Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 16 Issue 2(2024), Pages 31-39
https://doi.org/10.54671/BMAA-2024-2-3

AN EXAMPLE OF ALMOST COSYMPLECTIC 3-MANIFOLDS

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Abstract. In the present paper, we construct an example of almost cosymplectic 3-manifolds which is isomorphic to the group $E(1,1)$ of rigid motions of Minkowski 2-space.

1. Introduction

A $(2n+1)$-dimensional manifold M having the property that the structural group of its tangent bundle is reducible $U(n)\times\{1\}$ is called an almost contact manifold. Let M be almost contact metric manifold, i.e., M is a differentiable manifold and (φ,ξ,η,g) is an almost contact metric structure on M, formed by tensor fields φ,ξ,η of type $(1,1),(1,0),(0,1)$, respectively, and a Riemannian metric g such that

$$\begin{cases}\varphi^2=-I+\eta\otimes\xi,\ \varphi\xi=0,\ \eta\circ\varphi=0,\\ \eta(X)=g(X,\xi),\ g(\varphi X,\varphi Y)=g(X,Y)-\eta(X)\eta(Y).\end{cases} \tag{1.1}$$

As a consequence of the above relations we have

$$\eta(\xi)=1,\ \ d\eta(\xi,X)=0. \tag{1.2}$$

On such manifold we may always define a 2-form Φ by $\Phi(X,Y)=g(\varphi X,Y)$. A 2-form Φ is called the fundamental form of (φ,ξ,η,g).

We denote a vector field on $M\times\mathbb{R}$ by $(X,f\dfrac{d}{ds})$, where X is tangent to M, s is the coordinate on $\mathbb{R}$, and f is a $\mathcal{C}^\infty$ function on $M\times\mathbb{R}$. Define an almost complex structure on $M\times\mathbb{R}$ by $J(X,f\dfrac{d}{ds})=(\varphi X-f\xi,\eta(X)\dfrac{d}{ds})$, then $J^2=-I$ is easy checked. If now, J is integrable, we say that M is normal.

From (1.1) it is easily seen that $|\Phi|^2=\langle\Phi,\Phi\rangle=2n$ where $\langle\cdot,\cdot\rangle$ denotes the local scalar product induced by g. If the fundamental vector field of a contact metric structure is a Killing field with respect to its contact metric, the manifold is said to be almost Sasakian. If (M,φ,ξ,η,g) is Sasakian, then $\nabla\xi=\varphi$ and $\nabla_X\varphi=-g(X,\cdot)\xi+\eta\otimes X$. An almost contact metric manifold (M,φ,ξ,η,g) is called almost cosymplectic if both its fundamental form and contact form are closed,

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2000 *Mathematics Subject Classification.* 53B05, 53B12, 53C21, 53C25, 53C30.
Key words and phrases. Almost contact metric manifold; almost cosymplectic 3-manifold; Lie group.

Submitted December 23, 2023. Accepted May 25, 2024. Published June 12, 2024.
Communicated by U. C. De.

that is, if $d\Phi = 0$ and $d\eta = 0$ where d is the operator of exterior differentiation. For $d\eta = 0$, we obtain

$$\begin{cases} (\nabla_X \eta)Y = (\nabla_Y \eta)X, \\ \eta([\xi, X]) = \xi(\eta(X)). \end{cases} \tag{1.3}$$

As it is known, an almost contact metric structure is cosymplectic if and only if both $\nabla\eta$ and $\nabla\Phi$ vanish, where ∇ is the covariant differentiation with respect to g. If $(M, \varphi, \xi, \eta, g)$ is an almost contact metric manifold, we have the following statement:

$$\begin{cases} \nabla_\xi \varphi = 0, \\ \nabla_{\varphi X} \xi = -\varphi \nabla_X \xi. \end{cases} \tag{1.4}$$

In the present paper, we are interested in almost cosymplectic 3-manifolds. The article is organized as follows. In section 2, we provide a brief the notion of Levi-Civita connection. In section 3, we recall some results of almost contact metric manifolds. Example is also given. Section 4 deals with the study of connections and metrics properties on a contact metric manifold. In section 5, we study the distribution on almost cosymplectic manifolds. In section 6, we recall the local formalism due to Z. Olszak and establish the consequence. Finally in section 7, we present and study our example. We end the section with a characteristic theorem (Theorem 7.2). For more details to the group $E(1,1)$ of rigid motions of Minkowski 2-space, see [3].

2. Background on Levi-Civita Connection

2.1. **Linear Connections on a Manifold.** Let M be a real m-dimensional connected differentiable manifold of class $\mathcal{C}^\infty$. Let $\mathcal{C}^\infty(M)$ be the algebra of differentiable functions on M. A linear connection on M is a mapping

$$\nabla : \mathfrak{X}(M) \longrightarrow \mathrm{End}_{\mathbb{R}}\left[\mathfrak{X}(M)\right], X \longmapsto \nabla_X$$

satisfying the following conditions:

(i) $\nabla_{aX+Y}(Z) = a\nabla_X(Z) + \nabla_Y(Z)$,
(ii) $\nabla_X(aY + Z) = a\nabla_X(Y) + X(a) \cdot Y + \nabla_X(Z)$,

for any $a \in \mathcal{C}^\infty(M)$ and $X, Y, Z \in \mathfrak{X}(M)$. The operator ∇_X is called the covariant differentiation with respect to X.

We define the covariant differentiation of $a \in \mathcal{C}^\infty(M)$ with respect to X by

$$\nabla_X(a) = X(a). \tag{2.1}$$

Thus for any tensor S of type $(0, s)$ or $(1, s)$ we define the covariant derivative $\nabla_X S$ of S with respect to X by

$$(\nabla_X S)(X_1, \cdots, X_s) = \nabla_X(S(X_1, \cdots, X_s)) - \sum_{i=1}^{s} S(X_1, \cdots, \nabla_X X_i, \cdots, X_s), \tag{2.2}$$

for any $X_i \in \mathfrak{X}(M), i = 1, \cdots, s$.

The tensor field S is say to be parallel with respect to the linear connection ∇ if we have $\nabla_X S = 0$, for any $X \in \mathfrak{X}(M)$.

The torsion tensor T of linear connection ∇ is a tensor field T of type $(1, 2)$ defined by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y], \tag{2.3}$$

for any $X, Y \in \mathfrak{X}(M)$.

A torsion-free connection is linear connection with torsion tensor field zero. The curvature tensor R of linear connection ∇ is a tensor field of type $(1,3)$ defined by

$$R(X,Y)Z = [\nabla_X, \nabla_Y] Z - \nabla_{[X,Y]} Z, \tag{2.4}$$

for any $X, Y, Z \in \mathfrak{X}(M)$.

2.2. **The Levi-Civita Connection.** A tensor field g of type $(0,2)$ is said to be a Riemannian metric on M if the following conditions are fulfilled:

(i) g is symmetric, i.e., $g(X,Y) = g(Y,X)$, for any $X, Y \in \mathfrak{X}(M)$,
(ii) g is positive definite.

The manifold M endowed with a Riemannian metric is called a Riemannian manifold. Thus a linear connection ∇ on M is said to be Riemannian connection if the Riemannian metric g is parallel with respect to ∇, i.e., by (2.2) we have

$$X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z), \tag{2.5}$$

for any $X, Y, Z \in \mathfrak{X}(M)$.

The following theorem is well Know.

Theorem 2.1. *On a Riemannian manifold there exists one and only one torsion-free Riemannian connection.*

The Riemannian connection whose existence and uniqueness are stated in this theorem is called the Levi-Civita connection and it is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= X(g(Y,Z)) + Y(g(X,Z)) - Z(g(X,Y)) \\ &+ g([X,Y], Z) + g([Z,X], Y) + g([Z,Y], X), \end{aligned} \tag{2.6}$$

for any $X, Y, Z \in \mathfrak{X}(M)$.

This a Koszul's formula.

Proposition 2.2. *Let ∇ be the Levi-Civita connection and let R denote corresponding curvature. Then it holds that*

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0, \tag{2.7}$$

$$(\nabla_X R)(Y,Z,W) + (\nabla_Y R)(Z,X,W) + (\nabla_Z R)(X,Y,W) = 0, \tag{2.8}$$

for any $X, Y, Z, W \in \mathfrak{X}(M)$.

One is also able to derive the classical symmetries of the curvature tensor.

Proposition 2.3. *Let ∇ be the Levi-Civita connection and R denote corresponding curvature. Then it holds that*

$$R(X,Y,Z,W) = -R(Y,X,Z,W) = -R(X,Y,W,Z), \tag{2.9}$$

$$R(X,Y,Z,W) = R(Z,W,X,Y), \tag{2.10}$$

for any $X, Y, Z, W \in \mathfrak{X}(M)$.

3. On almost contact metric manifolds

Let S and r be the Ricci curvature tensor and the scalar curvature defined, respectively, by

$$S(X,Y)=\sum_{i=1}^{2n+1} g(R(e_i,X)Y,e_i), \quad r=\sum_{i=1}^{2n+1} S(e_i,e_i),$$

respectively, $\{e_i\}_{1\leq i\leq 2n+1}$ being an orthonormal frame with respect to g. In addition, the Ricci $*$-curvature tensor S^* and scalar $*$-curvature r^*, are given by

$$S^*(X,Y)=\sum_{i=1}^{2n+1} g(R(e_i,X)\varphi Y,\varphi e_i), \quad r^*=\sum_{i=1}^{2n+1} S^*(e_i,e_i).$$

Note that, in [4], the author found the following lemmas and theorems.

Lemma 3.1. [4] *Let (M,φ,ξ,η,g) is an almost contact metric manifold. If a fundamental 2-form Φ is closed, then*

$$\begin{aligned}&(\nabla_{\varphi X}\Phi)(\varphi Y,Z)+(\nabla_X\Phi)(Y,Z)-\eta(X)\left\{d\eta(\varphi Y,Z)+d\eta(Y,\varphi Z)\right\}\\ +\quad &\eta(Y)\left\{d\eta(\varphi Z,X)-\frac{1}{2}(\mathcal{L}_\xi g)(Z,\varphi X)\right\}+\eta(Z)\left\{d\eta(X,\varphi Y)-d\eta(\varphi X,Y)\right\}=0,\end{aligned}$$

where $\mathcal{L}_\xi$ is Lie derivative with respect to ξ.

Lemma 3.2. [4] *If (M,φ,ξ,η,g) is an almost cosymplectic manifold, we have*

$$(\nabla_{\varphi X}\varphi)\varphi Y+(\nabla_X\varphi)Y-\eta(Y)\nabla_{\varphi X}\xi=0.$$

Theorem 3.3. [4] *If a compact almost contact metric manifold satisfies*

$$\begin{cases} r-r^*-S(\xi,\xi)+\frac{1}{2}|\nabla\varphi|^2=0,\\ S(\xi,\xi)+|\nabla\xi|^2=0,\end{cases} \tag{3.1}$$

then it is almost cosymplectic.

Theorem 3.4. [4] *If an almost contact metric manifold (M,φ,ξ,η,g) fulfills the conditions (3.1), $\nabla_\xi\xi$ and the forms η and Φ are coclosed, then (M,φ,ξ,η,g) is almost cosymplectic.*

The products of an almost Kähler and a real line or a circle are the simplest examples of such manifolds.

Example. *Define a group operation in $\mathbb{R}^3$ by*

$$(t,x,y)*(s,u,v)=(t+s,exp(t)u+x,exp(-t)v+y),$$

then $(\mathbb{R}^3,*)$ is the solvable non-nilpotent Lie group. The following set of left-invariant vector fields forms an orthonormal basis for the corresponding Lie algebra: $e_0=\partial_t$, $e_1=exp(t)\partial_x$, $e_2=exp(-t)\partial_y$. According to this basis, one can construct almost contact metric structure (φ,ξ,η,g) on $\mathbb{R}^3$ as follows:

$$\begin{cases}\eta=dt,\ \xi=e_0,\ \varphi=exp(-2t)dx\otimes\partial_y-exp(2t)dy\otimes\partial_x,\\ g=dt\otimes dt+exp(-2t)dx\otimes dx+exp(2t)dy\otimes dy.\end{cases} \tag{3.2}$$

By (3.2) and $g(\nabla_{e_i}e_j,e_k)=-g(e_j,\nabla_{e_i}e_k)$, the non-zero components of the Levi-Civita connection are given by

$$\nabla_{e_1}e_1=e_0,\ \nabla_{e_1}e_0=-e_1,\ \nabla_{e_2}e_2=-e_0,\ \nabla_{e_2}e_0=-e_2.$$

Thus we obviously get $(\mathbb{R}^3, \varphi, \xi, \eta, g)$ *is an almost cosymplectic manifold.*

On the other hand, Kenmotsu studied in [2] another class of almost contact manifolds, defined by the following conditions on the associated almost contact structure

$$d\eta = 0,\ d\Phi = 2\eta \wedge \Phi. \tag{3.3}$$

A normal almost Kenmotsu manifold is said to be Kenmotsu manifold.
An almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ is said to be normal if the tensor field $[\varphi, \varphi] + d\eta \otimes \xi$ vanishes where

$$[\varphi, \varphi](X, Y) = [\varphi X, \varphi Y] - \varphi [X, \varphi Y] - \varphi [\varphi X, Y] + \varphi^2 [X, Y]. \tag{3.4}$$

A normal contact metric manifold is called Sasakian manifold. It is easily shown that the fundamental vector field of the Sasakian manifold is a Killing field. If M is almost cosymplectic, $d\eta = 0$, so the normality condition is given by the vanishing of the torsion tensor $[\varphi, \varphi]$, and in this case, M is said to be cosymplectic.

4. Connections and metrics properties on a contact metric manifold

When

$$d\eta = \Phi, \tag{4.1}$$

an almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ is said to be a contact metric manifold (for more details, see [1]). In this case, we have $i_\xi \Phi = \mathcal{L}_\xi \Phi = \mathcal{L}_\xi \eta = 0$.

Theorem 4.1. *On a contact metric manifold the integral curves of* ξ *are geodesics.*

Proof. For a contact metric structure we have

$$0 = \mathcal{L}_\xi \eta(X) = g(X, \nabla_\xi \xi),$$

so the integral curves of ξ are geodesics. □

If $(M, \varphi, \xi, \eta, g)$ is a contact metric manifold, there exists on $(M, \varphi, \xi, \eta, g)$ a unique connection ∇ satisfying $\nabla g = 0$.

Proposition 4.2. *The metric connection* ∇ *on a contact metric manifold* $(M, \varphi, \xi, \eta, g)$ *allows the following relations*

$$\begin{cases} \nabla_\xi \eta = 0; \\ \nabla_\xi (d\eta) = 0; \\ \nabla_\xi \varphi = 0. \end{cases}$$

Moreover,

$$\begin{cases} \eta(\nabla_X \xi) = 0; \\ (\nabla_X \eta)\xi = 0; \\ g(\nabla_X \xi, Y) = \dfrac{1}{2}(\mathcal{L}_\xi g)(X, Y); \end{cases}$$

for any $X, Y \in \mathfrak{X}(M)$.

Proof. By straightforward calculation. □

Corollary 4.3. *Let* $(M, \varphi, \xi, \eta, g)$ *be a contact metric manifold. Then the following assertions are equivalent:*

1) $\nabla \eta = 0$;

2) $\nabla\xi = 0$;
3) $\mathcal{L}_\xi g = 0$;
4) $\mathcal{L}_\xi \varphi = 0$.

We deduce the following assertion.

Theorem 4.4. *Let $(M, \varphi, \xi, \eta, g)$ be a contact metric manifold with metric connection ∇. Then the following assertions are equivalent:*

1) $\nabla\varphi = 0$;
2) $N_\varphi(X, Y) = [\varphi, \varphi](X, Y) + d\eta(X, Y)\xi$;
3) $\nabla(d\eta) = 0$.

5. Distribution on almost cosymplectic manifolds

Since $\xi = \eta^\sharp$ denotes the metric dual of the contact form η, consider the bilinear form $\sigma = \nabla\eta$ and the associated endomorphism $S := -\nabla\xi$. They are related by the formula $\sigma(X, Y) = g(SX, Y)$. As the form η is closed, we get for all vector fields X and Y on M:

$$0 = d\eta(X, Y) = -(\nabla_X\eta)Y + (\nabla_Y\eta)X = -\sigma(X, Y) + \sigma(Y, X),$$

thus showing that the $(0, 2)$-tensor σ is symmetric bilinear form, and correspondingly S is a symmetric endomorphism with respect to the metric g. From formula (2.14) from [4], we get

$$\begin{aligned} 0 &= \nabla\eta(X, Y) + \nabla\eta(\varphi X, \varphi Y) \\ &= \sigma(X, Y) + \sigma(\varphi X, \varphi Y) \\ &= g(SX, Y) + g(S\varphi X, \varphi Y) \\ &= g(SX, Y) - g(\varphi S\varphi X, Y) \\ &= g((S - \varphi S\varphi)X, Y), \end{aligned}$$

for all vector fields X and Y on M. Thus the tensor $S := -\nabla(\eta^\sharp)$ satisfies $S = \varphi S\varphi$. There is a local orthonormal $\{\xi, \varphi e_i, e_i\}$ on T_xM, called a φ-basis, such that $Se_i = \lambda_i e_i$, where $\lambda_0 = 0$, $\lambda_i = -\lambda$, $\lambda_{n+i} = \lambda\,(\lambda \geq 0)$. Hence $trace(S^2) = 2n\lambda^2$ at x. Thus $\lambda = \sqrt{\frac{1}{2n} trace(S^2)}$. Since $S\xi = 0$ and therefore also $S\varphi\xi = 0$, we thus get $\nabla_{\varphi\xi}\xi = \nabla_\xi\varphi\xi = \nabla_{\varphi\xi}\varphi\xi = \nabla_\xi\xi = 0$. In particular

$$[\xi, \varphi\xi] = 0. \tag{5.1}$$

We conclude that the distribution $\{\xi, \varphi\xi\}$ is integrable, and its integral leaves are totally geodesic and this implies that $[\mathcal{L}_\xi, \mathcal{L}_{\varphi\xi}] = \mathcal{L}_{[\xi,\varphi\xi]} = 0$.

6. Local formalism on almost cosymplectic 3-manifold

The material used in this section is due to Z. Olszak (for further details, we refer to [5] and references therein). On a neighborhood U of x on which there is a vector field e so that $\{e_0 = \xi, e_1 = e, e_2 = \varphi e\}$ is a local orthonormal frame field composed of eigenvectors of S. The components of the Levi-Civita connection are:

$$\begin{cases} \nabla_{e_0}e_0 = 0,\ \nabla_{e_1}e_0 = -\lambda e_2,\ \nabla_{e_2}e_0 = -\lambda e_1, \\ \nabla_{e_0}e_1 = ae_2,\ \nabla_{e_1}e_1 = -be_2,\ \nabla_{e_2}e_1 = \lambda e_0 + ce_2, \\ \nabla_{e_0}e_2 = -ae_1,\ \nabla_{e_1}e_2 = \lambda e_0 + be_1,\ \nabla_{e_2}e_2 = -ce_1, \end{cases} \tag{6.1}$$

where a, b, c are smooth functions.

As ∇ is torsion free, the Poisson brackets are given by:

$$[e_0, e_1] = (a+\lambda)e_2,\ [e_0, e_2] = (-a+\lambda)e_1,\ [e_1, e_2] = be_1 - ce_2. \tag{6.2}$$

From the Jacobi identity $[e_0, [e_1, e_2]] + [e_1, [e_2, e_0]] + [e_2, [e_0, e_1]] = 0$, we get the following

$$\begin{cases} e_1(\lambda) - e_0(b) - e_1(a) - c(a-\lambda) = 0, \\ e_2(\lambda) - e_0(c) + e_2(a) + b(a+\lambda) = 0. \end{cases} \tag{6.3}$$

Let $R_{X,Y} := [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ denote the curvature tensor of ∇ defined by (2.4). At $x \in M$, we obtain

$$\begin{cases} R_{e_0,e_0}e_i = R_{e_1,e_1}e_i = R_{e_2,e_2}e_i = 0,\ i = 0,1,2, \\ R_{e_0,e_1}e_0 = \lambda(2a+\lambda)e_1 - e_0(\lambda)e_2, \\ R_{e_0,e_1}e_1 = -\lambda(2a+\lambda)e_0 - (e_0(b) + e_1(a) + c(a+\lambda))e_2, \\ R_{e_0,e_1}e_2 = e_0(\lambda)e_0 + (e_0(b) + e_1(a) + c(a+\lambda))e_1, \\ R_{e_0,e_2}e_0 = -e_0(\lambda)e_1 + \lambda(-2a+\lambda)e_2, \\ R_{e_0,e_2}e_1 = e_0(\lambda)e_0 + (e_0(c) - e_2(a) + b(-a+\lambda))e_2, \\ R_{e_0,e_2}e_2 = \lambda(2a-\lambda)e_0 + (-e_0(c) + e_2(a) + b(a-\lambda)e_1, \\ R_{e_1,e_2}e_0 = -(e_1(\lambda) + 2\lambda c)e_1 + e_2(\lambda)e_2, \\ R_{e_1,e_2}e_1 = (e_1(\lambda) + 2\lambda c)e_0 + (e_1(c) + e_2(b) + b^2 + c^2 - \lambda^2)e_2, \\ R_{e_1,e_2}e_2 = -(e_2(\lambda) + 2b\lambda)e_0 - (e_1(c) + e_2(b) + b^2 + c^2 - \lambda^2)e_1. \end{cases}$$

The non-zero components of the Ricci curvature are:

$$\begin{cases} r_{00} = -2\lambda^2,\ r_{01} = -e_2(\lambda) - 2b\lambda, \\ r_{02} = -e_1(\lambda) - 2\lambda c,\ r_{12} = e_0(\lambda) \\ r_{11} = -e_1(c) - e_2(b) - b^2 - c^2 - 2a\lambda, \\ r_{22} = -e_1(c) - e_2(b) - b^2 - c^2 + 2a\lambda. \end{cases} \tag{6.4}$$

Corollary 6.1. *The scalar curvature r is given by:*

$$r = -2\left[e_1(c) + e_2(b) + b^2 + c^2 + \lambda^2\right]. \tag{6.5}$$

Theorem 6.2. [5] *Let $(M, \varphi, \xi, \eta, g)$ be a 3-dimensional almost cosymplectic manifold non-zero constant curvature. Then this manifold does not exist.*

We finish the present section to proving the following consequence.

Corollary 6.3. *In the above conditions, the following assertions are equivalent:*

(i) $S = \varphi S \varphi$;
(ii) $r_{11} = r_{22}$ *and* $r_{ij} = 0\,(i \neq j, i, j = 0,1,2)$;
(iii) $a = 0$, $e_0(\lambda) = 0$, $e_1(\lambda) = -2\lambda c$, $e_2(\lambda) = -2\lambda b$;
(iv) λ *is constant.*

7. An example of almost cosymplectic 3-manifolds

7.1. **Unimodular Lie groups.** A Lie group G is said to be unimodular if its left-invariant Haar measure is right invariant. Milnor gave an infinitesimal reformulation of unimodularity for 3-dimensional Lie groups. We recall it briefly here. Let $\mathcal{G}$ be a 3-dimensional oriented Lie algebra with an inner product $\langle \cdot, \cdot \rangle$. We define the vector product operation $\times : \mathcal{G} \times \mathcal{G} \longrightarrow \mathcal{G}$ as the skew-symmetric bilinear map which is uniquely determined by the following conditions for any $X, Y \in \mathcal{G}$:

$$\langle X, X \times Y \rangle = \langle Y, X \times Y \rangle = 0, \quad (7.1)$$
$$|X \times Y|^2 = \langle X, X \rangle \langle Y, Y \rangle - \langle X, Y \rangle^2, \quad (7.2)$$
$$det(X, Y, X \times Y) > 0 \quad (7.3)$$

if X and Y are linearly independent. On the other hand, the Lie bracket

$$[\cdot, \cdot] : \mathcal{G} \times \mathcal{G} \longrightarrow \mathcal{G}$$

is a skew-symmetric bilinear map. Comparing these two operations, we get a linear endomorphism $\mathrm{L}_{\mathcal{G}}$ which is uniquely determined by the formula

$$[X, Y] = \mathrm{L}_{\mathcal{G}}(X \times Y), \forall X, Y \in \mathcal{G}. \quad (7.4)$$

Now let G be an oriented 3-dimensional Lie group equipped with a left-invariant Riemannian metric. Then the metric induces an inner product on the Lie algebra $\mathcal{G}$. With respect to the orientation on $\mathcal{G}$ induced from G, the endomorphism field $\mathrm{L}_{\mathcal{G}}$ is uniquely determined. The unimodularity of G is characterized as follows.

Proposition 7.1. *Let G be an oriented 3-dimensional Lie group with a left-invariant Riemannian metric. Then G is unimodular if and only if the endomorphism $\mathrm{L}_{\mathcal{G}}$ is self-adjoint with respect to the metric.*

Proof. See [3]. □

Let G be a 3-dimensional unimodular Lie group with a left-invariant metric. Then there exists an orthonormal basis $\{e_i\}_{1 \le i \le 3}$ of the Lie algebra $\mathcal{G}$ such that

$$[e_1, e_2] = c_3 e_3, \quad [e_2, e_3] = c_1 e_1, \quad [e_3, e_1] = c_2 e_2, \quad c_i \in \mathbb{R}.$$

Three-dimensional unimodular Lie groups are classified in [3] as follows:

Signature of (c_1, c_2, c_3)	Simply connected Lie group	Property
$(+, +, +)$	SU_2	compact and simple
$(+, +, -)$	$\widetilde{SL_2\mathbb{R}}$	non-compact and simple
$(+, +, 0)$	$\widetilde{E(2)}$	solvable
$(+, -, 0)$	$E(1, 1)$	solvable
$(+, 0, 0)$	Heisenberg group Nil_3	nilpotent
$(0, 0, 0)$	$(\mathbb{R}^3, +)$	Abelian

Example. *The group*

$$E(1, 1) = \left\{ \begin{pmatrix} e^z & 0 & x \\ 0 & e^{-z} & y \\ 0 & 0 & 1 \end{pmatrix}, x, y, z \in \mathbb{R} \right\}$$

is the motion group of the Minkowski 2-plane.

7.2. **An example of almost cosymplectic 3-manifolds.** We consider the 3-dimensional manifold

$$Q = \left\{ (t, x, y) \in \mathbb{R}^3, t \in \mathbb{R}^* \right\}. \quad (7.5)$$

The left-translated vector fields $f_0 = \partial_t$, $f_1 = t\partial_x$, $f_2 = t^{-1}\partial_y$ are linearly independent at each point of Q. The dual coframe field is $\omega_0 = dt, \omega_1 = t^{-1}dx, \omega_2 = tdy$.

Define the left-invariant vector fields $u_0 = f_0, u_1 = \frac{1}{\sqrt{2}}(-f_1 + f_2), u_2 = \frac{1}{\sqrt{2}}(f_1 + f_2)$. This left-invariant frame field satisfies the commutation relations

$$[u_1, u_2] = 0, \; [u_2, u_0] = u_1, \; [u_0, u_1] = -u_2. \tag{7.6}$$

Let $e_i = \frac{u_i}{\alpha_i}, i = 0, 1, 2$, where α_i is a positive constant. Then Q is equipped with a left-invariant Riemannian metric such that $\{e_0, e_1, e_2\}$ is orthonormal.

Let g be a Riemannian metric on Q defined by $g(e_i, e_j) = \delta_{ij}$, $i, j = 0, 1, 2$, where δ_{ij} is the Kronecker symbol, that is, the form of the metric becomes

$$g = (\alpha_0)^2 \omega_0 \otimes \omega_0 + \frac{\alpha_1^2}{2}(-\omega_1 + \omega_2) \otimes (-\omega_1 + \omega_2) + \frac{\alpha_2^2}{2}(\omega_1 + \omega_2) \otimes (\omega_1 + \omega_2). \tag{7.7}$$

Define a 1-form η on Q by $\eta = \omega_0 := dt$, then $d\eta = 0$. Let φ be the $(1,1)$-tensor field defined by $\varphi e_0 = 0$, $\varphi e_1 = e_2$, $\varphi e_2 = -e_1$. Thus, we get

$$\varphi = \frac{\alpha_1^2}{2}(-\omega_1 + \omega_2) \otimes (f_1 + f_2) - \frac{\alpha_2^2}{2}(\omega_1 + \omega_2) \otimes (-f_1 + f_2). \tag{7.8}$$

The non-zero component of the fundamental 2-form Φ is given by

$$\Phi = -\omega_1 \wedge \omega_2. \tag{7.9}$$

Then $d\Phi = 0$. Thus, the quintuple $(Q, \varphi, \xi, \eta, g)$ is called almost cosymplectic manifold with fundamental 2-form Φ. In term of standard coordinates, since $\mathrm{Jac}(f_0, f_1, f_2) = 0$, we deduce that $f_1(\lambda) = f_2(\lambda) = 0$. By the formalism, we dedude that $f_0(\lambda) = 0$. So the eignvalue λ is constant. In these conditions a unique non-zero component of Ricci curvature is $r_{00} = -2\lambda^2$. Thus the scalar curvature is given by $r = -2\lambda^2$.

By the Koszul formula, we get $\nabla_{f_0} f_0 = 0$, $\nabla_{f_1} f_0 = -\lambda f_1$, $\nabla_{f_2} f_0 = \lambda f_2$.

By the formula $(\mathcal{L}_X \varphi)Y = [X, \varphi Y] - \varphi[X, Y]$, the components of the tensor field h are given by $h(f_0) = 0$, $h(f_1) = -\lambda f_1$, $h(f_2) = \lambda f_2$ and $trace(h) = 0$.

Theorem 7.2. *The almost cosymplectic* 3*-manifold* $(Q, \varphi, \xi, \eta, g)$ *is isomorphic to the group* $E(1,1)$ *of rigid motions of* Minkowski 2*-space.*

Acknowledgments. The author would like to thank the anonymous referee for his/her comments that helped us improve this article.

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