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BERTRAND AND MANNHEIM CURVES OF NORMAL-INTEGRAL AND BINORMAL-INTEGRAL CURVES IN EUCLIDEAN 3-SPACE

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Abstract. This paper examines Bertrand and Mannheim curves linked to normal-integral and binormal-integral curves in Euclidean 3-space. We derive the Frenet apparatus, curvature, and torsion properties for these curves. Our results demonstrate that the separation between corresponding points is invariant for Bertrand or Mannheim curves generated from normal-integral and binormal-integral curves. Additionally, the tangent vectors of normal-integral or binormal-integral Bertrand curves maintain a constant angle, whereas Mannheim curves lack this property. We establish precise criteria determining when normal-integral and binormal-integral curves constitute Bertrand or Mannheim curves. Finally, illustrative examples validate our theoretical findings.

1. Introduction

The geometric theory of special curves in Euclidean 3-space has developed significantly since Joseph Bertrand's seminal 1850 work on curve pairs sharing common principal normals [2]. Bertrand curves, characterized by a linear relation between curvature and torsion, have inspired extensive generalizations across diverse geometric settings including Lorentzian space [2], Galilean space [23, 25], and higher dimensions [3, 18, 21]. More recently, Mannheim curves, characterized by principal normal-binormal correspondences, have emerged as important generalizations in both Euclidean and non-Euclidean geometries [1, 7, 10–12, 24, 26].

Parallel developments in integral curve theory explore solutions to differential equations tangent to prescribed vector fields. For a vector field X and parametric curve $\alpha(t)$, the condition $\alpha'(t) = X(\alpha(t))$ defines an integral curve. Foundational work by Lee [20] established key connections to smooth flows, while subsequent research expanded to include: associated Frenet curves [4], Bishop frame applications [19], higher-dimensional extensions [6, 22], type-2 Bishop frame characterizations [27], novel frame constructions [15], Darboux q-vector fields [5] and quasi frame [8, 9, 13, 14, 16, 17]

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This work synthesizes these research directions by investigating Bertrand and Mannheim curves associated with normal-integral and binormal-integral curves in $\mathbb{E}^3$. We establish fundamental properties including:

(1) Frenet apparatus, curvature, and torsion relations
(2) Invariance of separation distances for Bertrand/Mannheim pairs
(3) Characterization of constant-angle conditions
(4) Classification criteria for normal/binormal-integral curves
(5) Explicit computational examples

Our results extend the classical theory while providing new geometric insights into the interplay between integral curves and their Bertrand/Mannheim partners.

2. Preliminaries

Consider a regular smooth curve $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ parameterized by arc length s. The curve has unit speed if:

$$\langle \alpha'(s), \alpha'(s) \rangle = 1,$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product. The Frenet-Serret frame consists of orthonormal vectors:

(1) Tangent vector: $T(s) = \alpha'(s)$.
(2) Principal normal: $N(s) = \alpha''(s)/\|\alpha''(s)\|$ $(\|\alpha''(s)\| \neq 0)$.
(3) Binormal: $B(s) = T(s) \times N(s)$.

Their evolution is governed by the Frenet-Serret equations:

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix},$$

with curvature $\kappa(s) = \|\alpha''(s)\|$ and torsion $\tau(s) = -\langle N'(s), B(s) \rangle$.

Definition 2.1. *A curve $\alpha(s)$ is a* Bertrand curve *if there exists $\beta(s)$ satisfying:*

$$N_\alpha(s) = N_\beta(s),$$
$$\beta(s) = \alpha(s) + cN_\alpha(s),$$

for some constant $c \neq 0$. Corresponding points maintain constant separation $|c|$, and their tangents form a constant angle θ [3, 23].

Definition 2.2. *[1, 24] A curve $\alpha(s)$ is a* Mannheim curve *if there exists $\gamma(s)$ such that $N_\alpha(s) = B_\gamma(s)$. The separation vector connects corresponding points along binormal directions.*

Definition 2.3. *For a non-degenerate curve $\alpha(s)$:*

(1) *Its* normal-integral curve *is $\beta(s) = \int N_\alpha(s)ds$ with $\beta'(s) = N_\alpha(s)$.*
(2) *Its* binormal-integral curve *is $\gamma(s) = \int B_\alpha(s)ds$ with $\gamma'(s) = B_\alpha(s)$.*

The Frenet apparatus of these integral curves exhibits elegant relations to the original curve's geometry [8,15,19]. In subsequent sections, we investigate Bertrand and Mannheim partners arising from these constructions.

3. Bertrand Curves of the integral curves in E^3

This section analyzes Bertrand curve pairs generated from normal-integral and binormal-integral curves in Euclidean 3-space. For each case, we derive Frenet frames, curvature, torsion, and establish classification criteria.

3.1. **Bertrand Curves of the normal-integral curves in E^3.**

In this subsection, we define the normal-integral curves and obtain the Frenet frame, the curvature, and the torsion of the normal-integral curves. In addition, we define the Bertrand curve of the normal-integral curve. Furthermore, the Frenet frame, the curvature, and the torsion of the Bertrand curves of normal-integral curves.

3.2. **Normal-Integral Bertrand Curves.**

Definition 3.1. *For a non-degenerate curve $\alpha(s)$ in E^3 with normal vector N, its* normal-integral curve *$\beta(s)$ is defined as:*

$$\beta(s) = \int N(s)ds \quad with \quad N(s) = \beta'(s). \tag{3.1}$$

Definition 3.2. *If two curves have the property that the tangent vectors of the two curves are perpendicular, then we call them an involute-evolute pair of curves. In other words, if the relation $< T, T^* >= 0$ is satisfied, where T and T^* are the tangent vectors of the two curves.*

Proposition 3.1. *For $\alpha(s)$ non-degenerate ($\alpha' \times \alpha'' \neq 0$), the Frenet apparatus of its normal-integral curve $\beta(s)$ satisfies:*

$$T_\beta(s) = N_\alpha(s),$$

$$N_\beta(s) = \frac{-\kappa_\alpha(s)}{\sqrt{\kappa_\alpha^2(s) + \tau_\alpha^2(s)}} T_\alpha(s) + \frac{\tau_\alpha(s)}{\sqrt{\kappa_\alpha^2(s) + \tau_\alpha^2(s)}} B_\alpha(s),$$

$$B_\beta(s) = \frac{\tau_\alpha(s)}{\sqrt{\kappa_\alpha^2(s) + \tau_\alpha^2(s)}} T_\alpha(s) + \frac{\kappa_\alpha(s)}{\sqrt{\kappa_\alpha^2(s) + \tau_\alpha^2(s)}} B_\alpha(s),$$

$$\kappa_\beta(s) = \sqrt{\kappa_\alpha^2(s) + \tau_\alpha^2(s)},$$

$$\tau_\beta(s) = \frac{\kappa_\alpha(s)\tau_\alpha^{'}(s) - \tau_\alpha(s)\kappa_\alpha^{'}(s)}{\sqrt{\kappa_\alpha^2(s) + \tau_\alpha^2(s)}}.$$

Corollary 3.1. *$\alpha(s)$ and its normal-integral curve $\beta(s)$ form an involute-evolute pair: $\langle T_\alpha, T_\beta \rangle = 0$.*

Definition 3.3. *A curve $\beta^*(s)$ is a* Bertrand mate *of $\beta(s)$ if $N_{\beta^*}(s) = N_\beta(s)$ at corresponding points.*

Theorem 3.1. *Asumme that $\kappa_\alpha^2 + \tau_\alpha^2 \neq 0$ and $\tau_\alpha \neq 0$. The Bertrand mate of normal-integral curve $\beta(s)$ admits the parametrization:*

$$\beta^*(s) = \beta(s) + \frac{c_1}{\sqrt{\kappa_\alpha^2 + \tau_\alpha^2}}(-\kappa_\alpha T_\alpha(s) + \tau_\alpha B_\alpha(s)),$$

where c_1 is a constant.

Proof. Given that $\beta(s)$ is a normal-integral curve of α, let $\beta^*(s)$ be a Bertrand curve of $\beta(s)$, such that:

$$\beta^*(s) = \beta(s) + \lambda(s)N_\beta(s),$$

where $\lambda(s)$ is a differentiable function. By taking the derivative concerning s and taking the inner product with $N_\beta(s)$, we can conclude that $\lambda = c_1$ is constant. Using the equations provided in Proposition 3.1, we find:

$$\beta^*(s) = \beta(s) + \frac{c_1}{\sqrt{\kappa_\alpha^2 + \tau_\alpha^2}}(-\kappa_\alpha T_\alpha(s) + \tau_\alpha B_\alpha(s)), \tag{3.2}$$

where c_1 is constant. □

Corollary 3.2. *If $\beta^*(s)$ is a Bertrand curve of $\beta(s)$, then the distance between $\beta(s)$ and $\beta^*(s)$ at corresponding points remains constant.*

Corollary 3.3. *The angle θ_1 between tangents T_β and T_{β^*} is constant.*

Proof. Define $\theta_1(s)$ as the angle between the tangents of the curves $\beta(s)$ and $\beta^*(s)$. This angle can be expressed as:

$$\cos\theta_1(s) = \langle T_{\beta^*}(s), T_\beta(s)\rangle.$$

By differentiating this expression with respect to s, we obtain:

$$\frac{d}{ds}\cos\theta_1(s) = \langle \kappa_{\beta^*} N_{\beta^*}(s), T_\beta(s)\rangle + \langle T_{\beta^*}, \kappa_\beta N_\beta\rangle,$$

which simplifies to zero, showing that θ_1 is constant.

□

Theorem 3.2. *Asumme that $\kappa_\alpha^2 + \tau_\alpha^2 \neq 0$ and $\tau_\alpha \neq 0$. A normal-integral curve $\beta(s)$ admits a Bertrand mate if and only if:*

$$c_1\nu\frac{\kappa_\alpha\tau_\alpha' - \tau_\alpha\kappa_\alpha'}{\sqrt{\kappa_\alpha^2 + \tau_\alpha^2}} + c_1\sqrt{\kappa_\alpha^2 + \tau_\alpha^2} = 1,$$

where $\nu = \cot\theta_1$ and c_1 is a non-zero constant.

Proof. Let $\beta(s)$ be a Bertrand curve of the normal-integral curve. By differentiating Equation (3.2), we obtain:

$$\frac{d\beta^*(s)}{ds} = (1 - c_1\kappa_\beta)T_\beta(s) + (c_1\tau_\beta)B_\beta(s). \tag{3.3}$$

Also, we can write

$$\frac{d\beta^*}{ds} = (\cos\theta)T_\beta + (\sin\theta)B_\beta,$$

where θ_1 is the angle between $\frac{d\beta^*}{ds}$ and T_β which is constant. Therefore, $\cos\theta_1 = 1 - c_1\kappa_\beta$ and $\sin\theta_1 = c_1\tau_\beta$.
We have $\cot\theta_1 = \frac{1-c_1\kappa_\beta}{c_1\tau_\beta} = \nu = constant$. Then we have:

$$\nu\tau_\beta + \kappa_\beta = \frac{1}{c_1}. \tag{3.4}$$

Hence we obtain:

$$c_1\nu\frac{\kappa_\alpha\tau_\alpha^{'} - \tau_\alpha\kappa_\alpha^{'}}{\sqrt{\kappa_\alpha^2 + \tau_\alpha^2}} + c_1\sqrt{\kappa_\alpha^2 + \tau_\alpha^2} = 1. \tag{3.5}$$

Conversely, suppose that Equations 3.4 and 3.5 are satisfied. By differentiating Equation 3.3 we can get

$$N_\beta^* = (\frac{1}{\kappa_\beta^*})(\frac{ds}{ds^*})[\frac{\nu\kappa_\beta}{\sqrt{\nu^2+1}} - \frac{\tau_\beta}{\sqrt{\nu^2+1}}]N_\beta.$$

Then, the curve $\beta^*(s)$ is a Bertrand curve of the normal-integral curve.

□

Theorem 3.3. *Asumme that $\kappa_\alpha^2 + \tau_\alpha^2 \neq 0$ and $\tau_\alpha \neq 0$. The Frenet frame of Bertrand mate $\beta^*(s)$ relates to α's frame via:*

$$\begin{aligned}
T_{\beta^*}(s) &= \frac{1}{\sqrt{\nu^2+1}}[\frac{\tau_\alpha}{\sqrt{\kappa_\alpha^2+\tau_\alpha^2}}T_\alpha + \nu N_\alpha + \frac{\kappa_\alpha}{\sqrt{\kappa_\alpha^2+\tau_\alpha^2}}B_\alpha],\\
N_\beta^*(s) &= \frac{1}{\sqrt{\kappa_\alpha^2+\tau_\alpha^2}}[-\kappa_\alpha T_\alpha + \tau_\alpha B_\alpha],\\
B_\beta^*(s) &= \frac{1}{\sqrt{\nu^2+1}}[\frac{\nu\tau_\alpha}{\sqrt{\kappa_\alpha^2+\tau_\alpha^2}}T_\alpha - N_\alpha + \frac{\nu\kappa_\alpha}{\sqrt{\kappa_\alpha^2+\tau_\alpha^2}}B_\alpha],
\end{aligned}$$

Proof. Let $\beta^*(s)$ be a Bertrand curve of $\beta(s)$, then from Equations (3.3) and (3.4), we have:

$$\frac{d\beta^*}{ds} = c_1\tau_\beta[\nu T_\beta + B_\beta], \tag{3.6}$$

Then we get

$$T_\beta^* = \frac{1}{\sqrt{\nu^2+1}}[\nu T_\beta + B_\beta].$$

By differentiating Equation (3.3), we obtain:

$$\frac{d^2\beta^*}{ds^2} = c_1\tau_\beta'\nu T_\beta + c_1\tau_\beta(\nu\kappa_\beta - \tau_\beta)N_\beta + c_1\tau_\beta'\nu B_\beta, \tag{3.7}$$

Taking the cross product of Equations 3.3 and Equation 3.7, we can obtain:

$$B_\beta^* = \frac{1}{\sqrt{\nu^2+1}}[-T_\beta + \nu B_\beta],$$

Also, we have $N_{\beta^*} = N_\beta$.

By substitution from the equations in Proposition 3.1, we obtain:

$$T_{\beta^*}(s) = \frac{1}{\sqrt{\nu^2+1}}[\frac{\tau_\alpha}{\sqrt{\kappa_\alpha^2+\tau_\alpha^2}}T_\alpha(s) + \nu N_\alpha(s) + \frac{\kappa_\alpha}{\sqrt{\kappa_\alpha^2+\tau_\alpha^2}}B_\alpha(s)], \tag{3.8}$$

$$N_{\beta^*}(s) = \frac{1}{\sqrt{\kappa_\alpha^2+\tau_\alpha^2}}[-\kappa_\alpha T_\alpha(s) + \tau_\alpha B_\alpha(s)], \tag{3.9}$$

$$B_{\beta^*}(s) = \frac{1}{\sqrt{\nu^2+1}}[\frac{\nu\tau_\alpha}{\sqrt{\kappa_\alpha^2+\tau_\alpha^2}}T_\alpha(s) - N_\alpha(s) + \frac{\nu\kappa_\alpha}{\sqrt{\kappa_\alpha^2+\tau_\alpha^2}}B_\alpha(s)], \tag{3.10}$$

□

Theorem 3.4. *Asumme that $\kappa_\alpha^2+\tau_\alpha^2\neq 0$ and $\tau_\alpha\neq 0$. The Curvature and torsion of β^* are given, respectively, by:*

$$\kappa_{\beta^*}(s)=\frac{\nu(\kappa_\alpha^2+\tau_\alpha^2)-(\kappa_\alpha\tau_\alpha^{'}-\tau_\alpha\kappa_\alpha^{'})}{c_1(1+\nu^2)(\kappa_\alpha\tau_\alpha^{'}-\tau_\alpha\kappa_\alpha^{'})},$$

$$\tau_{\beta^*}(s)=\frac{(\kappa_\alpha\tau'_\alpha-\tau_\alpha\kappa'_\alpha)(\kappa_\alpha^2+\tau_\alpha^2)(\nu^2-1)+\nu(\kappa_\alpha^2+\tau_\alpha^2)^2-\nu(\kappa_\alpha\tau'_\alpha-\tau_\alpha\kappa'_\alpha)^2}{(c_1\nu(\kappa_\alpha\tau'_\alpha-\tau_\alpha\kappa'_\alpha)(\kappa_\alpha^2+\tau_\alpha^2)-c_1(\kappa_\alpha\tau'_\alpha-\tau_\alpha\kappa'_\alpha)^2)(1+\nu^2)}.$$

Proof. From Equation (3.3) and (3.7), we have:

$$\kappa_{\beta^*}(s)=\frac{\nu\kappa_\beta-\tau_\beta}{c_1\tau_\beta(1+\nu^2)},$$

From the equations in Proposition 3.1, we obtain:

$$\kappa_{\beta^*}(s)=\frac{\nu(\kappa_\alpha^2+\tau_\alpha^2)-(\kappa_\alpha\tau_\alpha^{'}-\tau_\alpha\kappa_\alpha^{'})}{c_1(1+\nu^2)(\kappa_\alpha\tau_\alpha^{'}-\tau_\alpha\kappa_\alpha^{'})}.$$

By differentiating Equation (3.7), we obtain:

$$\begin{aligned}\frac{d^3\beta^*}{ds^3}=&(c_1\tau_\beta''\nu+c_1\tau_\beta^2\kappa_\beta-c_1\tau_\beta\nu\kappa_\beta^2)T_\beta+(2c_1\tau_\beta'\nu\kappa_\beta-3c_1\tau_\beta'\tau_\beta\\&+c_1\tau_\beta\nu\kappa_\beta')N_\beta+(c_1\tau_\beta^2\nu\kappa_\beta-c_1\tau_\beta^3+c_1\tau_\beta'')B_\beta.\end{aligned}$$

Then

$$\tau_{\beta^*}(s)=\frac{\tau_\beta\kappa_\beta(\nu^2-1)+\nu(\kappa_\beta^2-\tau_\beta^2)}{c_1\tau_\beta(\nu\kappa_\beta-\tau_\beta)((1+\nu^2)}.$$

From the equations in Proposition 3.1, we obtain:

$$\tau_{\beta^*}(s)=\frac{(\kappa_\alpha\tau'_\alpha-\tau_\alpha\kappa'_\alpha)(\kappa_\alpha^2+\tau_\alpha^2)(\nu^2-1)+\nu(\kappa_\alpha^2+\tau_\alpha^2)^2-\nu(\kappa_\alpha\tau'_\alpha-\tau_\alpha\kappa'_\alpha)^2}{(c_1\nu(\kappa_\alpha\tau'_\alpha-\tau_\alpha\kappa'_\alpha)(\kappa_\alpha^2+\tau_\alpha^2)-c_1(\kappa_\alpha\tau'_\alpha-\tau_\alpha\kappa'_\alpha)^2)(1+\nu^2)}.$$

□

3.3. **Bertrand Curves of the binormal-integral curves in E^3.** In this subsection, we define the binormal-integral curves and obtain the Frenet frame, the curvature, and the torsion of the binormal-integral curves. Also, we define the Bertrand curve of the binormal-integral curve. Furthermore, the Frenet frame, the curvature, and the torsion of the Bertrand curves of binormal-integral curves.

Definition 3.4. *For $\alpha(s)$ with binormal B, its* binormal-integral curve *$\gamma(s)$ is defined as:*

$$\gamma(s)=\int B(s)\ \mathrm{d}s \quad \& \ B(s)=\gamma'(s). \tag{3.11}$$

Proposition 3.2. *The Frenet apparatus of $\gamma(s)$ satisfies:*

$$\begin{aligned}T_\gamma&=B_\alpha,\\N_\gamma&=-N_\alpha,\\B_\gamma&=T_\alpha,\\\kappa_\gamma&=\tau_\alpha,\\\tau_\gamma&=\kappa_\alpha.\end{aligned}$$

Definition 3.5. *$\gamma^*(s)$ is a* Bertrand mate *of $\gamma(s)$ if $N_{\gamma^*}(s)=N_\gamma(s)$.*

Theorem 3.5. *The Bertrand mate of binormal-integral curve $\gamma(s)$ is:*

$$\gamma^*(s) = \gamma(s) + A_1 N_\alpha(s),$$

where A_1 is constant.

Proof. Let $\gamma(s)$ be a binormal-integral curve of α and $\gamma^*(s)$ be a Bertrand curve of γ. We can express the relation between the two curves as:

$$\gamma^*(s) = \gamma(s) + \lambda^*(s) N_\gamma(s), \tag{3.12}$$

where $\lambda^*(s)$ is a differentiable function. By differentiating with respect to s and using the condition that $N_{\gamma^*} = N_\gamma$, we conclude that $\lambda^*(s) = A_1$ is a constant. Substituting into the equation gives:

$$\gamma^*(s) = \gamma(s) + A_1 N_\alpha(s),$$

where A_1 is constant. □

Corollary 3.4. *The distance between $\gamma(s)$ and $\gamma^*(s)$ at corresponding points is constant.*

Corollary 3.5. *The angle θ_2 between tangents T_γ and T_{γ^*} is constant.*

Proof. Define $\theta_2(s)$ as the angle between the tangents of the curves $\gamma(s)$ and $\gamma^*(s)$. Since $\|T_\gamma\| = \|T_{\gamma^*}\| = 1$, the angle can be expressed as:

$$\cos\theta_2 = \langle T_{\gamma^*}, T_\gamma \rangle.$$

Differentiating this equation with respect to s, we obtain:

$$\frac{d}{ds}\cos\theta_2 = \langle \tau_{\gamma^*} N_{\gamma^*}, T_\gamma \rangle,$$

which equals zero. Thus, θ_2 is constant. □

Theorem 3.6. *A binormal-integral curve $\gamma(s)$ admits a Bertrand mate if and only if:*

$$\nu_2 \ \kappa_\alpha + \tau_\alpha = \frac{1}{A_1}, \tag{3.13}$$

where $\nu_2 = \cot\theta_2$ and A_1 is a non-zero constant.

Theorem 3.7. *The Frenet frame of $\gamma^*(s)$ relates to α's frame via:*

$$\begin{aligned} T_{\gamma^*} &= \frac{1}{\sqrt{\nu_2^2+1}} T_\alpha + \frac{\nu_2}{\sqrt{\nu_2^2+1}} B_\alpha, \\ N_{\gamma^*} &= -N_\alpha, \\ B_{\gamma^*} &= \frac{\nu_2}{\sqrt{\nu_2^2+1}} T_\alpha - \frac{1}{\sqrt{\nu_2^2+1}} B_\alpha, \end{aligned}$$

Proof. Let $\gamma^*(s)$ be a Bertrand curve of $\gamma(s)$, By differentiating with respect to s, we have:

$$\frac{d\gamma^*}{ds} = A_1 \tau_\gamma [\nu T_\gamma + B_\gamma]. \tag{3.14}$$

Then we get

$$T_{\gamma^*} = \frac{1}{\sqrt{\nu^2+1}} [T_\gamma + \nu B_\gamma].$$

By differentiating Equation (3.14), we obtain:

$$\frac{d^2\beta^*}{ds^2} = A_1\tau'_\gamma\nu T_\gamma + A\tau_\gamma(\nu\kappa_\gamma - \tau_\gamma)N_\gamma + A\tau'_\gamma B_\gamma, \tag{3.15}$$

Taking the cross product of Equations 3.14 and Equation 3.15, we can obtain:

$$B_{\gamma^*} = \frac{1}{\sqrt{\nu^2+1}}[\nu T_\gamma - B_\gamma],$$

Also, we have: $N_{\gamma^*} = N_\gamma$.
By substitution from the equations in Proposition 3.2, we obtain:

$$\begin{aligned} T_{\gamma^*} &= \frac{1}{\sqrt{\nu_2^2+1}}T_\alpha + \frac{\nu_2}{\sqrt{\nu_2^2+1}}B_\alpha, \\ N_{\gamma^*} &= -N_\alpha, \\ B_{\gamma^*} &= \frac{\nu_2}{\sqrt{\nu_2^2+1}}T_\alpha - \frac{1}{\sqrt{\nu_2^2+1}}B_\alpha. \end{aligned}$$

□

Theorem 3.8. *Let γ and γ^* be two Bertrand of the normal-integral curves, and assume that $\nu^2 \neq 1$, then κ_{γ^*} and τ_{γ^*} of the binormal-integral curve are given respectively by:*

$$\kappa_{\gamma^*} = \frac{(\nu\tau_\alpha - \kappa_\alpha)}{A_1\kappa_\alpha(\nu^2-1)},$$

$$\tau_{\gamma^*} = \frac{-A_1^2\kappa_\alpha^2(\nu\kappa''_\alpha - \nu\kappa_\alpha\tau_\alpha^2 + \kappa_\alpha^2\tau_\alpha) + \nu(\nu\kappa_\alpha^2\tau_\alpha - \kappa_\alpha^3 + \kappa''_\alpha)}{A_1^3\kappa_\alpha^4(\nu\tau_\alpha - \kappa_\alpha)((1+\nu^2)}.$$

Proof. From Equation (3.14 and 3.15), we have:

$$\kappa_{\gamma^*} = \frac{(\nu\kappa_\gamma - \tau_\gamma)}{A_1\tau_\gamma(\nu^2+1)},$$

From the equations in Proposition 3.2, we obtain:

$$\kappa_{\gamma^*} = \frac{(\nu\tau_\alpha - \kappa_\alpha)}{A_1\kappa_\alpha(\nu^2-1)},$$

By differentiating Equation (3.15) we obtain:

$$\frac{d^3\gamma^*}{ds^3} = A_1(\tau''_\gamma\nu - \kappa_\gamma^2\tau_\gamma\nu + \kappa_\gamma\tau_\gamma^2)T_\gamma + A_1(2\tau'_\gamma\nu\kappa_\gamma - 3\tau'_\gamma\tau_\gamma + \tau_\gamma\nu\kappa'_\gamma)N_\gamma + A_1(\tau_\gamma^2\nu\kappa_\gamma - \tau_\gamma^3 + \tau''_\gamma)B_\gamma.$$

Then

$$\tau_{\gamma^*} = \frac{-A_1^2\tau_\gamma^2(\nu\tau''_\gamma - \nu\tau_\gamma\kappa_\gamma^2 + \tau_\gamma^2\kappa_\gamma) + \nu(\nu\tau_\gamma^2\kappa_\gamma - \tau_\gamma^3 + \tau''_\gamma)}{A_1^3\tau_\gamma^4(\nu\kappa_\gamma - \tau_\gamma)((1+\nu^2)}.$$

From the equations in Proposition 3.2, we obtain:

$$\tau_{\gamma^*} = \frac{-A_1^2\kappa_\alpha^2(\nu\kappa''_\alpha - \nu\kappa_\alpha\tau_\alpha^2 + \kappa_\alpha^2\tau_\alpha) + \nu(\nu\kappa_\alpha^2\tau_\alpha - \kappa_\alpha^3 + \kappa''_\alpha)}{A_1^3\kappa_\alpha^4(\nu\tau_\alpha - \kappa_\alpha)((1+\nu^2)}.$$

□

4. Mannheim Curves of the integral curves in E^3

In this section, we investigate the Mannheim curves of the normal-integral curves and the binormal-integral curves. In each case, we introduce the Frenet vectors, curvature, and torsion of the Mannheim curves. Further, we give the necessary and sufficient condition of the curve to be the Mannheim curve.

4.1. Mannheim Curves of the normal-integral curves in E^3.

In this section, we define the Mannheim curves of the normal-integral curves. In each case, we introduce the Frenet vectors, curvature, and torsion of the Mannheim curves.

Proposition 4.1. *For $\alpha(s)$ non-degenerate ($\alpha' \times \alpha'' \neq 0$). Let a curve $\delta(s)$ be a normal-integral curve of the curve α, then the Frenet vectors, the curvature κ_δ and the torsion τ_δ of the curve δ in terms of the Frenet vectors, curvature and torsion of the curve α are given, respectively, by*

$$T_\delta = N_\alpha,$$
$$N_\delta = \frac{-\kappa_\alpha}{\sqrt{\kappa_\alpha^2+\tau_\alpha^2}}T_\alpha + \frac{\tau_\alpha}{\sqrt{\kappa_\alpha^2+\tau_\alpha^2}}B_\alpha,$$
$$B_\delta = \frac{\tau_\alpha}{\sqrt{\kappa_\alpha^2+\tau_\alpha^2}}T_\alpha + \frac{\kappa_\alpha}{\sqrt{\kappa_\alpha^2+\tau_\alpha^2}}B_\alpha,$$
$$\kappa_\delta = \sqrt{\kappa_\alpha^2+\tau_\alpha^2},$$
$$\tau_\delta = \frac{\kappa_\alpha \tau_\alpha^{'} - \tau_\alpha \kappa_\alpha^{'}}{\sqrt{\kappa_\alpha^2+\tau_\alpha^2}}.$$

Definition 4.1. *Let a curve δ be a nondegenerate curve normal-integral curve of the curve α. A curve δ^* is called a Mannheim curve for the normal-integral curve δ if the normal of the curve δ coincides with the binormal of the curve δ^* at the corresponding points i.e $B_\delta^* = N_\delta$.*

Theorem 4.1. *Let the curve δ be a nondegenerate curve normal-integral curve of the curve α. If $\delta^*(s)$ is a Mannheim curve of δ, then*

$$\delta^*(s) = \delta(s) + \frac{c_2}{\sqrt{\kappa_\alpha^2+\tau_\alpha^2}}(-\kappa_\alpha T_\alpha + \tau_\alpha B_\alpha),$$

where c_2 is constant.

Corollary 4.1. *Let the curve δ be a nondegenerate curve normal-integral curve of the curve α. If $\delta^*(s)$ is a Mannheim curve of δ, then the distance between δ and δ^* at corresponding points is constant.*

Corollary 4.2. *Let the curve δ be a nondegenerate curve normal-integral curve of the curve α. If $\delta^*(s)$ is a Mannheim curve of δ, then the angle between the tangents of δ and δ^* at corresponding points is not generally constant.*

Proof. Let $\theta_3(s)$ be the angle between the tangents of the two curves δ and δ^*, then we can write

$$\cos\theta_3 = < T_{\delta^*}, T_\delta > . \tag{4.1}$$

By differentiating Equation (4.1) , with respect to s, we have:

$$\frac{d}{ds}\cos\theta_3 =< \kappa_{\delta^*} N_{\delta^*} \frac{ds^*}{ds}, T_\delta >,$$

which is not zero. Therefore, θ_3 is not constant.

□

Theorem 4.2. *Let $\delta(s)$ be a nondegenerate curve normal-integral curve of the curve $\alpha(s)$, then $\delta^*(s)$ is a Mannheim curve of the curve $\delta(s)$ if and only if*

$$c_2(\kappa_\delta^2 + \tau_\delta^2) = \kappa_\delta,$$

and

$$\tau_\delta \kappa_\delta^{'} - \tau_\delta \kappa_\delta^{'} > 0.$$

where c_2 are non-zero constants.

Proof. Let $\delta^*(s)$ be a Mannheim curve of the normal-integral curve,

$$\delta^*(s) = \delta(s) + c_2 N_\delta. \tag{4.2}$$

By differentiating Equation (4.2), we obtain:

$$|\dot{\delta}| T_{\delta^*} = [1 - c_2\kappa_\delta]T_\delta + c_2\tau_\delta B_\delta, \tag{4.3}$$

where $|\dot{\delta}| = \frac{ds^*}{ds}$. Also, we have:

$$T_{\delta^*} = -(\sin\theta_4)T_\delta + (\cos\theta_4)B_\delta, \tag{4.4}$$

$$N_{\delta^*} = (\cos\theta_4)T_\delta + (\sin\theta_4)B_\delta, \tag{4.5}$$

where θ_4 is the angle between T_{δ^*} and B_δ which is constant. Therefore, $|\dot{\delta}|\cos\theta_3 = c_2\tau_\delta$ and $-|\dot{\delta}|\sin\theta_4 = 1 - c_2\kappa_\delta$.
Then we have:

$$c_2\tau_\delta \sin\theta_4 + (1 - c_2\kappa_\delta)\cos\theta_4 = 0. \tag{4.6}$$

By differentiating Equation (4.4), we have:

$$\kappa_{\delta^*} N_{\delta^*} |\dot{\delta}| = -\cos\theta_4\theta_4' T_\delta - [\tau_\delta\cos\theta_4 + \kappa_\delta\sin\theta_4]N_\delta - \sin\theta_4\theta_4' B_\delta. \tag{4.7}$$

and by inner product of $B_{\delta^*} = N_\delta$ with Equation 4.7, we obtain:

$$\tau_\delta\cos\theta_4 + \kappa_\delta\sin\theta_4 = 0. \tag{4.8}$$

From Equations (4.6 and 4.8), we have:

$$c_2(\kappa_\delta^2 + \tau_\delta^2) = \kappa_\delta, \tag{4.9}$$

By differentiating Equation (4.8), we have:

$$\theta_4' = \frac{\tau_\delta\kappa_\delta' - \tau_\delta'\kappa_\delta}{\kappa_\delta^2 + \tau_\delta^2}.$$

We obtain that θ_4' is negative from Equations (4.5 and 4.7). Then,

$$\tau_\delta\kappa_\delta^{'} - \tau_\delta^{'}\kappa_\delta > 0. \tag{4.10}$$

Conversely, suppose that Equations 4.9 and 4.10 are satisfied. By differentiating Equation 4.2 and by using Equation 4.9 we can get:

$$|\dot{\delta}| T_{\delta^*} = \frac{c_2\tau_\delta}{\kappa_\delta}[\tau_\delta T_\delta + \kappa_\delta B_\delta]. \tag{4.11}$$

By differentiating Equation 4.11 and by using Equation 4.10, we can get that the curve $\delta^*(s)$ is a Mannheim curve of the normal-integral curve. □

Theorem 4.3. *Let the curve δ be a normal-integral curve of the curve α. Assume that $1 - c_2\kappa_\delta > 0$. If $\delta^*(s)$ is a Mannheim curve of $\delta(s)$, then the Frenet frame $\{T_{\delta^*}, N_{\delta^*}, B_{\delta^*}\}$ of the curve $\delta^*(s)$ is given in terms of the Frenet frame of the curve α by*

$$\begin{aligned} T_{\delta^*} &= \frac{1}{\sqrt{1 - c_2\kappa_\delta}}[(1 - c_2\kappa_\delta)T_\delta + c_2\tau_\delta B_\delta], \\ N_{\delta^*} &= \frac{1}{\sqrt{1 - c_2\kappa_\delta}}[c_2\tau_\delta T_\delta + (c_2\kappa_\delta - 1)B_\delta], \\ B_{\delta^*} &= N_\delta, \end{aligned}$$

where

$$\begin{aligned} T_\delta &= N_\alpha, \\ N_\delta &= \frac{-\kappa_\alpha}{\sqrt{\kappa_\alpha^2 + \tau_\alpha^2}}T_\alpha + \frac{\tau_\alpha}{\sqrt{\kappa_\alpha^2 + \tau_\alpha^2}}B_\alpha, \\ B_\delta &= \frac{\tau_\alpha}{\sqrt{\kappa_\alpha^2 + \tau_\alpha^2}}T_\alpha + \frac{\kappa_\alpha}{\sqrt{\kappa_\alpha^2 + \tau_\alpha^2}}B_\alpha, \\ \kappa_\delta &= \sqrt{\kappa_\alpha^2 + \tau_\alpha^2}, \\ \tau_\delta &= \frac{\kappa_\alpha\tau_\alpha^{'} - \tau_\alpha\kappa_\alpha^{'}}{\sqrt{\kappa_\alpha^2 + \tau_\alpha^2}}. \end{aligned}$$

Theorem 4.4. *Let δ and δ^* be two Mannheim of the normal-integral curves, then the curvature and the torsion of the normal-integral curve are given respectively by:*

$$\kappa_{\delta^*} = \frac{\kappa_\delta}{c_2\tau_\delta}\frac{\kappa_\delta\tau_\delta' - \tau_\delta\kappa_\delta'}{(\kappa_\delta^2 + \tau_\delta^2)^{3/2}},$$

$$\tau_{\delta^*} = \frac{\kappa_\delta^2 + \tau_\delta^2}{\tau_\delta},$$

where

$$\begin{aligned} \kappa_\delta &= \sqrt{\kappa_\alpha^2 + \tau_\alpha^2}, \\ \tau_\delta &= \frac{\kappa_\alpha\tau_\alpha^{'} - \tau_\alpha\kappa_\alpha^{'}}{\sqrt{\kappa_\alpha^2 + \tau_\alpha^2}}. \end{aligned}$$

4.2. **Mannheim Curves of the binormal-integral curves in E^3.** In this section, we define the Mannheim curves of the binormal-integral curves in E^3. In each case, we introduce the Frenet vectors, curvature, and torsion of the Mannheim curves.

Proposition 4.2. *For $\alpha(s)$ non-degenerate ($\alpha' \times \alpha'' \neq 0$) in E^3. Let a curve σ be a binormal-integral curve of the curve α, then the Frenet vectors, the curvature κ_σ and the torsion τ_σ of the curve σ in terms of the Frenet vectors, curvature and torsion of the curve α are given, respectively, by*

$$\begin{aligned} T_\sigma &= B_\alpha, \\ N_\sigma &= -N_\alpha, \\ B_\sigma &= T_\alpha, \end{aligned}$$

$$\kappa_\sigma = \tau_\alpha$$

$$\tau_\sigma = \kappa_\alpha.$$

Definition 4.2. *Let a curve σ be a binormal-integral curve of the curve α. A curve σ^* is called a Mannheim curve for the binormal-integral curve σ if the principal normal of the curve σ coincides with the binormal of the curve σ^* at the corresponding points, i.e $B^*_\sigma = N_\sigma$.*

Theorem 4.5. *Let the curve σ be a binormal-integral curve of the curve α. If $\sigma^*(s)$ is a Mannheim curve of σ, then*

$$\sigma^*(s) = \sigma(s) - A_2 N_\alpha,$$

where A_2 is constant.

Corollary 4.3. *Let the curve σ be a binormal-integral curve of the curve α. If $\sigma^*(s)$ is a Mannheim curve of σ, then the distance between σ and σ^* at corresponding points is constant.*

Corollary 4.4. *Let the curve σ be a binormal-integral curve of the curve α. If $\sigma^*(s)$ is a Mannheim curve of σ, then the angle between the tangents of σ and σ^* at corresponding points is not constant.*

Theorem 4.6. *Let $\sigma(s)$ be a Mannheim curve of the binormal-integral curve. Then $\sigma(s)$ is a Mannheim curve if and only if*

$$A_2(\kappa_\sigma^2 + \tau_\sigma^2) = \kappa_\sigma,$$

and

$$\tau_\sigma \kappa_\sigma^{'} - \tau_\sigma \kappa_\sigma^{'} > 0.$$

where A_2 are non-zero constants.

Theorem 4.7. *Let the curve σ be a binormal-integral curve of the curve α. If $\sigma^*(s)$ is a Mannheim curve of $\sigma(s)$, then the Frenet frame $\{T_{\sigma^*}, N_{\sigma^*}, B_{\sigma^*}\}$ of the curve $\sigma^*(s)$ is given in terms of the Frenet frame of the curve α by*

$$\begin{aligned} T_{\sigma^*} &= \frac{1}{\sqrt{\nu_2^2+1}} T_\alpha + \frac{\nu_2}{\sqrt{\nu_2^2+1}} B_\alpha, \\ N_{\sigma^*} &= -N_\alpha, \\ B_{\sigma^*} &= \frac{\nu_2}{\sqrt{\nu_2^2+1}} T_\alpha - \frac{1}{\sqrt{\nu_2^2+1}} B_\alpha, \end{aligned}$$

Theorem 4.8. *Let σ and σ^* be two Mannheim of the normal-integral curves, then the curvature and the torsion of the binormal-integral curve are given respectively by:*

$$\kappa_{\sigma^*} = \frac{(\nu\tau_\alpha - \kappa_\alpha)}{A_2 \kappa_\alpha (\nu^2 - 1)},$$

$$\tau_{\sigma^*} = \frac{-A_2^2\kappa_\alpha^2(\nu\kappa_\alpha'' - \nu\kappa_\alpha\tau_\alpha^2 + \kappa_\alpha^2\tau_\alpha) + \nu(\nu\kappa_\alpha^2\tau_\alpha - \kappa_\alpha^3 + \kappa_\alpha'')}{A_2^3\kappa_\alpha^4(\nu\tau_\alpha - \kappa_\alpha)((1+\nu^2)}.$$

5. Example

Let's consider the following unit speed curve given in Figure 1:

$$\alpha(s) = \left(\frac{1}{\sqrt{2}}\cos(s), \frac{1}{\sqrt{2}}sin(s), \frac{s}{\sqrt{2}}\right), \tag{5.1}$$

This is a circular helix, which is a non-planar curve.

5.1. **Frenet Frame.** Let's compute T, N, and B using equation (2.1):

$$T(s) = \alpha'(s) = \left(-\frac{1}{\sqrt{2}}\sin(s), \frac{1}{\sqrt{2}}\cos(s), \frac{1}{\sqrt{2}}\right), \tag{5.2}$$

$$\alpha''(s) = \left(-\frac{1}{\sqrt{2}}\cos(s), -\frac{1}{\sqrt{2}}\sin(s), 0\right), \tag{5.3}$$

$$N(s) = \frac{\alpha''(s)}{|\alpha''(s)|} = (-\cos(s), -\sin(s), 0), \tag{5.4}$$

$$B(s) = T(s) \times N(s) = \left(\frac{1}{\sqrt{2}}\sin(s), -\frac{1}{\sqrt{2}}\cos(s), \frac{1}{\sqrt{2}}\right). \tag{5.5}$$

5.2. **N(s)-direction Curve.** The N(s)-direction curve $\beta(s)$ is given in Figure 2:

$$\begin{aligned}\beta(s) &= \int N(s)ds \\ &= \int (-\cos(s), -\sin(s), 0)ds \\ &= (-\sin(s), \cos(s), 0) + C_1,\end{aligned} \tag{5.6}$$

where C_1 is a constant vector.

5.3. **B(s)-direction Curve.** The B(s)-direction curve $\gamma(s)$ is given Figure 3 by:

$$\begin{aligned}\gamma(s) &= \int B(s)ds \\ &= \int \left(\frac{1}{\sqrt{2}}\sin(s), -\frac{1}{\sqrt{2}}\cos(s), \frac{1}{\sqrt{2}}\right)ds \\ &= \left(-\frac{1}{\sqrt{2}}\cos(s), -\frac{1}{\sqrt{2}}\sin(s), \frac{s}{\sqrt{2}}\right) + C_2\end{aligned} \tag{5.7}$$

where C_2 is a constant vector.

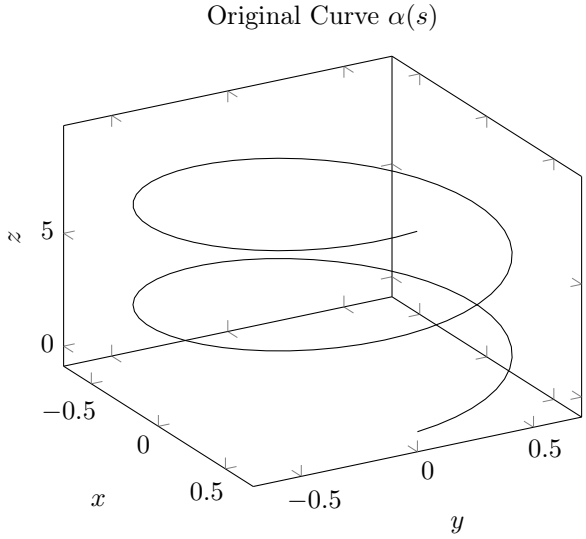

FIGURE 1. The original circular helix $\alpha(s)$

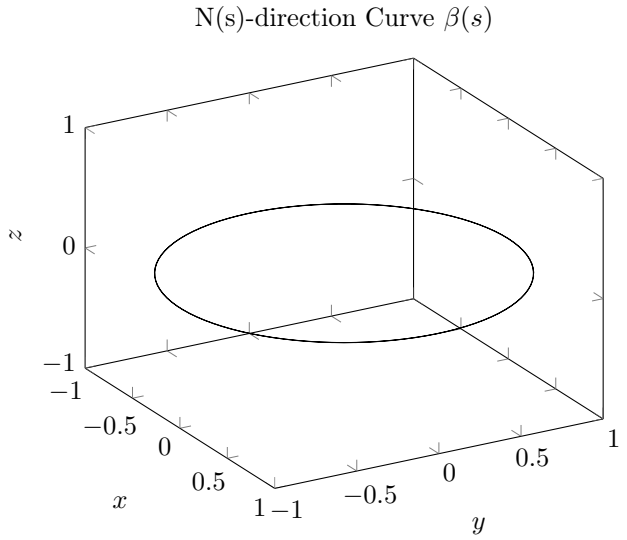

FIGURE 2. The N(s)-direction curve $\beta(s)$

CONCLUSION

In conclusion, this study enhances the comprehension of Bertrand and Mannheim curves as they pertain to normal-integral and binormal-integral curves in Euclidean 3-space. It has been shown that the distances between corresponding points on these curves remain unchanged, and distinctions have been identified in the behavior of angles between tangents for Bertrand and Mannheim curves. The research further

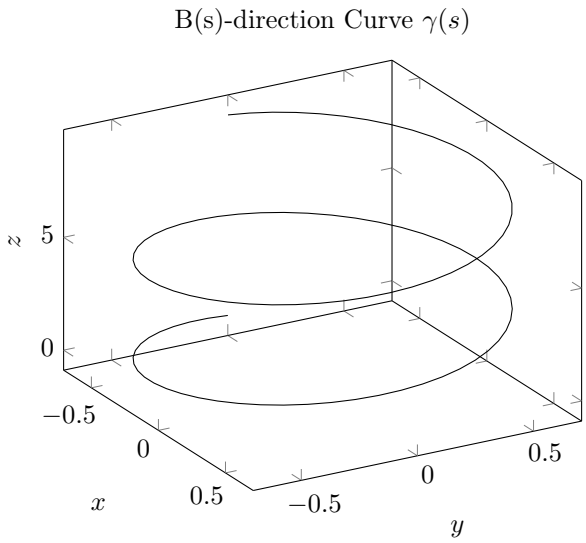

FIGURE 3. The B(s)-direction curve $\gamma(s)$

provides crucial criteria for determining whether normal-integral and binormal-integral curves can be classified as Bertrand or Mannheim curves, contributing to the broader understanding of these geometric entities.

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