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# ON BOUNDEDNESS AND COMPACTNESS OF THE HADAMARD-TYPE FRACTIONAL OPERATORS IN MORREY SPACES

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Abstract. The objective of this work is to establish the boundedness and compactness of Hadamard-type fractional integral operators in classical Morrey spaces. To prove the boundedness and compactness of these operators, we use respectively some results of [7] and Theorem [3, Th. 1.12 ] which is based on well-known Freche-Kolmogorov theorem. Moreover , we deduce the boundedness and the compactness of Hadamard fractional integral operators in the same spaces.

## 1. Introduction

The classical Morrey spaces were introduced by Charles Morrey in 1938, in connection with the investigation of the properties of solutions of quasi-linear differential equations. Also they are applied in functional analysis and other fields of mathematics. In the last decades, the boundedness and compactness of integral operators in Lebesgue spaces, Morrey spaces and Morrey-type spaces have been actively studied for example ( see [1],[2],[3],[5],[6] and [7]).

**Definition 1.1.** *Let $\Omega \subset \mathbb{R}^n$ be a Lebesgue measurable set, $0 < p \leq \infty, 0 \leq \lambda \leq \frac{n}{p}$. The Morrey space $M_p^{\lambda}(\Omega)$, is the space of all functions $f$ Lebesgue measurable on $\Omega$ for which*

$$||f||_{M_p^{\lambda}(\Omega)} = \sup_{t\in\Omega, r>0} r^{-\lambda}||f||_{L_p(B(t,r)\cap\Omega)} < \infty, \tag{1.1}$$

*where $B(t,r)$ is the open ball in $\mathbb{R}^n$ centered at the point $t$ of radius $r > 0$.*

*If $\lambda = 0$, then*

$$M_p^0(\Omega) = L_p(\Omega). \tag{1.2}$$

*If $\lambda = \frac{n}{p}$, then*

$$M_p^{\frac{n}{p}}(\Omega) = L_\infty(\Omega). \tag{1.3}$$

*If $p = \infty$, then (1.2) coincides with (1.3).*

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*If $\lambda < 0$ or $\lambda > \frac{n}{p}$, then the space $M_p^\lambda(\Omega)$ consists only of functions $f$ equivalent to 0 on $\Omega$.*

*If $0 < \lambda < \frac{n}{p}$, $M_p^\lambda(\Omega)$ is a quasi-Banach space (Banach space if $1 \le p \le \infty$).*

We use the following definition (see [4]).

**Definition 1.2.** *Let $\Omega = (a, b), (0 \le a < b \le \infty)$ be a finite or infinite interval of the half-axis $\mathbb{R}^+$. We consider the left-sided and right-sided integrals of fractional order $\alpha \in \mathbb{C}$ $(\mathcal{R}(\alpha) > 0)$ defined by*

$$\big(H_{a+}^\alpha f\big)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \big(\log \frac{x}{s}\big)^{\alpha-1} \frac{f(s)}{s} ds, \quad (a < x < b) \tag{1.4}$$

*and*

$$\big(H_{b-}^\alpha f\big)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \big(\log \frac{s}{x}\big)^{\alpha-1} \frac{f(s)}{s} ds, \quad (a < x < b), \tag{1.5}$$

*respectively.*

*The fractional integrals, more general than those in* (1.4) *and* (1.5) *with $\delta \ge 0$, are defined by*

$$\big(H_{a+,\delta}^\alpha f\big)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \Big(\frac{s}{x}\Big)^\delta \big(\log \frac{x}{s}\big)^{\alpha-1} \frac{f(s)}{s} ds, \quad (a < x < b) \tag{1.6}$$

*and*

$$\big(H_{b-,\delta}^\alpha f\big)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \Big(\frac{x}{s}\Big)^\delta \big(\log \frac{s}{x}\big)^{\alpha-1} \frac{f(s)}{s} ds, \quad (a < x < b). \tag{1.7}$$

## 2. Main results

The following lemma was proved in [7].

**Lemma 2.1.** *Let $a \in \mathbb{R}^n, b \in \overline{\mathbb{R}^n}, -\infty < a_i < b_i \le \infty, i = 1, \dots, n$ and $Q(a, b) = \{x \in \mathbb{R}^n, a_i < x_i < b_i, i = 1, \dots, n\}$, and $0 < p \le \infty, 0 \le \lambda \le \frac{n}{p}$. Then*

$$\|f\|_{L_p(Q(a,y))} \le |y - a|^\lambda \|f\|_{M_p^\lambda(Q(a,b))} \tag{2.1}$$

*for any parallelepiped $Q(a, b)$ and for any $y \in Q(a, b)$.*

By putting $\Omega = (a, b) = \{x \in \mathbb{R}, 0 < a < x < b < \infty\}$ in Lemma 2.1, we deduce the following inequality which is useful in the proofs of main theorems

$$||f||_{L_p(a,x)} \le (x - a)^\lambda ||f||_{M_p^\lambda(a,b)} \tag{2.2}$$

for any $x \in (a, b)$.

**Theorem 2.2.** *Let $1 < p \le \infty, 0 < q \le \infty, 0 \le \lambda \le \frac{1}{p}, 0 \le \mu \le \frac{1}{q}, \frac{1}{p} < \alpha < 1$, $\delta \ge 0$. Then $H_{a+,\delta}^\alpha$ is bounded from $M_p^\lambda(a, b)$ to $M_q^\mu(a, b)$.*

*Proof.* Let $f \in M_p^{\lambda}(a,b)$, $x \in B(t,r) \cap (a,b) \neq \emptyset$, $t \in (a,b), r > 0$. By using Hlder's inequality, we obtain

$$\begin{aligned}
\left|\left(H^{\alpha}_{a+,\delta} f\right)(x)\right| &= \left|\frac{1}{\Gamma(\alpha)} \int_a^x \left(\frac{s}{x}\right)^{\delta} \left(\log \frac{x}{s}\right)^{\alpha-1} \frac{f(s)}{s} ds\right| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_a^x \left| \left(\frac{s}{x}\right)^{\delta} \left(\log \frac{x}{s}\right)^{\alpha-1} \frac{f(s)}{s}\right| ds \\
&\leq \frac{1}{\Gamma(\alpha)} \left[\int_a^x \left( \left(\frac{s}{x}\right)^{\delta} \left(\frac{1}{s}\right) \left(\log \frac{x}{s}\right)^{\alpha-1}\right)^{\frac{p}{p-1}} ds\right]^{\frac{p-1}{p}} ||f||_{L_p(a,x)} \\
&= \frac{1}{\Gamma(\alpha)} \left[\int_a^x \left(\frac{s}{x}\right)^{\frac{\delta p}{p-1}} s^{\frac{p}{1-p}} \left(\log \frac{x}{s}\right)^{(\alpha-1)\frac{p}{p-1}} ds\right]^{\frac{p-1}{p}} ||f||_{L_p(a,x)}.
\end{aligned}$$

We set

$$I = \left[\int_a^x \left(\frac{s}{x}\right)^{\frac{\delta p}{p-1}} s^{\frac{p}{1-p}} \left(\log \frac{x}{s}\right)^{(\alpha-1)\frac{p}{p-1}} ds\right]^{\frac{p-1}{p}}.$$

By change of variable $s = xe^{-\tau}$, we get

$$\begin{aligned}
I &= \left(\int_0^{\log \frac{x}{a}} e^{-\delta\left(\frac{p}{p-1}\right)\tau} \left(xe^{-\tau}\right)^{\frac{p}{1-p}} \tau^{(\alpha-1)\frac{p}{p-1}} xe^{-\tau} d\tau\right)^{\frac{p-1}{p}} \\
&\leq \left(\int_0^{\log \frac{x}{a}} x^{\left(\frac{p}{1-p}+1\right)} e^{\tau\left(\frac{p}{p-1}-1\right)} \tau^{(\alpha-1)\frac{p}{p-1}} d\tau\right)^{\frac{p-1}{p}} \\
&= \left(\int_0^{\log \frac{x}{a}} x^{\left(\frac{1}{1-p}\right)} e^{\tau\left(\frac{1}{p-1}\right)} \tau^{(\alpha-1)\frac{p}{p-1}} d\tau\right)^{\frac{p-1}{p}} \\
&\leq x^{\left(\frac{1}{1-p}\right)\left(\frac{p-1}{p}\right)} \left(e^{\left(\frac{1}{p-1}\right)\log \frac{x}{a}}\right)^{\frac{p-1}{p}} \left(\int_0^{\log \frac{x}{a}} \tau^{(\alpha-1)\frac{p}{p-1}} d\tau\right)^{\frac{p-1}{p}} \\
&= x^{-\frac{1}{p}} \left(\frac{x}{a}\right)^{\frac{1}{p}} \left(\int_0^{\log \frac{x}{a}} \tau^{(\alpha-1)\frac{p}{p-1}} d\tau\right)^{\frac{p-1}{p}} \\
&= a^{-\frac{1}{p}} \left(\int_0^{\log \frac{x}{a}} \tau^{(\alpha-1)\frac{p}{p-1}} d\tau\right)^{\frac{p-1}{p}}.
\end{aligned}$$

Since $\frac{1}{p} < \alpha < 1$, we get $(\alpha-1)\frac{p}{p-1} + 1 > 0$ and we arrive at

$$\begin{aligned}
\left(\int_0^{\log \frac{x}{a}} \tau^{(\alpha-1)\frac{p}{p-1}} d\tau\right)^{\frac{p-1}{p}} &= \left(\frac{p-1}{\alpha p-1}\right)^{\frac{p-1}{p}} \left(\log \frac{x}{a}\right)^{\left(\frac{\alpha p-1}{p-1}\right)\left(\frac{p-1}{p}\right)} \\
&\leq \left(\frac{p-1}{\alpha p-1}\right)^{\frac{p-1}{p}} \left(\log \frac{b}{a}\right)^{\alpha-\frac{1}{p}}.
\end{aligned}$$

Thus

$$|(H^{\alpha}_{a+,\delta} f)(x)| \leq \frac{a^{-\frac{1}{p}}}{\Gamma(\alpha)} \left(\frac{p-1}{\alpha p-1}\right)^{\frac{p-1}{p}} \left(\log \frac{b}{a}\right)^{\alpha-\frac{1}{p}} ||f||_{L_p(a,x)} = C_1 ||f||_{L_p(a,x)},$$

where

$$C_1 = \frac{a^{-\frac{1}{p}}}{\Gamma(\alpha)} \left(\frac{p-1}{\alpha p-1}\right)^{\frac{p-1}{p}} \left(\log \frac{b}{a}\right)^{\alpha-\frac{1}{p}}.$$

By using inequality (2.2), we obtain

$$|(H^{\alpha}_{a+,\delta}f)(x)| \leq C_1(x-a)^{\lambda}||f||_{M^{\lambda}_p(a,b)} \leq C_1(b-a)^{\lambda}||f||_{M^{\lambda}_p(a,b)},$$

thus

$$|(H^{\alpha}_{a+,\delta}f)(x)| \leq C_1(b-a)^{\lambda}||f||_{M^{\lambda}_p(a,b)}. \tag{2.3}$$

Integrating (2.3) over $B(t,r)\cap(a,b)$, we get

$$||H^{\alpha}_{a+,\delta}f||_{L_q(B(t,r)\cap(a,b))} \leq C_1(b-a)^{\lambda}||1||_{L_q(B(t,r)\cap(a,b))}||f||_{M^{\lambda}_p(a,b)}.$$

We have

$$||1||_{L_q(B(t,r)\cap(a,b))} \leq \begin{cases} (2r)^{\frac{1}{q}} & : \ 0<r<b-a, \\ (b-a)^{\frac{1}{q}} & : \ r\geq b-a. \end{cases} \tag{2.4}$$

Now, we distinguish two cases.

If $0<r<b-a$, then

$$\begin{aligned} r^{-\mu}||H^{\alpha}_{a+,\delta}f||_{L_q(B(t,r)\cap(a,b))} &\leq r^{-\mu}(2r)^{\frac{1}{q}}C_1(b-a)^{\lambda}||f||_{M^{\lambda}_p(a,b)} \\ &= 2^{\frac{1}{q}}C_1 r^{-\mu+\frac{1}{q}}(b-a)^{\lambda}||f||_{M^{\lambda}_p(a,b)} \\ &\leq 2^{\frac{1}{q}}C_1(b-a)^{-\mu+\frac{1}{q}}(b-a)^{\lambda}||f||_{M^{\lambda}_p(a,b)} \\ &= C_2(b-a)^{\lambda+\frac{1}{q}-\mu}||f||_{M^{\lambda}_p(a,b)}, \end{aligned}$$

where $C_2 = 2^{\frac{1}{q}}C_1$. Consequently

$$\sup_{t\in(a,b),0<r<b-a} r^{-\mu}||H^{\alpha}_{a+,\delta}f||_{L_q(B(t,r)\cap(a,b))} \leq C_2(b-a)^{\lambda+\frac{1}{q}-\mu}||f||_{M^{\lambda}_p(a,b)}. \tag{2.5}$$

If $r\geq b-a$, then

$$\begin{aligned} r^{-\mu}||H^{\alpha}_{a+,\delta}f||_{L_q(B(t,r)\cap(a,b))} &\leq r^{-\mu}(b-a)^{\frac{1}{q}}C_1(b-a)^{\lambda}||f||_{M^{\lambda}_p(a,b)} \\ &\leq C_1(b-a)^{-\mu}(b-a)^{\lambda+\frac{1}{q}}||f||_{M^{\lambda}_p(a,b)} \\ &= C_1(b-a)^{\lambda+\frac{1}{q}-\mu}||f||_{M^{\lambda}_p(a,b)}. \end{aligned}$$

Then

$$\sup_{t\in(a,b),r\geq b-a} r^{-\mu}||H^{\alpha}_{a+,\delta}f||_{L_q(B(t,r)\cap(a,b))} \leq C_1(b-a)^{\lambda+\frac{1}{q}-\mu}||f||_{M^{\lambda}_p(a,b)}. \tag{2.6}$$

We have

$$||H^{\alpha}_{a+,\delta}f||_{M^{\mu}_q(a,b)}$$

$$= \max\left\{\sup_{t\in(a,b),0<r<b-a} r^{-\mu}||H^{\alpha}_{a+,\delta}f||_{L_q(B(t,r)\cap(a,b))}, \sup_{t\in(a,b),r\geq b-a} r^{-\mu}||H^{\alpha}_{a+,\delta}f||_{L_q(B(t,r)\cap(a,b))}\right\}.$$

Finally, by (2.5) and (2.6), we get

$$||H^{\alpha}_{a+,\delta}f||_{M^{\mu}_q(a,b)} \leq C_2(b-a)^{\lambda+\frac{1}{q}-\mu}||f||_{M^{\lambda}_p(a,b)},$$

it follows that

$$||H^{\alpha}_{a+,\delta}f||_{M^{\mu}_q(a,b)} \leq C_3||f||_{M^{\lambda}_p(a,b)},$$

where $C_3 = C_2(b-a)^{\lambda+\frac{1}{q}-\mu}$.

The proof is complete. □

**Remark.** *$H^{\alpha}_{b-,\delta}$ is a bounded operator from $M^{\lambda}_p(a,b)$ to $M^{\mu}_q(a,b)$ (the proof is similar to that of Theorem 2.2).*

**Remark.** *If we put $\delta = 0$, we get the boundedness of Hadamard operators $H^{\alpha}_{a+}, H^{\alpha}_{b-}$ from $M^{\lambda}_{p}(a,b)$ to $M^{\mu}_{q}(a,b)$.*

In [3] the Morrey space $L^{p,\lambda_1}(\mathbb{R}^n)$ is defined as follows

$$L^{p,\lambda_1}(\mathbb{R}^n) = \big\{ f \in L^p_{loc} : ||f||_{L^{p,\lambda_1}(\mathbb{R}^n)} = \sup_{t\in\mathbb{R}^n, r>0} r^{\frac{-\lambda_1}{p}} ||f||_{L_p(B(t,r))} < \infty \big\},$$

where $1 \le p < \infty$, $0 < \lambda_1 < n$, $n \ge 1$. If $\lambda = \frac{\lambda_1}{p}$, $0 < \lambda < \frac{n}{p}$, we have $L^{p,\lambda_1}(\mathbb{R}^n) = M^{\lambda}_{p}(\mathbb{R}^n)$.

We will work on the following subspace of $M^{\lambda}_{p}(\mathbb{R})$

$$M^{\lambda}_{p,a,b}(\mathbb{R}) = \big\{ f \in M^{\lambda}_{p}(\mathbb{R}) : f = 0 \text{ on } \mathbb{R}\backslash(a,b) \big\}. \tag{2.7}$$

Let for $-\infty < a < b < \infty$, $\gamma_{a,b} : \ M^{\lambda}_{p}(a,b) \to M^{\lambda}_{p,a,b}(\mathbb{R})$,

$$\gamma_{a,b}(f) = \tilde{f} = \begin{cases} f, & \text{on } (a,b), \\ 0 & \text{on } \mathbb{R}\backslash(a,b). \end{cases} \tag{2.8}$$

**Proposition 2.3.** *Let $0 < p \le \infty$, $0 \le \lambda \le \frac{1}{p}$,$-\infty < a < b < \infty$. Then*

(1) *$\gamma_{a,b}$ is bijective.*

(2)

$$\max\Big\{ (b-a)^{-\lambda} ||f||_{L_p(a,b)}, ||f||_{M^{\lambda}_{p}(a,b)} \Big\} \le ||\gamma_{a,b}(f)||_{M^{\lambda}_{p,a,b}(\mathbb{R})} \le \max\Big\{ (b-a)^{-\lambda} ||f||_{L_p(a,b)}, 2^{\lambda} ||f||_{M^{\lambda}_{p}(a,b)} \Big\}. \tag{2.9}$$

*Proof.* (1) Suppose that $f, g \in M^{\lambda}_{p}(a,b)$ are such that $\gamma(f) = \tilde{f} = \gamma(g) = \tilde{g}$ on $M^{\lambda}_{p,a,b}(a,b)$, since $\tilde{f} = \tilde{g}$ a.e on $\mathbb{R}$, thus $f = g$ on $M^{\lambda}_{p}(a,b)$, and $\gamma_{a,b}$ is injective. $\forall \tilde{f} \in M^{\lambda}_{p,a,b}(a,b)\, \exists f \in M^{\lambda}_{p}(a,b)$ such that $\gamma(f) = \tilde{f}$, then $\gamma_{a,b}$ is surjective. We deduce that $\gamma_{a,b}$ is bijective.

(2) Note that

$$||\gamma_{a,b}(f)||_{M^{\lambda}_{p,a,b}(\mathbb{R})} = \sup_{-\infty<t<\infty} \sup_{r>0} r^{-\lambda} ||\gamma_{a,b}(f)||_{L_p((t-r,t+r)\cap(a,b))} = \max\big\{ I_1, I_2, I_3, I_4, I_5 \big\},$$

where

$$I_1 = \sup_{t\le a} \sup_{a-t<r<b-t} r^{-\lambda} ||f||_{L_p(a,t+r)},$$

$$I_2 = \sup_{t\le a} \sup_{r\ge b-t} r^{-\lambda} ||f||_{L_p(a,b)},$$

$$I_3 = \sup_{a<t<b} \sup_{r>0} r^{-\lambda} ||f||_{L_p((t-r,t+r)\cap(a,b))},$$

$$I_4 = \sup_{t\ge b} \sup_{t-b<r<t-a} r^{-\lambda} ||f||_{L_p(t-r,b)},$$

$$I_5 = \sup_{t\ge b} \sup_{r\ge t-a} r^{-\lambda} ||f||_{L_p(a,b)}.$$

By inequality (2.2)

$$\begin{aligned} I_1 &\le \sup_{t\le a} \sup_{a-t<r<b-t} \left( \frac{t-a}{r} + 1 \right)^{\lambda} ||f||_{M^{\lambda}_{p}(a,b)} \\ &= \sup_{t\le a} 2^{\lambda} ||f||_{M^{\lambda}_{p}(a,b)} = 2^{\lambda} ||f||_{M^{\lambda}_{p}(a,b)}. \end{aligned}$$

Clearly,

$$I_2 = \sup_{t\le a}(b-t)^{-\lambda}||f||_{L_p(a,b)} = (b-a)^{-\lambda}||f||_{L_p(a,b)}$$

and

$$I_3 = ||f||_{M_p^\lambda(a,b)}.$$

Moreover, for similar reasons

$$I_4 \le \sup_{t\ge b}\sup_{t-b<r<t-a}\left(\frac{b-t}{r}+1\right)^{\lambda}||f||_{M_p^\lambda(a,b)} = 2^\lambda||f||_{M_p^\lambda(a,b)}$$

and

$$I_5 = (b-a)^{-\lambda}||f||_{L_p(a,b)}.$$

Consequently, inequality (2.9) follows. □

By putting $n=1$ in Theorem [3, Th. 1.12 ], we deduce the following Lemma.

**Lemma 2.4.** *Let $1 \le q < \infty$ and $0 < \mu < \frac{1}{q}$. Suppose the subset $D$ in $M_q^\mu(\mathbb{R})$ satisfies the following conditions:*

(1) *Uniform norm boundedness*

$$\sup_{f\in D}||f||_{M_q^\mu(\mathbb{R})} < \infty. \tag{2.10}$$

(2) *Uniform translation continuity*

$$\lim_{y\to 0^+}\sup_{f\in D}||f(\cdot+y)-f(\cdot)||_{M_q^\mu(\mathbb{R})} = 0. \tag{2.11}$$

(3)

$$\lim_{\theta\to\infty}\sup_{f\in D}||f\chi_{E_\theta}||_{M_q^\mu(\mathbb{R})} = 0, \tag{2.12}$$

*uniformly in $f\in D$, where $E_\theta = \{x\in\mathbb{R} : |x| > \theta\}$. Then $D$ is a strongly pre-compact set in $M_q^\mu(\mathbb{R})$.*

**Theorem 2.5.** *Let $1 < p \le \infty, 1 \le q < \infty, 0 \le \lambda \le \frac{1}{p}, \frac{1}{2}(1+\frac{1}{p}) < \alpha < 1, 0 \le \delta \le 1, 0 < \mu < \frac{1}{q}$ .*

*Let $f \in M_{p,a,b}^\lambda(\mathbb{R})$, $0 < a < b < \infty$, then $A$ is a compact operator from $M_{p,a,b}^\lambda(\mathbb{R})$ to $M_{q,a,b}^\mu(\mathbb{R})$, where*

$$Af(x) = \frac{\chi_{(a,b)}(x)}{\Gamma(\alpha)}\int_a^x\left(\frac{s}{x}\right)^\delta\left(\log\frac{x}{s}\right)^{\alpha-1}\frac{f(s)}{s}ds, \quad x\in\mathbb{R}.$$

*Proof.* For any $x\in\mathbb{R}$, $Af(x) = \chi_{(a,b)}(x)H_{a+,\delta}^\alpha f(x)$. By assumption, for every $p>1$, we have $\frac{1}{p} < \frac{1}{2}(1+\frac{1}{p})$, hence $\frac{1}{p} < \alpha < 1$. According to (2.7), Proposition 2.3 and boundedness of $H_{a+,\delta}^\alpha$, we get

$$\begin{aligned}||Af||_{M_{q,a,b}^\mu(\mathbb{R})} &\le \max\left\{(b-a)^{-\lambda}||H_{a+,\delta}^\alpha f||_{L_q(a,b)}, 2^\lambda||H_{a+,\delta}^\alpha f||_{M_q^\mu(a,b)}\right\}\\ &\le \max\left\{C_3(b-a)^{-\lambda}||f||_{L_p(a,b)}, 2^\lambda C_3||f||_{M_p^\lambda(a,b)}\right\}\\ &\le 2^\lambda C_3||f||_{M_{p,a,b}^\lambda(\mathbb{R})},\end{aligned} \tag{2.13}$$

where $C_3 > 0$ is independent of $f\in M_{p,a,b}^\lambda(\mathbb{R})$.

Let $F$ be an arbitrary bounded set in $M_{p,a,b}^\lambda(\mathbb{R})$, then there exists $M>0$ such that $||f||_{M_{p,a,b}^\lambda(\mathbb{R})} \le M$ for every $f\in F$.

(1) By using (2.13), we have $||Af||_{M^{\mu}_{q,a,b}(\mathbb{R})} \leq 2^{\lambda} C_3 M$, for every $f \in F$.
Finally

$$\sup_{h \in A(F)} ||h||_{M^{\mu}_{q,a,b}(\mathbb{R})} \leq 2^{\lambda} C_3 M < \infty.$$

a) By definition of $A$, for all $x \notin (a,b), y > 0$ sufficiently small such that $x+y \notin (a,b)$, we have

$$|Af(x+y) - Af(x)| = 0. \tag{2.14}$$

b) For $x \in (a,b), y > 0$, sufficiently small such that $x+y \in (a,b)$, it follows that

$$\begin{aligned}
|Af(x+y) - Af(x)| &= \left| \frac{1}{\Gamma(\alpha)} \int_a^{x+y} \left(\frac{s}{x+y}\right)^{\delta} \left(\log \frac{x+y}{s}\right)^{\alpha-1} \frac{f(s)}{s} ds - \frac{1}{\Gamma(\alpha)} \int_a^{x} \left(\frac{s}{x}\right)^{\delta} \left(\log \frac{x}{s}\right)^{\alpha-1} \frac{f(s)}{s} ds \right| \\
&= \frac{1}{\Gamma(\alpha)} \left| \int_a^{x} \left(\frac{s}{x+y}\right)^{\delta} \left(\log \frac{x+y}{s}\right)^{\alpha-1} \frac{f(s)}{s} ds + \int_x^{x+y} \left(\frac{s}{x+y}\right)^{\delta} \left(\log \frac{x+y}{s}\right)^{\alpha-1} \frac{f(s)}{s} ds \right. \\
&\quad \left. - \int_a^{x} \left(\frac{s}{x}\right)^{\delta} \left(\log \frac{x}{s}\right)^{\alpha-1} \frac{f(s)}{s} ds \right| \\
&\leq \frac{1}{\Gamma(\alpha)} \left| \int_a^{x} \left(\frac{s}{x+y}\right)^{\delta} \left(\log \frac{x+y}{s}\right)^{\alpha-1} \frac{f(s)}{s} - \left(\frac{s}{x}\right)^{\delta} \left(\log \frac{x}{s}\right)^{\alpha-1} \frac{f(s)}{s} ds \right| \\
&\quad + \frac{1}{\Gamma(\alpha)} \left| \int_x^{x+y} \left(\frac{s}{x+y}\right)^{\delta} \left(\log \frac{x+y}{s}\right)^{\alpha-1} \frac{f(s)}{s} ds \right| \\
&= J_1 + J_2.
\end{aligned}$$

Thus

$$\begin{aligned}
J_1 &= \frac{1}{\Gamma(\alpha)} \left| \int_a^{x} \left(\frac{s}{x+y}\right)^{\delta} \left(\log \frac{x+y}{s}\right)^{\alpha-1} \frac{f(s)}{s} - \left(\frac{s}{x}\right)^{\delta} \left(\log \frac{x}{s}\right)^{\alpha-1} \frac{f(s)}{s} ds \right| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_a^{x} \left| \left( \frac{1}{\left(\frac{x+y}{s}\right)^{\delta} \left(\log \frac{x+y}{s}\right)^{1-\alpha}} - \frac{1}{\left(\frac{x}{s}\right)^{\delta} \left(\log \frac{x}{s}\right)^{1-\alpha}} \right) \left( \frac{f(s)}{s} \right) \right| ds \\
&= \frac{1}{\Gamma(\alpha)} \int_a^{x} \left| \left( \frac{\left(\frac{x}{s}\right)^{\delta} \left(\log \frac{x}{s}\right)^{1-\alpha} - \left(\frac{x+y}{s}\right)^{\delta} \left(\log \frac{x+y}{s}\right)^{1-\alpha}}{\left(\frac{x}{s}\right)^{\delta} \left(\log \frac{x}{s}\right)^{1-\alpha} \left(\frac{x+y}{s}\right)^{\delta} \left(\log \frac{x+y}{s}\right)^{1-\alpha}} \right) \left( \frac{f(s)}{s} \right) \right| ds \\
&\leq \frac{1}{\Gamma(\alpha)} \int_a^{x} \left( \frac{\left(\frac{x+y}{s}\right)^{\delta} \left(\log \frac{x+y}{s}\right)^{1-\alpha} - \left(\frac{x}{s}\right)^{\delta} \left(\log \frac{x}{s}\right)^{1-\alpha}}{\left(\frac{x}{s}\right)^{2\delta} \left(\log \frac{x}{s}\right)^{2(1-\alpha)}} \right) \left( \frac{|f(s)|}{s} \right) ds.
\end{aligned}$$

Now, we distinguish two cases.

a) If $0 < \delta \leq 1$, we use the well-known inequality $(x+y)^{\beta} \leq x^{\beta} + y^{\beta}$ if $x, y \geq 0$, $0 < \beta \leq 1$, thus

$$\begin{aligned}
J_1 &\leq \frac{1}{\Gamma(\alpha)} \int_a^{x} \left( \frac{\left[ \left(\frac{x}{s}\right)^{\delta} + \left(\frac{y}{s}\right)^{\delta} \right] \left( \left(\log \frac{x+y}{s}\right)^{1-\alpha} \right) - \left(\frac{x}{s}\right)^{\delta} \left(\log \frac{x}{s}\right)^{1-\alpha}}{\left(\frac{x}{s}\right)^{2\delta} \left(\log \frac{x}{s}\right)^{2(1-\alpha)}} \right) \left( \frac{|f(s)|}{s} \right) ds \\
&= \frac{1}{\Gamma(\alpha)} \int_a^{x} \left( \frac{\left(\frac{x}{s}\right)^{\delta} \left[ \left(\log \frac{x+y}{s}\right)^{1-\alpha} - \left(\log \frac{x}{s}\right)^{1-\alpha} \right] + \left(\frac{y}{s}\right)^{\delta} \left(\log \frac{x+y}{s}\right)^{1-\alpha}}{\left(\frac{x}{s}\right)^{2\delta} \left(\log \frac{x}{s}\right)^{2(1-\alpha)}} \right) \left( \frac{|f(s)|}{s} \right) ds.
\end{aligned}$$

By using the following inequality $|x^\beta - y^\beta| \leq |x-y|^\beta$ if $x, y > 0$ and by putting $\beta = 1-\alpha$, we get

$$J_1 \leq \frac{1}{\Gamma(\alpha)} \int_a^x \left( \frac{\left(\frac{x}{s}\right)^\delta \left(\log \frac{x+y}{x}\right)^{1-\alpha} + \left(\frac{y}{s}\right)^\delta \left(\log \frac{x+y}{s}\right)^{1-\alpha}}{\left(\frac{x}{s}\right)^{2\delta} \left(\log \frac{x}{s}\right)^{2(1-\alpha)}} \right) \left( \frac{\left|f(s)\right|}{s} \right) ds$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_a^x \left[ \left(\frac{x}{s}\right)^\delta \left(\frac{y}{x}\right)^{1-\alpha} + \left(\frac{y}{s}\right)^\delta \left(\log \frac{x+y}{s}\right)^{1-\alpha} \right] \left(\frac{s}{x}\right)^{2\delta} \left(\log \frac{x}{s}\right)^{2(\alpha-1)} \frac{\left|f(s)\right|}{s} ds$$

$$= \frac{1}{\Gamma(\alpha)} \int_a^x \left[ x^{\delta+\alpha-1} y^{1-\alpha} s^{-\delta} + y^\delta s^{-\delta} \left(\log \frac{x+y}{s}\right)^{1-\alpha} \right] \left(\frac{s}{x}\right)^{2\delta} \left(\log \frac{x}{s}\right)^{2(\alpha-1)} \frac{\left|f(s)\right|}{s} ds$$

$$\leq \frac{\max(a^{\delta+\alpha-1}, b^{\delta+\alpha-1}) y^{1-\alpha} a^{-\delta} + y^\delta a^{-\delta} \left(\log \frac{b}{a}\right)^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \left(\frac{s}{x}\right)^{2\delta} \left(\log \frac{x}{s}\right)^{2(\alpha-1)} \frac{\left|f(s)\right|}{s} ds.$$

Now, we conclude that

$$J_1 \leq C_4 (y^{1-\alpha} + y^\delta) \int_a^x \left(\frac{s}{x}\right)^{2\delta} \left(\log \frac{x}{s}\right)^{2(\alpha-1)} \frac{\left|f(s)\right|}{s} ds, \tag{2.15}$$

where $C_4 = \dfrac{\max\left(\max(a^{\delta+\alpha-1}, b^{\delta+\alpha-1}) a^{-\delta}, a^{-\delta} \left(\log \frac{b}{a}\right)^{1-\alpha}\right)}{\Gamma(\alpha)}$.

By applying classical Hlder's inequality, we obtain

$$J_1 \leq C_4 (y^{1-\alpha} + y^\delta) \left[ \int_a^x \left(\frac{s}{x}\right)^{\frac{2\delta p}{p-1}} s^{\frac{p}{1-p}} \left(\log \frac{x}{s}\right)^{2(\alpha-1)\frac{p}{p-1}} ds \right]^{\frac{p-1}{p}} ||f||_{L_p(a,x)}. \tag{2.16}$$

Let

$$J_1' = \left[ \int_a^x \left(\frac{s}{x}\right)^{\frac{2\delta p}{p-1}} s^{\frac{p}{1-p}} \left(\log \frac{x}{s}\right)^{2(\alpha-1)\frac{p}{p-1}} ds \right]^{\frac{p-1}{p}}.$$

By change of variable $s = xe^{-\tau}$, we get

$$\begin{aligned}
J_1' &= \left( \int_0^{\log \frac{x}{a}} e^{\left(\frac{-2\delta p}{p-1}\right)\tau} \left(xe^{-\tau}\right)^{\frac{p}{1-p}} \tau^{2(\alpha-1)\frac{p}{p-1}} xe^{-\tau} d\tau \right)^{\frac{p-1}{p}} \\
&\leq \left( \int_0^{\log \frac{x}{a}} x^{\left(\frac{p}{1-p}+1\right)} e^{\tau\left(\frac{p}{p-1}-1\right)} \tau^{2(\alpha-1)\frac{p}{p-1}} d\tau \right)^{\frac{p-1}{p}} \\
&= \left( \int_0^{\log \frac{x}{a}} x^{\left(\frac{1}{1-p}\right)} e^{\tau\left(\frac{1}{p-1}\right)} \tau^{2(\alpha-1)\frac{p}{p-1}} d\tau \right)^{\frac{p-1}{p}} \\
&\leq x^{\left(\frac{1}{1-p}\right)\left(\frac{p-1}{p}\right)} \left( e^{\left(\frac{1}{p-1}\right)\log \frac{x}{a}} \right)^{\frac{p-1}{p}} \left( \int_0^{\log \frac{x}{a}} \tau^{2(\alpha-1)\frac{p}{p-1}} d\tau \right)^{\frac{p-1}{p}} \\
&= x^{-\frac{1}{p}} \left(\frac{x}{a}\right)^{\frac{1}{p}} \left( \int_0^{\log \frac{x}{a}} \tau^{2(\alpha-1)\frac{p}{p-1}} d\tau \right)^{\frac{p-1}{p}} \\
&= a^{-\frac{1}{p}} \left( \int_0^{\log \frac{x}{a}} \tau^{2(\alpha-1)\frac{p}{p-1}} d\tau \right)^{\frac{p-1}{p}}.
\end{aligned}$$

From $\frac{1}{2}\big(1+\frac{1}{p}\big) < \alpha$, it is follows that $\frac{p}{p-1}(1+\frac{1}{p}) - \frac{2p}{p-1} < \frac{2\alpha p}{p-1} - \frac{2p}{p-1}$, consequently $\frac{p}{p-1}\big[\frac{1}{p}-1\big] + \frac{p-1}{p-1} < 2(\alpha-1)\frac{p}{p-1}+1$, then $2(\alpha-1)\frac{p}{p-1}+1>0$. Therefore

$$\begin{aligned}\left(\int_0^{\log\frac{x}{a}} \tau^{2(\alpha-1)\frac{p}{p-1}} d\tau\right)^{\frac{p-1}{p}} &= \left(\frac{p-1}{2\alpha p-p-1}\right)^{\frac{p-1}{p}} \left(\log\frac{x}{a}\right)^{\left(\frac{2\alpha p-p-1}{p-1}\right)\left(\frac{p-1}{p}\right)} \\ &\leq \left(\frac{p-1}{2\alpha p-p-1}\right)^{\frac{p-1}{p}} \left(\log\frac{b}{a}\right)^{2\alpha-1-\frac{1}{p}}.\end{aligned}$$

From (2.16), we deduce

$$J_1 \leq C_4(y^{1-\alpha}+y^{\delta})a^{-\frac{1}{p}}\left(\frac{p-1}{2\alpha p-p-1}\right)^{\frac{p-1}{p}} \left(\log\frac{b}{a}\right)^{2\alpha-1-\frac{1}{p}} ||f||_{L_p(a,x)} = C_5(y^{1-\alpha}+y^{\delta})||f||_{L_p(a,x)},$$

where

$$C_5 = C_4 a^{-\frac{1}{p}}\left(\frac{p-1}{2\alpha p-p-1}\right)^{\frac{p-1}{p}} \left(\log\frac{b}{a}\right)^{2\alpha-1-\frac{1}{p}}.$$

By inequality (2.2), we obtain

$$J_1 \leq C_5(y^{1-\alpha}+y^{\delta})(x-a)^{\lambda}||f||_{M_p^{\lambda}(a,b)} \leq C_5 M(y^{1-\alpha}+y^{\delta})(b-a)^{\lambda}. \qquad (2.17)$$

By applying Hlder's inequality, we estimate $J_2$

$$\begin{aligned} J_2 &= \frac{1}{\Gamma(\alpha)}\left|\int_x^{x+y}\left(\frac{s}{x+y}\right)^{\delta}\left(\log\frac{x+y}{s}\right)^{\alpha-1}\frac{f(s)}{s}ds\right| \\ &\leq \frac{1}{\Gamma(\alpha)}\int_x^{x+y}\left|\left(\frac{s}{x+y}\right)^{\delta}\left(\log\frac{x+y}{s}\right)^{\alpha-1}\frac{f(s)}{s}\right|ds \\ &\leq \frac{1}{\Gamma(\alpha)}\left[\int_x^{x+y}\left(\left(\frac{s}{x+y}\right)^{\delta}\left(\frac{1}{s}\right)\left(\log\frac{x+y}{s}\right)^{\alpha-1}\right)^{\frac{p}{p-1}}ds\right]^{\frac{p-1}{p}}||f||_{L_p(x,x+y)} \\ &= \frac{1}{\Gamma(\alpha)}\left[\int_x^{x+y}\left(\frac{s}{x+y}\right)^{\frac{\delta p}{p-1}} s^{\frac{p}{1-p}}\left(\log\frac{x+y}{s}\right)^{(\alpha-1)\frac{p}{p-1}}ds\right]^{\frac{p-1}{p}}||f||_{L_p(x,x+y)}.\end{aligned}$$

Let

$$J_2' = \left[\int_x^{x+y}\left(\frac{s}{x+y}\right)^{\frac{\delta p}{p-1}} s^{\frac{p}{1-p}}\left(\log\frac{x+y}{s}\right)^{(\alpha-1)\frac{p}{p-1}}ds\right]^{\frac{p-1}{p}}.$$

By change of variable $s = (x+y)e^{-\tau}$, we obtain

$$\begin{aligned}
J_2' &= \left(\int_0^{\log\frac{x+y}{x}} e^{\frac{-\delta p}{p-1}\tau}\big((x+y)e^{-\tau}\big)^{\left(\frac{p}{1-p}\right)}\tau^{(\alpha-1)\frac{p}{p-1}}(x+y)e^{-\tau}d\tau\right)^{\frac{p-1}{p}}\\
&\le \left(\int_0^{\log\frac{x+y}{x}} (x+y)^{\left(\frac{p}{1-p}+1\right)}e^{\tau\left(\frac{p}{p-1}-1\right)}\tau^{(\alpha-1)\frac{p}{p-1}}d\tau\right)^{\frac{p-1}{p}}\\
&= \left(\int_0^{\log\frac{x+y}{x}} (x+y)^{\left(\frac{1}{1-p}\right)}e^{\tau\left(\frac{1}{p-1}\right)}\tau^{(\alpha-1)\frac{p}{p-1}}d\tau\right)^{\frac{p-1}{p}}\\
&\le (x+y)^{\left(\frac{1}{1-p}\right)\left(\frac{p-1}{p}\right)}\left(e^{\left(\frac{1}{p-1}\right)\log\frac{x+y}{x}}\right)^{\frac{p-1}{p}}\left(\int_0^{\log\frac{x+y}{x}}\tau^{(\alpha-1)\frac{p}{p-1}}d\tau\right)^{\frac{p-1}{p}}\\
&= (x+y)^{\frac{-1}{p}}\left(\frac{x+y}{x}\right)^{\frac{1}{p}}\left(\int_0^{\log\frac{x+y}{x}}\tau^{(\alpha-1)\frac{p}{p-1}}d\tau\right)^{\frac{p-1}{p}}\\
&\le a^{-\frac{1}{p}}\left(\int_0^{\log\frac{x+y}{x}}\tau^{(\alpha-1)\frac{p}{p-1}}d\tau\right)^{\frac{p-1}{p}}.
\end{aligned}$$

Since $\frac{1}{p} < \alpha < 1$, $(\alpha-1)\frac{p}{p-1} + 1 > 0$. Consequently

$$\begin{aligned}
\left(\int_0^{\log\frac{x+y}{x}}\tau^{(\alpha-1)\frac{p}{p-1}}d\tau\right)^{\frac{p-1}{p}} &= \left(\frac{p-1}{\alpha p-1}\right)^{\frac{p-1}{p}}\left(\log\frac{x+y}{x}\right)^{\left(\frac{\alpha p-1}{p-1}\right)\left(\frac{p-1}{p}\right)}\\
&\le \left(\frac{p-1}{\alpha p-1}\right)^{\frac{p-1}{p}}\left(\frac{y}{x}\right)^{\frac{\alpha p-1}{p}}\\
&\le \left(\frac{p-1}{\alpha p-1}\right)^{\frac{p-1}{p}}\left(\frac{y}{a}\right)^{\alpha-\frac{1}{p}}.
\end{aligned}$$

Thus

$$J_2 \le \frac{J_2'}{\Gamma(\alpha)}||f||_{L_p(x,x+y)} \le \frac{a^{-\frac{1}{p}}}{\Gamma(\alpha)}\left(\frac{p-1}{\alpha p-1}\right)^{\frac{p-1}{p}}\left(\frac{y}{a}\right)^{\alpha-\frac{1}{p}}||f||_{L_p(x,x+y)}.$$

Since $x, x+y \in (a,b)$, we obtain $(x,x+y)\cap(a,b) = (x,x+y)$, then $(x,x+y) = B(x+\frac{y}{2},\frac{y}{2})$, where $B(x+\frac{y}{2},\frac{y}{2})$ is the open ball in $\mathbb{R}$ centered at the point $x+\frac{y}{2}$ of radius $\frac{y}{2}$.

Note that

$$\begin{aligned}
||f||_{M_p^\lambda(a,b)} &= \sup_{t\in(a,b),r>0} r^{-\lambda}||f||_{L_p(B(t,r)\cap(a,b))}\\
&\ge r^{-\lambda}||f||_{L_p\left(B(t,r)\cap(a,b)\right)\Big|_{t=x+\frac{y}{2},r=\frac{y}{2}}}\\
&= \left(\frac{y}{2}\right)^{-\lambda}||f||_{L_p\left((a,b)\cap B(x+\frac{y}{2},\frac{y}{2})\right)}\\
&= 2^\lambda y^{-\lambda}||f||_{L_p(x,x+y)},
\end{aligned}$$

hence

$$J_2 \le \frac{2^{-\lambda}a^{-\alpha}}{\Gamma(\alpha)}\left(\frac{p-1}{\alpha p-1}\right)^{\frac{p-1}{p}} y^{\alpha-\frac{1}{p}+\lambda}||f||_{M_p^\lambda(a,b)} \le C_6 M y^{\tilde{\lambda}}, \tag{2.18}$$

where $C_6 = \dfrac{2^{-\lambda}a^{-\alpha}}{\Gamma(\alpha)}\left(\dfrac{p-1}{\alpha p-1}\right)^{\frac{p-1}{p}}$ and $\tilde{\lambda} = \alpha - \dfrac{1}{p} + \lambda$ .

Taking in account (2.17) and (2.18), we arrive at

$$\begin{aligned} |Af(x+y) - Af(x)| &\leq M\chi_{(a,b)}(x)\Big(C_5(y^{1-\alpha}+y^{\delta})(b-a)^{\lambda} + C_6 y^{\tilde{\lambda}}\Big) \\ &\leq C_7 M\chi_{(a,b)}(x)\Big((y^{1-\alpha}+y^{\delta})(b-a)^{\lambda} + y^{\tilde{\lambda}}\Big), \end{aligned}$$

where $C_7 = \max(C_5, C_6)$. Then

$$\begin{aligned} ||Af(\cdot+y) - Af(\cdot)||_{L_q(B(t,r))} &\leq C_7 M||\chi_{(a,b)}||_{L_q(B(t,r))}\Big((y^{1-\alpha}+y^{\delta})(b-a)^{\lambda} + y^{\tilde{\lambda}}\Big) \\ &= C_7 M||1||_{L_q(B(t,r)\cap(a,b))}\Big((y^{1-\alpha}+y^{\delta})(b-a)^{\lambda} + y^{\tilde{\lambda}}\Big). \end{aligned}$$

Next, we distinguish two cases. By (2.4), we get

(1) If $0 < r < b-a$, then

$$\begin{aligned} r^{-\mu}||Af(\cdot+y) - Af(\cdot)||_{L_q(B(t,r))} &\leq r^{-\mu}(2r)^{\frac{1}{q}} C_7 M\Big((y^{1-\alpha}+y^{\delta})(b-a)^{\lambda} + y^{\tilde{\lambda}}\Big) \\ &= 2^{\frac{1}{q}} C_7 r^{\frac{1}{q}-\mu} M\Big((y^{1-\alpha}+y^{\delta})(b-a)^{\lambda} + y^{\tilde{\lambda}}\Big) \\ &\leq C_8 (b-a)^{\frac{1}{q}-\mu} M\Big((y^{1-\alpha}+y^{\delta})(b-a)^{\lambda} + y^{\tilde{\lambda}}\Big), \end{aligned}$$

where $C_8 = 2^{\frac{1}{q}} C_7$.

Thus

$$\sup_{t\in\mathbb{R}, 0<r<b-a} r^{-\mu}||Af(\cdot+y) - Af(\cdot)||_{L_q(B(t,r))} \leq C_8 (b-a)^{\frac{1}{q}-\mu} M\Big((y^{1-\alpha}+y^{\delta})(b-a)^{\lambda} + y^{\tilde{\lambda}}\Big). \tag{2.19}$$

(2) If $r \geq b-a$, then

$$\begin{aligned} r^{-\mu}||Af(\cdot+y) - Af(\cdot)||_{L_q(B(t,r))} &\leq r^{-\mu}(b-a)^{\frac{1}{q}} C_7 M\Big((y^{1-\alpha}+y^{\delta})(b-a)^{\lambda} + y^{\tilde{\lambda}}\Big) \\ &\leq C_7 (b-a)^{\frac{1}{q}-\mu} M\Big((y^{1-\alpha}+y^{\delta})(b-a)^{\lambda} + y^{\tilde{\lambda}}\Big). \end{aligned}$$

Thus

$$\sup_{t\in\mathbb{R}, r\geq b-a} r^{-\mu}||Af(\cdot+y) - Af(\cdot)||_{L_q(B(t,r))} \leq C_7 M (b-a)^{\frac{1}{q}-\mu}\Big((y^{1-\alpha}+y^{\delta})(b-a)^{\lambda} + y^{\tilde{\lambda}}\Big). \tag{2.20}$$

We have

$$||Af(\cdot+y) - Af(\cdot)||_{M^{\mu}_{q,a,b}(\mathbb{R})}$$

$$= \max\left\{ \sup_{t\in\mathbb{R}, 0<r<b-a} r^{-\mu}||Af(\cdot+y) - Af(\cdot)||_{L_q(B(t,r))}, \sup_{t\in\mathbb{R}, r\geq b-a} r^{-\mu}||Af(\cdot+y) - Af(\cdot)||_{L_q(B(t,r))} \right\}.$$

Consequently, by (2.19) and (2.20), we obtain

$$\begin{aligned} ||Af(\cdot+y) - Af(\cdot)||_{M^{\mu}_{q,a,b}(\mathbb{R})} &\leq C_8 (b-a)^{\frac{1}{q}-\mu} M\Big((y^{1-\alpha}+y^{\delta})(b-a)^{\lambda} + y^{\tilde{\lambda}}\Big) \\ &= C_9(y^{1-\alpha}+y^{\delta}) + C_{10} y^{\tilde{\lambda}} \\ &\leq \max(C_9, C_{10})\big(y^{1-\alpha}+y^{\delta}+y^{\tilde{\lambda}}\big), \end{aligned}$$

where $C_9 = C_8 M(b-a)^{\lambda+\frac{1}{q}-\mu}$ and $C_{10} = C_8 M(b-a)^{\frac{1}{q}-\mu}$. Then

$$\lim_{y\to 0^+} \sup_{Af\in A(F)} ||Af(\cdot+y) - Af(\cdot)||_{M^{\mu}_{q,a,b}(\mathbb{R})} = 0,$$

uniformly in $Af \in A(F)$.

b) Now we will deal with the case $\delta = 0$, by using the inequality $|x^\beta - y^\beta| \leq |x-y|^\beta$, $x, y > 0$, $0 < \beta < 1$, we get

$$\begin{aligned} J_1 &= \frac{1}{\Gamma(\alpha)} \int_a^x \left[ \left( \log \frac{x+y}{s} \right)^{1-\alpha} - \left( \log \frac{x}{s} \right)^{1-\alpha} \right] \left( \log \frac{x}{s} \right)^{2(\alpha-1)} \frac{\big|f(s)\big|}{s} ds \\ &\leq \frac{1}{\Gamma(\alpha)} \int_a^x \left( \log \frac{x+y}{x} \right)^{1-\alpha} \left( \log \frac{x}{s} \right)^{2(\alpha-1)} \frac{\big|f(s)\big|}{s} ds \\ &\leq \frac{1}{\Gamma(\alpha)} \left( \frac{y}{x} \right)^{1-\alpha} \int_a^x \left( \log \frac{x}{s} \right)^{2(\alpha-1)} \frac{\big|f(s)\big|}{s} ds \\ &\leq \frac{1}{\Gamma(\alpha)} \left( \frac{y}{a} \right)^{1-\alpha} \int_a^x \left( \log \frac{x}{s} \right)^{2(\alpha-1)} \frac{\big|f(s)\big|}{s} ds. \end{aligned}$$

Thus

$$J_1 \leq C_{11} y^{1-\alpha} \int_a^x \left( \log \frac{x}{s} \right)^{2(\alpha-1)} \frac{\big|f(s)\big|}{s} ds, \tag{2.21}$$

where $C_{11} = \dfrac{a^{\alpha-1}}{\Gamma(\alpha)}$.

By the same proof as previously mentioned, we can obtain

$$||Af(\cdot+y) - Af(\cdot)||_{M^{\mu}_{q,a,b}(\mathbb{R})} \leq C_{12}\big(y^{1-\alpha} + y^{\tilde{\lambda}}\big),$$

where $C_{12} > 0$. Thus

$$\lim_{y\to 0^+} \sup_{Af\in A(F)} ||Af(\cdot+y) - Af(\cdot)||_{M^{\mu}_{q,a,b}(\mathbb{R})} = 0,$$

uniformly in $Af \in A(F)$.

(3) Let $E_\theta = \big\{x \in \mathbb{R},\ |x| > \theta\big\}$, thus $(Af)\chi_{E_\theta} \equiv 0$ for any $f \in F$ and $\theta > b$. Then

$$\sup_{Af\in A(F)} ||(Af)\chi_{E_\theta}||_{M^{\mu}_{q,a,b}(\mathbb{R})} = \sup_{Af\in A(F)} \sup_{t\in\mathbb{R},\, r>0} r^{-\mu} ||(Af)\chi_{E_\theta}||_{(B(t,r))} = 0,$$

and

$$\lim_{\theta\to\infty} \sup_{Af\in A(F)} ||(Af)\chi_{E_\theta}||_{M^{\mu}_{q,a,b}(\mathbb{R})} = 0,$$

uniformly in $Af \in A(F)$.

By Lemma 2.4 we conclude that $A(F)$ is strongly pre-compact in $M^{\mu}_{q,a,b}(\mathbb{R})$. Since $F$ is bounded, then $A$ is a compact operator from $M^{\lambda}_{p,a,b}(\mathbb{R})$ to $M^{\mu}_{q,a,b}(\mathbb{R})$ . □

Let the operator $A^*$ defined as follows

$$A^* f(x) = \frac{\chi_{(a,b)}(x)}{\Gamma(\alpha)} \int_x^b \left( \frac{x}{s} \right)^{\delta} \left( \log \frac{s}{x} \right)^{\alpha-1} \frac{f(s)}{s} ds, \quad x \in \mathbb{R}.$$

**Corollary 2.6.** *If the operator $A^*$ satisfies the assumptions of Theorem 2.5, then $A^*$ is a compact operator from $M^{\lambda}_{p,a,b}(\mathbb{R})$ to $M^{\mu}_{q,a,b}(\mathbb{R})$.*

The proof is similar to that of Theorem 2.5.
Let the operators $T$ and $T^*$ are such that $T, T^* : M^{\lambda}_{p,a,b}(\mathbb{R}) \to M^{\mu}_{q,a,b}(\mathbb{R})$, where

$$Tf(x) = \frac{\chi_{(a,b)}(x)}{\Gamma(\alpha)} \int_a^x \Big(\log \frac{x}{s}\Big)^{\alpha-1} \frac{f(s)}{s} ds, \quad x \in \mathbb{R},$$

and

$$T^*f(x) = \frac{\chi_{(a,b)}(x)}{\Gamma(\alpha)} \int_x^b \Big(\log \frac{s}{x}\Big)^{\alpha-1} \frac{f(s)}{s} ds, \quad x \in \mathbb{R}.$$

**Remark.** *By setting $\delta = 0$ in Theorem 2.5, we conclude that $T$ and $T^*$ are compact operators from $M^{\lambda}_{p,a,b}(\mathbb{R})$ to $M^{\mu}_{q,a,b}(\mathbb{R})$ .*

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