated to the computation of the α-, β-, and γ- duals of the space $\left|\bar{A}_{(r,s,t,u)}\right|(p)$ and characterizations of certain matrix operators on the series space $\left|\bar{A}_{(r,s,t,u)}\right|(p)$. Before presenting the main theorems concerning these dual spaces, we give some lemmas that are required for the proofs.

Lemma 3.1. [7] ***(i)*** *Let* $1 < p_k \leq H < \infty$ *for all* $k \in \mathbb{N}$. *Then,* $A = (a_{nk}) \in (\ell(p), \ell_1)$ *if and only if there exists an integer* $B > 1$ *such that*

$$\sup_{N \in \mathcal{F}} \sum_k \left| \sum_{n \in N} a_{nk} B^{-1} \right|^{q_k} < \infty.$$

(ii) *Let* $0 < p_k \leq 1$ *for all* $k \in \mathbb{N}$. *Then,* $A = (a_{nk}) \in (\ell(p), \ell_1)$ *if and only if*

$$\sup_{N \in \mathcal{F}} \sup_{k \in \mathbb{N}} \left| \sum_{n \in N} a_{nk} \right|^{p_k} < \infty.$$

Lemma 3.2. [18] ***(i)*** *Let* $1 < p_k \leq H < \infty$ *for all* $k \in \mathbb{N}$. *Then,* $A = (a_{nk}) \in (\ell(p), \ell_\infty)$ *if and only if there exists an integer* $B > 1$ *such that*

$$\sup_{n \in \mathbb{N}} \sum_k \left| a_{nk} B^{-1} \right|^{q_k} < \infty. \tag{3.1}$$

(ii) *Let* $0 < p_k \leq 1$ *for all* $k \in \mathbb{N}$. *Then,* $A = (a_{nk}) \in (\ell(p), \ell_\infty)$ *if and only if*

$$\sup_{n,k \in \mathbb{N}} |a_{nk}|^{p_k} < \infty. \tag{3.2}$$

Lemma 3.3. [18] *Let* $0 < p_k \leq H < \infty$ *for all* $k \in \mathbb{N}$. *Then,* $A = (a_{nk}) \in (\ell(p), c)$ *if and only if* (3.1), (3.2) *and*

$$\lim_{n \to \infty} a_{nk} = \gamma_k, (k \in \mathbb{N})$$

are satisfied.

Let $B \in \{n \in \mathbb{N} : n \geq 2\}$ and define the sets $\psi_1(p), \psi_2(p), \psi_3(p), \psi_4(p)$ and $\psi_5(p)$ as follows:

$$\psi_1(p) = \left\{ a = (a_n) \in \omega : \sup_{N \in \mathcal{F}} \sup_{j \in \mathbb{N}} \left| \sum_{n \in N} a_n \sum_{k=j}^{n} \tilde{b}_{nk} u_j^{-1/q_j} \right|^{p_j} < \infty \right\}, \tag{3.3}$$

$$\psi_2(p)=\bigcup_{B>1}\left\{a=(a_n)\in\omega:\ \sup_{N\in\mathcal{F}}\sum_j\left|\sum_{n\in N}a_n\sum_{k=j}^{n}\tilde{b}_{nk}u_j^{-1/q_j}B^{-1}\right|^{q_j}<\infty\right\}, \tag{3.4}$$

$$\psi_3(p)=\left\{a=(a_n)\in\omega:\ \sup_{m,j\in\mathbb{N}}\left|\sum_{n=j}^{m}a_n\sum_{k=j}^{n}\tilde{b}_{nk}u_j^{-1/q_j}\right|^{p_j}<\infty\right\}, \tag{3.5}$$

$$\psi_4(p)=\bigcup_{B>1}\left\{a=(a_n)\in\omega:\sup_{m\in\mathbb{N}}\sum_{j=1}^{m}\left|\sum_{n=j}^{m}a_n\sum_{k=j}^{n}\tilde{b}_{nk}u_j^{-1/q_j}B^{-1}\right|^{q_j}<\infty\right\}, \tag{3.6}$$

$$\psi_5(p)=\left\{a=(a_n)\in\omega:\ \lim_{m\to\infty}\sum_{n=j}^{m}a_n\sum_{k=j}^{n}\tilde{b}_{nk}u_j^{-1/q_j}=\gamma_j,\quad j\in\mathbb{N}\right\}. \tag{3.7}$$

Theorem 3.4. *Let the sets $\psi_1(p)$ and $\psi_2(p)$ be defined as in (3.3) and (3.4), respectively. Then, the following statements hold:*

(i) *Let $0<p_k\leq 1$ for all $k\in\mathbb{N}$. Then,*

$$\left\{\left|\bar{A}_{(r,s,t,u)}\right|(p)\right\}^{\alpha}=\psi_1(p).$$

(ii) *Let $1<p_k\leq H<\infty$ for all $k\in\mathbb{N}$. Then,*

$$\left\{\left|\bar{A}_{(r,s,t,u)}\right|(p)\right\}^{\alpha}=\psi_2(p).$$

Proof. We provide the proof for case (ii) only, as case (i) can be established through an analogous argument.

Let $p_k>1$, $x=(x_n)\in\left|\bar{A}_{(r,s,t,u)}\right|(p)$ and $a=(a_n)\in\omega$. Now, consider $\varphi=(\varphi_n)=\tilde{A}(x)$ and

$$y_n=u_n^{1/q_n}(\varphi_n-\varphi_{n-1})$$

for $n\geq 0$, $\varphi_{-1}=0$, then, $y=(y_n)\in\ell(p)$ and

$$\varphi_n=\sum_{k=0}^{n}u_k^{-1/q_k}y_k.$$

Then, by employing relation (2.5), we can write that

$$x_n=\sum_{k=0}^{n}\tilde{b}_{nk}\varphi_k=\sum_{k=0}^{n}\tilde{b}_{nk}\sum_{j=0}^{k}u_j^{-1/q_j}y_j.$$

So, we calculate that

$$a_nx_n=a_n\sum_{k=0}^{n}\tilde{b}_{nk}\sum_{j=0}^{k}u_j^{-1/q_j}y_j=\sum_{j=0}^{n}a_n\left(u_j^{-1/q_j}\sum_{k=j}^{n}\tilde{b}_{nk}\right)y_j \tag{3.8}$$

$$=\sum_{j=0}^{n}\mu_{nj}y_j=\mu_n(y),\ \ (n\in\mathbb{N})$$

where the matrix $\mu=(\mu_{nj})$ is defined by

$$\mu_{nj}=\begin{cases}a_nu_j^{-1/q_j}\sum_{k=j}^{n}\tilde{b}_{nk},\ 0\leq j\leq n,\\ 0,\quad j>n\end{cases} \tag{3.9}$$

for all $n, j \in \mathbb{N}$. Consequently, it follows directly from (3.8) that $ax = (a_n x_n) \in \ell_1$ whenever $x = (x_n) \in \left|\bar{A}_{(r,s,t,u)}\right|(p)$ if and only if $\mu y \in \ell_1$ whenever $y = (y_n) \in \ell(p)$. This implies that $a = (a_n) \in \left\{\left|\bar{A}_{(r,s,t,u)}\right|(p)\right\}^{\alpha}$ if and only if $\mu \in (\ell(p), \ell_1)$, where the matrix $\mu = (\mu_{nj})$ is defined as in (3.9). Therefore, by applying Part (i) of Lemma 3.1, we conclude that $\left\{\left|\bar{A}_{(r,s,t,u)}\right|(p)\right\}^{\alpha} = \psi_2(p)$ which completes the proof.

□

Theorem 3.5. *Let the sets $\psi_3(p)$ and $\psi_4(p)$ be defined as in (3.5) and (3.6), respectively. Then, the following statements hold:*

(i) *Let $0 < p_k \leq 1$ for all $k \in \mathbb{N}$. Then,*

$$\left\{\left|\bar{A}_{(r,s,t,u)}\right|(p)\right\}^{\gamma} = \psi_3(p).$$

(ii) *Let $1 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$. Then,*

$$\left\{\left|\bar{A}_{(r,s,t,u)}\right|(p)\right\}^{\gamma} = \psi_4(p).$$

Proof. We again restrict our proof to case (ii), as case (i) follows from a similar line of reasoning.

Let $p_k > 1$, and suppose $x = (x_n) \in \left|\bar{A}_{(r,s,t,u)}\right|(p)$ and $a = (a_n) \in \omega$. By utilizing equality (3.8), we calculate that

$$\sum_{n=0}^{m} a_n x_n = \sum_{n=0}^{m} a_n \sum_{j=0}^{n} \left(u_j^{-1/q_j} \sum_{k=j}^{n} \tilde{b}_{nk} \right) y_j \tag{3.10}$$

$$= \sum_{j=0}^{m} \sum_{n=j}^{m} a_n \left(u_j^{-1/q_j} \sum_{k=j}^{n} \tilde{b}_{nk} \right) y_j = \sum_{j=0}^{m} h_{mj} y_j = H_m(y)$$

where the matrix $H = (h_{mj})$ is defined by

$$h_{mj} = \begin{cases} \sum\limits_{n=j}^{m} a_n \left(u_j^{-1/q_j} \sum_{k=j}^{n} \tilde{b}_{nk} \right), & 0 \leq j \leq m, \\ 0, & j > m. \end{cases} \tag{3.11}$$

From relation (3.10), it follows that $ax = (a_n x_n) \in bs$ whenever $x = (x_n) \in \left|\bar{A}_{(r,s,t,u)}\right|(p)$ if and only if $Hy \in \ell_\infty$ whenever $y \in \ell(p)$. This establishes that $a = (a_n) \in \left\{\left|\bar{A}_{(r,s,t,u)}\right|(p)\right\}^{\gamma}$ if and only if $H \in (\ell(p), \ell_\infty)$, where the matrix $H = (h_{mj})$ is defined as in (3.11). Therefore, by combining the condition (3.10) with the result of Part (i) of Lemma 3.2, we conclude that $\left\{\left|\bar{A}_{(r,s,t,u)}\right|(p)\right\}^{\gamma} = \psi_4(p)$, which completes the proof.

□

Theorem 3.6. *Let the sets $\psi_3(p)$, $\psi_4(p)$ and $\psi_5(p)$ be defined as in (3.5), (3.6), and (3.7), respectively. Then, the following statements hold:*

(i) *Let $0 < p_k \leq 1$ for all $k \in \mathbb{N}$. Then,*

$$\left\{\left|\bar{A}_{(r,s,t,u)}\right|(p)\right\}^{\beta} = \psi_3(p) \cap \psi_5(p).$$

(ii) *Let $1 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$. Then,*

$$\left\{\left|\bar{A}_{(r,s,t,u)}\right|(p)\right\}^{\beta} = \psi_4(p) \cap \psi_5(p).$$

Proof. Employing an analogous approach to that used in the proof of Theorem 3.5, we observe from equality (3.10) that $ax = (a_n x_n) \in cs$ whenever $x = (x_n) \in \left|\bar{A}_{(r,s,t,u)}\right|(p)$ if and only if $Hy \in c$ whenever $y \in \ell(p)$, where the matrix $H = (h_{mj})$ is defined as in (3.11). This demonstrates that $a = (a_n) \in \left\{\left|\bar{A}_{(r,s,t,u)}\right|(p)\right\}^{\beta}$ if and only if $H \in (\ell(p), c)$. Consequently, by applying Lemma 3.3, we conclude that the β-duals of $\left|\bar{A}_{(r,s,t,u)}\right|(p)$ are given by $\left\{\left|\bar{A}_{(r,s,t,u)}\right|(p)\right\}^{\beta} = \psi_3(p) \cap \psi_5(p)$ for $0 < p_k \leq 1$ and $\left\{\left|\bar{A}_{(r,s,t,u)}\right|(p)\right\}^{\beta} = \psi_4(p) \cap \psi_5(p)$ for $1 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$, respectively.

□

We now proceed to the direct characterizations of the matrix classes $\left(\left|\bar{A}_{(r,s,t,u)}\right|(p), \lambda\right)$ and $\left(\lambda, \left|\bar{A}_{(r,s,t,u)}\right|(p)\right)$, where λ is an arbitrary sequence space.

Theorem 3.7. *Let λ be an arbitrary sequence space. Then, $A = (a_{nk}) \in \left(\left|\bar{A}_{(r,s,t,u)}\right|(p), \lambda\right)$ if and only if $A_n \in \left\{\left|\bar{A}_{(r,s,t,u)}\right|(p)\right\}^{\beta}$ for all $n \in \mathbb{N}$ and $G \in (\ell(p), \lambda)$, where the infinite matrix $G = (g_{nj})$ is defined by*

$$g_{nj} = \sum_{r=j}^{\infty} a_{nr} \left(u_j^{-1/q_j} \sum_{k=j}^{r} \tilde{b}_{rk} \right)$$

for all $n, j \in \mathbb{N}$.

Proof. Let λ be an arbitrary sequence space. Recalling that the spaces $\left|\bar{A}_{(r,s,t,u)}\right|(p)$ and $\ell(p)$ are linearly isomorphic, assume that $A \in \left(\left|\bar{A}_{(r,s,t,u)}\right|(p), \lambda\right)$ and let $y \in \ell(p)$. Define the matrix $V(p) = (v_{nk}(p))$ by $F(p) \circ \tilde{A} = V(p)$. Then, the elements of the product matrix $F(p)\left(\tilde{A}\right)$ are computed by

$$v_{nk}(p) = \sum_{j=k}^{\infty} f_{nj}(p)\, \tilde{a}_{jk} = f_{n,n-1}(p)\, \tilde{a}_{n-1,k} + f_{nn}(p)\, \tilde{a}_{nk} = u_n^{1/q_n} \left(\tilde{a}_{nk} - \tilde{a}_{n-1,k}\right). \tag{3.12}$$

Also, we calculate the elements of the matrix product $GV(p)$ by

$$\begin{aligned} (GV(p))_{nk} &= \sum_{j=k}^{\infty} g_{nj} v_{jk}(p) \\ &= \sum_{j=k}^{\infty} \sum_{i=j}^{\infty} a_{ni} \left(\sum_{r=j}^{i} \tilde{b}_{ir} u_j^{-1/q_j} \right) u_j^{1/q_j} \left(\tilde{a}_{jk} - \tilde{a}_{j-1,k}\right) \\ &= a_{nk}. \end{aligned}$$

This establishes that $GV(p)$ exists and $A_n \in \left\{\left|\bar{A}_{(r,s,t,u)}\right|(p)\right\}^{\beta}$, which implies that $G_n \in \{\ell(p)\}^{\beta}$ for each $n \in \mathbb{N}$. As a result, the transformation Gy exists for each

$y \in \ell(p)$ and we have that

$$\begin{aligned}\sum_{k=0}^{\infty} g_{nk} y_k &= \sum_{k=0}^{\infty}\sum_{j=k}^{\infty} a_{nj} \sum_{r=k}^{j} \tilde{b}_{jr} u_k^{-1/q_k} \times \left(\sum_{j=0}^{k} u_k^{1/q_k} \left(\tilde{a}_{kj} - \tilde{a}_{k-1,j} \right) x_j \right) \\ &= \sum_{k=0}^{\infty}\sum_{j=k}^{\infty} a_{nj} \left(v_{jk}(p) \right)^{-1} \times \left(\sum_{j=0}^{k} v_{kj}(p)\, x_j \right) \\ &= \sum_{k=0}^{\infty} a_{nk} x_k.\end{aligned}$$

Therefore, the relation $Gy = Ax$ holds, which demonstrates that $G \in (\ell(p), \lambda)$.

Conversely, assume that $A_n \in \left\{ \left| \bar{A}_{(r,s,t,u)} \right|(p) \right\}^{\beta}$ for all $n \in \mathbb{N}$, that $G \in (\ell(p), \lambda)$ and let $x \in \left| \bar{A}_{(r,s,t,u)} \right|(p)$. Under these conditions, the transformation Ax exists. So, we derive from the equality

$$\begin{aligned}\sum_{k=0}^{m} a_{nk} x_k &= \sum_{k=0}^{m} a_{nk} \sum_{j=0}^{k} \left(\sum_{r=j}^{k} \tilde{b}_{kr} u_j^{-1/q_j} \right) y_j \\ &= \sum_{k=0}^{m} g_{mk}^{(n)} y_k,\end{aligned}$$

where the matrix $G^{(n)} = \left(g_{mk}^{(n)} \right)$ is defined for each n by

$$g_{mk}^{(n)} = \begin{cases} a_{nj} \sum\limits_{j=k}^{m} \left(\sum\limits_{r=k}^{j} \tilde{b}_{jr} u_k^{-1/q_k} \right), & 0 \le k \le m \\ 0, & k > m \end{cases}$$

that $Ax = Gy$, as $m \to \infty$. This result confirms that $A \in \left(\left| \bar{A}_{(r,s,t,u)} \right|(p), \lambda \right)$, which completes the proof.

□

Theorem 3.8. *Let λ be an arbitrary sequence space. Then, $A \in \left(\lambda, \left| \bar{A}_{(r,s,t,u)} \right|(p) \right)$ if and only if $C \in (\lambda, \ell(p))$, where $C = (c_{nk})$ is defined by*

$$c_{nk} = \sum_{j=0}^{n} u_n^{1/q_n} \left(\tilde{a}_{nj} - \tilde{a}_{n-1,j} \right) a_{jk}.$$

Proof. Let $y \in \lambda$ and consider the partial sum

$$\sum_{k=0}^{m} c_{nk} y_k = \sum_{j=0}^{n} u_n^{1/q_n} \left(\tilde{a}_{nj} - \tilde{a}_{n-1,j} \right) \sum_{k=0}^{m} a_{jk} y_k. \tag{3.13}$$

Taking the limit as $m \to \infty$ in (3.13), we obtain the relation $(Cy)_n = \left(V(p)\,(Ay) \right)_n$, where $V(p) = (v_{nk}(p)) = F(p) \circ \tilde{A}$ is defined as in equality (3.12). Consequently, it follows that $Ay \in \left| \bar{A}_{(r,s,t,u)} \right|(p)$ whenever $y \in \lambda$ if and only if $Cy \in \ell(p)$ whenever $y \in \lambda$. This establishes the desired result and concludes the proof.

□

If $p_n = p$ for every $n \in \mathbb{N}$, we have following results, where q denotes the conjugate exponent satisfying $1/p + 1/q = 1$, with $1/q = 0$ when $p = 1$.

Corollary 3.9. *Let* $1 \leq p < \infty$ *and the matrix* $\tilde{B} = \left(\tilde{b}_{nk}\right)_{n,k=0}^{\infty}$ *is given as in* (2.4)*. In this case,*
a) $A = (a_{nk}) \in (\left|\bar{A}_{(r,s,t,u)}\right|_1, \ell_\infty)$ *if and only if*

$$\lim_{m\to\infty} \sum_{j=i}^{m} a_{nj} \sum_{k=i}^{j} \tilde{b}_{jk} \quad \textit{exists for all } n, i \in \mathbb{N}, \tag{3.14}$$

$$\sup_{m,i\in\mathbb{N}} \left| \sum_{j=i}^{m} a_{nj} \sum_{k=i}^{j} \tilde{b}_{jk} \right| < \infty \ , \ \textit{for all } n \in \mathbb{N}, \tag{3.15}$$

and

$$\sup_{n,i\in\mathbb{N}} \left| \sum_{k=i}^{\infty} a_{nk} \sum_{j=i}^{k} \tilde{b}_{kj} \right| < \infty \ . \tag{3.16}$$

b) $A = (a_{nk}) \in (\left|\bar{A}_{(r,s,t,u)}\right|_1, c)$ *if and only if* (3.14), (3.15) *and* (3.16) *hold, and*

$$\lim_{n\to\infty} \sum_{k=i}^{\infty} a_{nk} \sum_{j=i}^{k} \tilde{b}_{kj} \quad \textit{exists for all } i \in \mathbb{N}.$$

Corollary 3.10. *Let* $1 < p < \infty$ *and the matrix* $\tilde{B} = \left(\tilde{b}_{nk}\right)_{n,k=0}^{\infty}$ *is given as in* (2.4)*. In this case,*
a) $A = (a_{nk}) \in (\left|\bar{A}_{(r,s,t,u)}\right|_p, \ell_\infty)$ *if and only if* (3.14) *holds, and*

$$\sup_{m\in\mathbb{N}} \sum_{i=0}^{m} \left| u_i^{-1/q} \sum_{j=i}^{m} a_{nj} \sum_{k=i}^{j} \tilde{b}_{jk} \right|^q < \infty \quad \textit{for all } n \in \mathbb{N}, \tag{3.17}$$

and

$$\sup_{n\in\mathbb{N}} \sum_{i=0}^{\infty} \left| u_i^{-1/q} \sum_{k=i}^{\infty} a_{nk} \sum_{j=i}^{k} \tilde{b}_{kj} \right|^q < \infty \ . \tag{3.18}$$

b) $A = (a_{nk}) \in (\left|\bar{A}_{(r,s,t,u)}\right|_p, c)$ *if and only if* (3.14), (3.17) *and* (3.18) *hold, and*

$$\lim_{n\to\infty} \sum_{k=i}^{\infty} a_{nk} \sum_{j=i}^{k} \tilde{b}_{kj} \quad \textit{exists for all } i \in \mathbb{N}.$$

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Canan Hazar Güleç
Department of Mathematics, Pamukkale University, Denizli, Türkiye
E-mail address: gchazar@pau.edu.tr