Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 16 Issue 2(2024), Pages 15-30
https://doi.org/10.54671/BMAA-2024-2-2

MULTIPLIERS ON VECTOR-VALUED L^1-SPACES FOR SOME NONCOMMUTATIVE HYPERGROUPS

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Abstract. Let G be a locally compact hypergroup and K be a compact subhypergroup of G such that (G, K) is a Gelfand pair. For a Banach algebra A, $L^1(G, A)$ denotes the A-valued L^1-space on G. In this paper, we extend the generalized Wendel theorem to $L^1(G, A)^\natural$, the subspace of $K-$invariant functons of $L^1(G, A)$, when A is a Banach algebra with identity.

1. Introducion

The theory of multipliers first appears in harmonic analysis in connection with the theory of summability for Fourier series ([12]). Since then, the notion has been employed in many areas of harmonic analysis, such as the study of properties of the Fourier transformation and its extensions, the characterization of group algebras, the characterization of convolution in terms of its mapping properties. The theory of multipliers is becoming more used in other areas of pure and applied mathematics including the theory of Banach algebras, the theory of singular integrals and fractional integration, interpolation theory, stochastic processes, the theory of semi-groups of operators, partial differential equations, signal theory (study of invariant linear systems), time-frequency analysis. In Fourier analysis, a multiplier operator is a type of linear operator or transformation of functions. These operators act on a function by altering its Fourier transform. Specifically, they multiply the Fourier transform of a function by a specified function known as the symbol. In signal theory, the multipliers are the filters and the symbol is the filter's frequency response.

One way of obtaining some insight into the multiplier problem is to examine the situation for the group algebras. In his paper [22], Wendel has established the following result:

Theorem. Let G be a locally compact group and let $T : L^1(G) \longrightarrow L^1(G)$, be a bounded linear operator. Then the following statements are equivalent.
(i) T commutes with translations
(ii) $T(f * g) = Tf * g, \forall f, g \in L^1(G)$

2000 *Mathematics Subject Classification.* Primary 43A62, 43A22; Secondary 20N20.
Key words and phrases. Hypergroups, Gelfand pair, multipliers, Vector-valued L^1spaces .

Submitted April 20, 2024. Published June 3, 2024.
Communicated by P. Jorgensen.

(iii) $T(f*\nu) = Tf*\nu, \forall f \in L^1(G), \forall \nu \in M_b(G)$

(iv) There exists a unique measure $\mu \in M_b(G)$ *such that* $Tf = \mu * f$, $\forall f \in L^1(G)$.

Wendel's result have been extended in several manners by many authors (see [2],[15],[17]). During the last decades, the multipliers for hypergroups have also been studied. The pionners W. C. Connett and A. L. Schwartz (see [4], [5], [6]) were interested in the topic of multipliers for ultraspherical series and Jacobi expansions. Their work is strongly connected to multipliers for polynomial hypergroups generated by the ultraspherical polynomials. Also, the "multiplier criteria of Hormander type for Jacobi expansions" published in 1980 by G. Gasper and W. Trebels [8] have a strong correlation to the characterization of multipliers defined on the Jacobi hypergroup. In 1982, R. Lasser [13] generalized Wendel's theorem to commutative hypergroups.

Let us note that hypergroups generalize locally compact groups where the convolution of two Dirac measures is a Dirac measure. A hypergroup is a locally compact Hausdorff space equipped with a convolution product which maps two Dirac measures to a probability measure with compact support. The notion of hypergroup has been sufficiently studied (see for example [7, 11, 16, 18]). Harmonic analysis and probability theory on commutative hypergroups are well-developped and many results from group theory remain valid (see [1]). When G is a commutative hypergroup, the convolution algebra $M_c(G)$ consisting of measures with compact support on G is commutative. A typical example of commutative hypergroup is the double coset $G//K$ when G is a locally compact group, K is a compact subgroup of G such that (G, K) is a Gelfand pair.

In their paper [17], Sarma and al. have extended Wendel's theorem to a A-valued L^1-spaces on a locally compact hypergroup G, where A is a Banach algebra with a bounded identity. In the same paper [17], the authors have proved that if G is an abelian hypergroup, then the assertions in Wendel's theorem are equivalent to the following statements:

(v) There exists a unique measure $\mu \in M_b(G, A^{**})$ *such that* $\widehat{Tf} = \widehat{\mu}\widehat{f}$, *for any* $f \in L^1(G, A^{**})$, *where* A^{**} *is the double dual of* A.

(vi) There exists a unique function $\varphi \in L^\infty(\widehat{G}, A^{**})$ *such that* $\widehat{Tf} = \varphi\widehat{f}$, *for any* $f \in L^1(G, A^{**})$.

This result establishes a characterization of multipliers of $L^1(G, A)$ in terms of Fourier transform. The proof of this characterization is based on the commutativity of $L^1(G, A)$ which is due to the commutativity of G. But in general, the hypergroups involved in applications are not commutative and, its well-known that $L^1(G, A)$ is not commutative if G is noncommutative. Thus, our main concern in this paper is to prove characterazations *(v)* and *(vi)* when the hypergroup G is not abelian. Specifically, we consider the case where the hypergroup G is not commutative but admitting a compact subhypergroup K leading to a commutative subalgebra of $L^1(G, A)$.

The goal of this paper is to extend the result of Sarma and al. to locally compact hypergroups (non-abelian) admitting a compact subhypergroup K such that (G, K) is a Gelfand pair. In the next section, we give notations and setup useful for the remainder of this paper. In section 3, after defining the Arens multiplication in the algebra A, we show that any left multiplier of A can be extended into a left multiplier of A^{**}, the double dual of A. We then establish that the algebras $L^1(G, A)$

is isomorphic to a certain algebra $L^1\left(G,\widetilde{A}\right)$, also that the algebra $L^1(G//K,A)$ is isomorphic to the subalgebra $L^1(G,A)^\natural$ of $L^1(G,A)$ consisting of K- invariant elements. Relying on these results, we prove our main result, namely the characterization of a multiplier by means of the Fourier transform of a bounded K-invariant measure on G.

2. Notations and preliminaries

Let G be a locally compact space. We denote by:
- $C(G)$ (resp. $M(G)$) the space of continuous complex-valued functions (resp. the space of Radon measures) on G,
- $C_b(G)$ (resp. $M_b(G)$) the space of bounded continuous functions (resp. the space of bounded Radon measures) on G,
- $\mathcal{K}(G)$ (resp. $M_c(G)$) the space of continuous functions (resp. the space of Radon measures) with compact support on G,
- $\mathfrak{C}(G)$ the space of compact subspaces of G,
- δ_x the point measure at $x\in G$,
- $\mathrm{supp}(\mu)$, the support of the measure μ

Let us note that the topology on $M(G)$ is the cône topology [11] and the topology on $\mathfrak{C}(G)$ is the topology of Michael [14].

Definition 1. *G is said to be a hypergroup if the following assumptions are satisfied.*
(H1) There is a binary operator $*$ named convolution on $M_b(G)$ under which $M_b(G)$ is an associative algebra such that:
i) the mapping $(\mu,\nu)\longmapsto\mu*\nu$ is continuous from $M_b(G)\times M_b(G)$ in $M_b(G)$.
ii) For all $x,y\in G$, $\delta_x*\delta_y$ is a measure of probability with compact support.
iii) the mapping: $(x,y)\longmapsto supp(\delta_x*\delta_y)$ is continuous from $G\times G$ in $\mathfrak{C}(G)$.
(H2) There is a unique element e (called neutral element) in G such that $\delta_x*\delta_e=\delta_e*\delta_x=\delta_x$ for all $x\in G$.
(H3) There is an involutive homeomorphism: $x\longmapsto\overline{x}$ from G in G, named involution, such that:
i) $(\delta_x*\delta_y)^-=\delta_{\overline{y}}*\delta_{\overline{x}}$, for all $x,y\in G$ with $\mu^-(f)=\mu(f^-)$ where $f^-(x)=f(\overline{x}),\forall f\in C(G)$ and $\mu\in M(G)$.
ii) For all $x,y,z\in G$, $z\in supp(\delta_x*\delta_y)$ if and only if $x\in supp(\delta_z*\delta_{\overline{y}})$.

For two subsets A and B of G, $A*B=\displaystyle\bigcup_{x\in A,y\in B}\mathrm{supp}(\delta_x*\delta_y)$.
For $x,y\in G$, $\{x\}*\{y\}$ is denoted by $x*y$.
The hypergroup G is commutative if $\delta_x*\delta_y=\delta_y*\delta_x$ for all $x,y\in G$.
For $x,y\in G$ and for $f\in C(G)$,

$$f(x*y)=(\delta_x*\delta_y)(f)=\int_G f(z)d(\delta_x*\delta_y)(z).$$

The convolution of two measures μ,ν in $M_b(G)$ is defined by:

$$(\mu*\nu)(f)=\int_G\int_G(\delta_x*\delta_y)(f)d\mu(x)d\nu(y)=\int_G\int_G f(x*y)d\mu(x)d\nu(y),\forall f\in C(G)$$

For μ in $M_b(G)$, $\mu^*=(\overline{\mu})^-$. So $M_b(G)$ is a $*$-Banach algebra.

Definition 2. *$H \subset G$ is a subhypergroup of G if the following conditions are satisfied.*
1. H is non empty and closed in G,
2. $\forall x \in H, \overline{x} \in H$,
3. $\forall x, y \in H$, $supp(\delta_x * \delta_y) \subset H$.

Let us now give some examples of hypergroups.

Example 1. *Any topological locally compact group G is a hypergroup. The concolution is defined by*

$$\delta_x * \delta_y = \delta_{xy}, x, y \in G$$

Example 2. *Let K be a compact subgroup of a locally compact group G. Let dk be a normalized Haar measure of K and dx be a left Haar mleasure of G with e the neutral element. It is well- known that $G//K = \{KxK, x \in G\}$ is locally compact topological space with the quotient topology. The operator*

$$\delta_{KxK} * \delta_{KyK} = \int_K \delta_{KxkyK} dk$$

defines hypergroup structure on $G//K$. The identity is $KeK = K$.
For $x \in G, \overline{KxK} = Kx^{-1}K$. A left Haar measure on $G//K$ is defined by $m = \int_K \delta_{KxK} dx$.

Example 3. *Let H be a compact group with a normalized measure σ and G be a locally compact group. Let*

$$\begin{array}{ccc} G \times H & \longrightarrow & G \\ (x, h) & \longmapsto & x^h \end{array}$$

be a continuous action of H on G such that for any $h \in H$, the mapping: $x \longmapsto x^h$ is an automorphism of G. We consider the quotient topology on $G^H = \left\{x^H, x \in G\right\}$. The mapping:

$$\delta_{x^H} * \delta_{y^H} = \int_H \delta_{(x^h y)^H} d\sigma(h) = \int_H \delta_{(xy^t)^H} d\sigma(t)$$

defines hypergroup structure on G^H with identity $1^H = \{1\}$, where 1 is the neural element of G. If $x \in G$, $\overline{x^H} = (x^{-1})^H$.

Example 4. ***(Polynomial hypergroups-Jacobi polynomial hypergroups).*** *Let G be a countable set with the discrete topologie and $d \in \mathbb{N}$. Let us consider $\mathcal{S} = \{Q_x : x \in G\}$ a set of polynomials in $\mathbb{C}[z_1, ..., z_d]$. For any $n \in \mathbb{Z}_+$, let set $\mathcal{P}_n = \{Q \in \mathbb{C}[z_1, ..., z_d] : d^\circ Q \leq n\}$; $\mathcal{B}_n = \{Q \in S : Q \in \mathcal{P}_n\}$. Let make the assumption that $\mathcal{B}_n$ is a basis of $\mathcal{P}_n$. Then for $x, y \in G$*

$$Q_x Q_y = \sum_{w \in G} C(x, y, w) Q_w$$

with $C(x, y, w) \in \mathbb{C}$. For $x, y \in G$, define the convolution by

$$\delta_x * \delta_y (\{w\}) = C(x, y, w); w \in G.$$

This convolotion defines a hypergroup structure on G. $(G, *)$ is called a polynomial hypergroup (in d variables).

A Jacobi polynomial hypergroup is a polynomial hypergroup in 1 where $G=\mathbb{Z}_+$ and Q_n is a normalized Jacobi polynomial $Q_n=Q_n^{\alpha,\beta}$ defined by

$$Q_n^{\alpha,\beta}(x)=\frac{(-1/2)^n}{(\alpha+1)^n}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}\left[(1-x)^{n+\alpha}(1+x)^{n+\beta}\right]$$

with (α,β) belongs to the parameter set

$$V=\left\{\begin{array}{c}(\alpha',\beta')\in\mathbb{R}^2:\alpha'\geq\beta'>-1,(\alpha'+\beta'+1)(\alpha'+\beta'+4)^2(\alpha'+\beta'+6)\geq\\ (\alpha'-\beta')^2\left[(\alpha'+\beta'+1)^2-7(\alpha'+\beta'+1)-24\right]\end{array}\right\}$$

such that

$$Q_m^{\alpha,\beta}Q_n^{\alpha,\beta}=\sum_{k\in\mathbb{Z}_+}C(m,n,k)\,Q_k^{\alpha,\beta}$$

with $C(m,n,k)\geq 0$ for m,n,k in $\mathbb{Z}_+$. For more details see [1].

Let us now consider a locally compact hypergroup G provided with a left Haar measure μ_G and K a compact subhypergroup of G, with a normalized Haar measure ω_K. For $x\in G$, the double coset of x with respect to K is $K*\{x\}*K=\{k_1*x*k_2;k_1,k_2\in K\}$. We write simply KxK for a double coset. All double cosets form a partition of G and the quotient topology with respect to the corresponding equivalence relation equips the double cosets space $G//K$ with a locally topology ([1], page 53). The natural mapping $p_K:G\longrightarrow G//K$ defined by

$$p_K(x)=KxK,x\in G \tag{1}$$

is an open surjective continuous mapping. A function $f\in C(G)$ is said to be invariant by K or $K-$ *invariant* if $f(k_1*x*k_2)=f(x)$ for all $x\in G$ and for all $k_1,k_2\in K$. We denote by $C^\natural(G)$, (resp. $\mathcal{K}^\natural(G)$) the space of continuous functions (resp. continuous functions with compact support) which are $K-$invariant. For $f\in C^\natural(G)$, one defines the function $\widetilde{f}$ on $G//K$ by

$$\widetilde{f}(KxK)=f(x)\forall x\in G. \tag{2}$$

$\widetilde{f}$ is well-defined and it is continuous on $G//K$. Conversely, for all continuous function φ on $G//K$, the function $f=\varphi\circ p_K$ belongs to $C^\natural(G)$. One has the obvious consequence that the mapping

$$f\longmapsto\widetilde{f} \tag{3}$$

sets up a topological isomorphism between the topological vector spaces $C^\natural(G)$ and $C(G//K)$ (see [19, 21]). So, for any f in $C^\natural(G)$, $f=\widetilde{f}\circ p_K$. Otherwise, we consider the K-projection $f\longmapsto f^\natural$ (by identifying $f^\natural$ and $\widetilde{f^\natural}$) from $C(G)$ into $C(G//K)$ where for $x\in G, f^\natural(x)=\int_K\int_K f(k_1*x*k_2)d\omega_K(k_1)d\omega_K(k_2)$. If $f\in\mathcal{K}(G)$, then $f^\natural\in\mathcal{K}(G//K)$. For a measure $\mu\in M(G)$, one defines $\mu^\natural$ by

$$\mu^\natural(f)=\mu(f^\natural)\text{ for }f\in\mathcal{K}(G).$$

μ is said to be $K-$invariant if $\mu^\natural=\mu$ and we denote by $M^\natural(G)$ the set of all those measures. Considering these above definitions, one defines a hypergroup operation on $G//K$ by

$$\delta_{KxK}*\delta_{KyK}(\widetilde{f})=\int_K f(x*k*y)d\omega_K(k)\text{ (see [19] and [1]).} \tag{4}$$

This defines uniquely the convolution $(KxK) * (KyK)$ on $G//K$. The involution is defined by: $\overline{KxK} = K\overline{x}K$ and the neutral element is K. Let us put $m = \int_G \delta_{KxK} d\mu_G(x)$, m is a left Haar measure on $G//K$. We say that (G, K) is a Gelfand pair if the convolution algebra $M_c(G//K)$ is commutative. $M_c(G//K)$ is topologically isomorphic to $M_c^\natural(G)$. Considering the convolution product on $\mathcal{K}(G)$ defined by

$$f * g(x) = \int_G f(x * \overline{y}) g(y) d\mu_G(y),$$

$\mathcal{K}(G)$ is a convolution algebra and $\mathcal{K}^\natural(G)$ is a subalgebra of $\mathcal{K}(G)$.Thus (G, K) is a Gelfand pair if and only if $\mathcal{K}^\natural(G)$ is commutative ([9], theorem 3.2.2).

In this section, we suppose that (G, K) is a Gelfand pair.
Let $\widehat{G}$ be the dual space of the hypergroup G. $\widehat{G}$ is the set of complex-valued continuous, bounded function ϕ on G such that:
(i) ϕ is K- invariant,
(ii) $\phi(e) = 1$,
(iii) $\int_K \phi(x * k * y) dw_K(k) = \phi(x)\phi(y)$ $\forall x, y \in G$,
(iv) $\phi(\overline{x}) = \overline{\phi(x)}$ $\forall x \in G$.
Equipped with the topology of uniform convergence on compacta, $\widehat{G}$ is a locally compact Hausdorff space and the function $\mathbf{1} : x \longmapsto 1$ belongs to $\widehat{G}$.

For μ belongs to $M_b(G)$, the Fourier transform of μ is defined by

$$\widehat{\mu}(\phi) = \int_G \phi(\overline{x}) d\mu(x),\ \phi \in \widehat{G}. \tag{5}$$

$\widehat{\mu} \in C_b(\widehat{G})$ and the map:

$$\begin{array}{cc} M_b(G) & \longrightarrow C_b(\widehat{G}) \\ \mu & \longmapsto \widehat{\mu} \end{array}$$

is a continuous linear operator.
By identifying $f \in L^1(G)$ and $f\mu_G$, we have

$$\widehat{f}(\phi) = \int_G \phi(\overline{x}) f(x) d\mu_G(x) \forall \phi \in \widehat{G}. \tag{6}$$

For $\beta \in M_b(\widehat{G})$, the inverse Fourier transform of β is defined by

$$\overset{\vee}{\beta}(x) = \int_{\widehat{G}} \phi(x) d\beta(\phi),\ x \in G.$$

By identifying $\varphi \in L^1(\widehat{G})$ and $\varphi\pi$ (where π is the Plancherel measure), we have $\overset{\vee}{\varphi}(x) = \int_{\widehat{G}} \phi(x)\varphi(\phi) d\pi(\phi)$, $x \in G$.
For more details on the Fourier transform, see [10].

3. Characterization of vector-valued multipliers

3.1. **Arens multiplication on the double dual of a Banach algebra.**

Let A be a complex Banach algebra. Denote by A^* (resp. A^{**}) the first (resp. the second) conjugate space of A.
Let us define the following operations.
If $f \in A^*$ and $a \in A$, define $f \circledast a \in A^*$ by

$$f \circledast a(b) = f(ab),\ \forall b \in A. \tag{7}$$

If $F \in A^{**}$ and $f \in A^*$, define $F \circledast f \in A^*$ by

$$F \circledast f(a) = F\left(f \circledast a\right),\ \forall a \in A. \tag{8}$$

If $F, H \in A^{**}$, define $F \odot H \in A^{**}$ by

$$F \odot H(f) = F\left(H \circledast f\right),\ \forall f \in A^*. \tag{9}$$

$\odot$ is called the Arens multiplication and the double dual A^{**} is a Banach algebra with respect to the Arens multiplication (see [20]). $(A^{**}, \odot)$ has an identity if and only if A has a bounded approximate identity (see [20], p.270).
For a belongs to A, let us consider the mapping $\widetilde{a} : A^* \longrightarrow \mathbb{C}$ defined by

$$\widetilde{a}(f) = f(a),\ \forall f \in A^*.$$

$\widetilde{a} \in A^{**}$ and the mapping

$$\begin{aligned} \Pi : \quad A &\longrightarrow A^{**} \\ a &\longmapsto \widetilde{a} \end{aligned}$$

is an isometric algebras isomorphism of A into $(A^{**}, \odot)$ (see [3]). By setting $\Pi(A) = \widetilde{A}$, we have that $\widetilde{A}$ is a subalgebra of $(A^{**}, \odot)$.

Let A be a complex Banach algebra with a bounded approximate identity. A *left multiplier* of A is a bounded linear operator T on A such that

$$T(ab) = (Ta)b,\ \forall a, b \in A.$$

The space of the left multipliers of A denoted by $M_l(A)$ is a Banach algebra.
A Characterization of $M_l(A)$ is given by the following proposition.

Proposition 1. *Every left multilplier of A can be extended as a left multiplier of A^{**}.*

Proof. For $T \in M_l(A)$, let us define the operators below.
- Define T^* by $T^* f(a) = \widetilde{Ta}(f)$ for $f \in A^*$ and $a \in A$.
- Define T^{**} by $T^{**}F\left(f\right) = F\left(T^* f\right)$ for $F \in A^{**}$ and $f \in A^*$.
It is clear that T^* (resp. T^{**}) is a linear operator on A^* (resp. A^{**}). For $a \in A$

and $f \in A^*$, we have

$$\begin{aligned} f \circledast Ta\,(b) &= f(T(ab)) \\ &= \widetilde{T(ab)}(f) \\ &= T^* f(ab) \\ &= T^* f \circledast a(b). \end{aligned}$$

Thus,

$$f \circledast Ta = T^* f \circledast a, \forall a \in A, f \in A^*. \tag{10}$$

Also,

$$T^*\,(H \circledast f) = H \circledast T^* f, \text{ for } f \in A^* \text{and } H \in A^{**}. \tag{11}$$

In fact, for $a \in A$, we have

$$\begin{aligned} T^*\,(H \circledast f)\,(a) &= \widetilde{Ta}(H \circledast f) \\ &= H \circledast f(Ta) \\ &= H\,(f \circledast Ta) \\ &= H\,(T^* f \circledast a) \\ &= H \circledast T^* f(a), \end{aligned}$$

where the fourth equality is due to (10).
Furthermore, for $f \in A^*$ and $F, H \in A^{**}$, we have

$$\begin{aligned} T^{**}(F \odot H)\,(f) &= F \odot H\,(T^* f) \\ &= F\,(H \circledast T^* f) \\ &= F\,(T^*\,(H \circledast f)) \\ &= T^{**} F\,(H \circledast f) \\ &= T^{**} F \odot H\,(f)\,, \end{aligned}$$

where the third equality is due to (11). Thus, $T^{**} \in M_l(A^{**})$.

□

Let us now suppose that A has an identity e_A with $\|e_A\|_A = 1$. Then $(A^{**}, \odot)$ has an identity E such that $\|E\|_{A^{**}} = 1$ (see [20]).
Let note that

$$E \circledast f = f, \text{ for any } f \in A^*. \tag{12}$$

In fact, for all $a \in A$, we have

$$\begin{aligned} f(a) &= \widetilde{a}(f) \\ &= \widetilde{a} \odot E\,(f) \\ &= \widetilde{a}\,(E \circledast f) \\ &= E \circledast f(a). \end{aligned}$$

Let us consider the mapping $\Upsilon : M_l(A) \longrightarrow A^{**}$ defined by

$$\Upsilon(T) = T^{**} E.$$

Υ is linear, and let us note $\Upsilon(M_l(A))$ by $A^{(m)}$. We have the following properties.

Proposition 2. *Let A be commutative and T belongs to $M_l(A)$. Then the following statements are true.*
(i) $T^{**}E \circledast f = T^* f$, $\forall\ f \in A^*$.
(ii) $T^{**}\widetilde{a} = \widetilde{a} \odot T^{**}E$, $\forall a \in A$.
(iii) $\widetilde{a} \odot T^{**}E \in \widetilde{A}$, $\forall a \in A$.
(iv) Υ *is an isometry of* $M_l(A)$ *into* A^{**}.
(v) $A^{(m)} = \widetilde{A}$.

Proof. (i) Lest us note that

$$T^*\left(f \circledast a\right) = T^* f \circledast a,\ f \in A^*,\ a \in A.$$

In fact, for any b belongs to A, we have

$$\begin{aligned} T^*\left(f \circledast a\right)(b) \quad &= \left(f \circledast a\right)(Tb) \\ &= f(aTb) \\ &= f(Tba) \\ &= f(T(ba)) \\ &= f(T(ab)) \\ &= T^* f(ab) \\ &= T^* f \circledast a(b). \end{aligned}$$

Thus, for all $a \in A$, we have

$$\begin{aligned} T^{**}E \circledast f(a) \quad &= T^{**}E(f \circledast a) \\ &= E\left(T^*\left(f \circledast a\right)\right) \\ &= E\left(T^* f \circledast a\right) \\ &= E\left(f \circledast Ta\right) \\ &= E \circledast f(Ta) \\ &= f(Ta) \\ &= T^* f(a), \end{aligned}$$

where the fourth (resp. sixth) equality is due to (10) (resp. (12))
(ii) For any $f \in A^*$, we have

$$\begin{aligned} T^{**}\widetilde{a}(f) \quad &= \widetilde{a}(T^* f) \\ &= \widetilde{a}(T^{**}E \circledast f) \\ &= \widetilde{a} \odot T^{**}E(f), \end{aligned}$$

where the second equality is due to (i). Thus $T^{**}\widetilde{a} = \widetilde{a} \odot T^{**}E$.
(iii) For all $f \in A^*$, we have

$$\begin{aligned} \widetilde{a} \odot T^{**}E(f) \quad &= T^{**}\widetilde{a}(f) \\ &= \widetilde{a}(T^* f) \\ &= T^* f(a) \\ &= \widetilde{Ta}(f), \end{aligned}$$

that is $\widetilde{a} \odot T^{**}E = \widetilde{Ta} \in \widetilde{A}$.

(iv) Let us note that $\|T^*\| = \|T\|$. Indeed,

$$\begin{aligned}
\|T^*\| &= \sup_{\|f\|_{A^*}\leq 1} (\|T^*f\|_{A^*}) \\
&= \sup_{\|f\|_{A^*}\leq 1} \left(\sup_{\|a\|_A\leq 1} |T^*f(a)| \right) \\
&= \sup_{\|a\|_A\leq 1} \left(\sup_{\|f\|_{A^*}\leq 1} \left|\widetilde{Ta}(f)\right| \right) \\
&= \sup_{\|a\|_A\leq 1} (\left\|\widetilde{Ta}\right\|_{A^{**}}) \\
&= \sup_{\|a\|_A\leq 1} (\|Ta\|_A), \text{ since } \Pi \text{ is isometric} \\
&= \|T\|.
\end{aligned}$$

Now, let $f \in A^*$. We have

$$\|T^*f\|_{A^*} = \|T^{**}E \circledast f\|_{A^*} \leq \|T^{**}E\|_{A^{**}} \|f\|_{A^*},$$

so $\|T^{**}E\|_{A^{**}} \geq \|T^*\|$.
Moreover, we have

$$|T^{**}E(f)| = |E(T^*f)| \leq \|E\|_{A^{**}} \|T^*f\|_{A^*} = \|T^*f\|_{A^*} \leq \|T^*\| \|f\|_{A^*},$$

that is $\|T^{**}E\|_{A^{**}} \leq \|T^*\|$. Thus, $\|T^{**}E\|_{A^{**}} = \|T^*\| = \|T\|$.
(v) For $a \in A$, let define L_a by $L_a(b) = ab$, $\forall b \in A$. It is clear that $L_a \in M_l(A)$.
Let $f \in A^*$. We have

$$L_a^*f(b) = f(ab) = f \circledast a(b), \forall b \in A,$$

that is $L_a^*f = f \circledast a$. So we have

$$\begin{aligned}
L_a^{**}E(f) &= E(L_a^*f) = E(f \circledast a) \\
&= E \circledast f(a) \\
&= f(a) \\
&= \widetilde{a}(f),
\end{aligned}$$

and $\Upsilon(L_a) = L_a^{**}E = \widetilde{a}$. It follows that $\widetilde{A} = \{\Upsilon(L_a); a \in A\} = \Upsilon(\{L_a; a \in A\}) \subset \Upsilon(M_l(A)) = A^{(m)}$. Also $A^{(m)} \subset \widetilde{A}$. In fact, for any $T \in M_l(A)$, we have

$$T^{**}E = E \odot T^{**}E = \widetilde{e_A} \odot T^{**}E \in \widetilde{A} \text{ by (iii)}.$$

Thus $A^{(m)} = \widetilde{A}$ and the proof is complete. □

3.2. **Multipliers on Vector-valued $\mathbf{L^1}$-spaces.**

Let G be a locally compact hypergroup and A be a complex Banach algebra with an identity e_A with norm one. G is endowed with a left Haar measure μ_G. Let K be a compact subhypergroup of G endowed with a normalized Haar measure ω_K. Let us recall the following notations.
$L^1(G, A)$ is the Banach space of A-valued Bochner integrable functions on G; $M(G, A)$ denotes the Banach space of A-valued Borel measures on G; $M_b(G, A)$

denotes the Banach space of A-valued Borel bounded measures on G and $C_0(G,A)$ denotes the Banach space of A-valued continuous functions of G, vanishing at infinity under the norm $|||F|||_\infty = \sup_{x\in G} \|F(x)\|$.
The convolution of two $A-$valued functions f and g on G, when it makes sense is defined by

$$f*g(x) = \int_G f(x*y)g(\overline{y})d\mu_G(y),\ x\in G$$

and the convolution of a $A-$valued function f and a $A-$valued measure ν on G, when it makes sense is defined by

$$\nu * f(x) = \int_G f(\overline{y}*x)d\nu(y) \text{ and } f*\nu(x) = \int_G f(x*\overline{y})d\nu(y),\ x\in G.$$

Under convolutions, $L^1(G,A)$, $M_b(G,A)$, and $C_0(G,A)$ become a $L^1(G)$-Banach module.

Let us recall the following properties.

Lemma 1. [17] *$L^1(G,A)$ is a closed two-side ideal in $M_b(G,A)$ via the mapping: $f \longmapsto \mu_f$, where $\mu_f(E) = \int_E f(x)d\mu_G(x)$ for every Borel set E.*

Lemma 2. [17] *$L^1(G,A) = L^1(G)\widehat{\otimes}A$, where $\widehat{\otimes}$ is the projective tensor product.*

Lemma 3. [17] *$M(G,A^{**})$ is isometrically isomorphic to the dual of $C_0(G)\overset{\vee}{\otimes}A^*$, where $\overset{\vee}{\otimes}$ is the injective tensor product.*

For $f\in L^1(G,A)$, (resp.$\mu\in M_b(G,A)$) the Fourier transform is defined as in formula (6) (resp. (5)), when (G,K) is a Gelfand pair.
Let f and g be two functions from a locally compact space X into $\widetilde{A}$. The product fg of f and g is defined by

$$fg(x) = f(x)\odot g(x)\ \forall x\in G. \tag{7}$$

It is clear that fg stays a function from X into $\widetilde{A}$ and $\|fg(x)\|_{\widetilde{A}} \le \|f(x)\|_{\widetilde{A}}\|g(x)\|_{\widetilde{A}}$, $\forall x\in X$.

We can identify $L^1(G,A)$ and $L^1\left(G,\widetilde{A}\right)$ by the following theorem.

Theorem 1. *The mapping $\psi : L^1(G,A) \longrightarrow L^1(G,\widetilde{A})$ defined by $\psi(f) = \Pi\circ f$ is a Banach isomorphism. Moreover, ψ is an isometry.*

Proof. If $f\in L^1(G,A)$, then $\psi(f)\in L^1(G,\widetilde{A})$ and

$$\begin{aligned}\int_G \|\psi(f)(x)\|_{\widetilde{A}}\,d\mu_G(x) &= \int_G \left\|\widetilde{f(x)}\right\|_{\widetilde{A}} d\mu_G(x)\\ &= \int_G \|f(x)\|_A\,d\mu_G(x).\\ &= \|f\|_1.\end{aligned}$$

Let $f,g\in L^1(G,A)$ and $\alpha,\beta\in\mathbb{C}$. For any $x\in G$, we have

$$\begin{aligned}\psi(\alpha f+\beta g)(x) &= \widetilde{(\alpha f+\beta g)}(x)\\ &= \alpha \widetilde{f(x)+\beta g}(x)\\ &= \alpha\widetilde{f(x)}+\beta\widetilde{g(x)}\\ &= \alpha\psi(f)(x)+\beta\psi(g)(x)\\ &= (\alpha\psi(f)+\beta\psi(g))(x).\end{aligned}$$

Thus, ψ is linear. Moreover, for $x\in G$ and $\varphi\in A^*$, we have

$$\begin{aligned}[\psi(f)*\psi(g)(x)](\varphi) &= \int_G \widetilde{f(x*y)}\odot\widetilde{g(\overline{y})}(\varphi)\,d\mu_G(y)\\ &= \int_G \widetilde{f(x*y)}\left(\widetilde{g(\overline{y})}\circledast\varphi\right)d\mu_G(y)\\ &= \int_G \widetilde{g(\overline{y})}\circledast\varphi\,(f(x*y))\,d\mu_G(y)\\ &= \int_G \widetilde{g(\overline{y})}\,(\varphi\circledast f(x*y))\,d\mu_G(y)\\ &= \int_G \varphi\circledast f(x*y)\,(g(\overline{y}))\,d\mu_G(y)\\ &= \int_G \varphi\,(f(x*y)\,g(\overline{y}))\,d\mu_G(y)\\ &= \varphi\left(\int_G f(x*y)\,g(\overline{y})\,d\mu_G(y)\right)\\ &= \varphi(f*g(x))\\ &= \widetilde{f*g(x)}(\varphi)\\ &= [\psi(f*g)(x)](\varphi),\end{aligned}$$

that is $\psi(f*g)=\psi(f)*\psi(g)$.
The surjection comes from the fact that Π is an isometric bijection, and the proof is complete. $\square$

A function $f\in L^1(G,A)$ is said to be K-invariant if $f(k_1*x*k_2)=f(x)$, for all $k_1,k_2\in K$ and all $x\in G$. The K-projection is the mapping $P_K: f\longmapsto f^\natural$ where $f^\natural(x)=\int_K\int_K f(k_1*x*k_2)d\omega_K(k_1)d\omega_K(k_2)$. For any $f\in L^1(G,A)$, $f^\natural\in L^1(G,A)$ and since the measure ω_K is invariant, then $f^\natural$ is K-invariant.
As in the case of $L^1(G)$, we will denote by $L^1(G,A)^\natural$ the subspace of $L^1(G,A)$ containing K-invariant elements.
We also denote by $M(G,A)^\natural$ the subspace of $M(G,A)$ containing the measures μ such that $\mu\left(f^\natural\right)=\mu(f)$ for any $f\in L^1(G,A)$.

By the definition of ψ, $f\in L^1(G,A)^\natural \iff \psi(f)\in L^1\left(G,\widetilde{A}\right)^\natural$.

We can also identify $L^1(G//K,A)$ and $L^1(G,A)^\natural$ when G is unimodular by the following theorem.

Theorem 2. *The algebras $L^1(G//K,A)$ and $L^1(G,A)^\natural$ are isometrically isomorphic.*

Proof. Let us consider the mapping: $f \longmapsto \widetilde{f}$ defined by (3). For $f \in L^1(G,A)^\natural$, we have

$$\int\limits_{G//K} \left\| \widetilde{f}(KxK) \right\|_A dm(KxK) = \int\limits_G \|f(x)\|_A \, d\mu_G(x) = \|f\|_1 ,$$

so, $\widetilde{f} \in L^1(G//K,A)$ and $\left\|\widetilde{f}\right\|_1 = \|f\|_1$.
Moreover, let $\widetilde{f}, \widetilde{g} \in L^1(G//G,A)$. For any $KxK \in G//K$, we have

$$\begin{aligned}
\widetilde{f} * \widetilde{g}(KxK) &= \int\limits_{G//K} \widetilde{f}(K\overline{y}K)\, \widetilde{g}(KyK * KxK)\, dm(KyK) \\
&= \int\limits_G f(\overline{y}) \int\limits_K g(y * k * x)\, d\omega_K(k)\, d\mu_G(y) \quad (\text{ see } (4)) \\
&= \int\limits_K \left(\int\limits_G f(\overline{y})\, g(y * k * x)\, d\mu_G(y) \right) d\omega_K(k) \\
&= \int\limits_K \left(\int\limits_G f(k * \overline{y})\, g(y * x)\, d\mu_G(y) \right) d\omega_K(k) \\
&= \int\limits_G f(\overline{y})\, g(y * x)\, d\mu_G(y) \text{ since } f \text{ is } K\text{- invariant} \\
&= f * g(x) \\
&= \widetilde{f * g}(KxK).
\end{aligned}$$

Thus, $\widetilde{f * g} = \widetilde{f} * \widetilde{g}$, and the proof is complete. ☐

Relying on the results above, we give the following theorem which is our main result.

Theorem 3. *Let G be a locally compact hypergroup; let K be a compact subhypergroup of G such that (G,K) is a Gelfand pair, and A be a commutative Banach algebra with identity. If $T : L^1(G,A)^\natural \longrightarrow L^1(G,A)^\natural$ is a bounded linear operator, then the following statements are equivalent.*
(i) $T(f * g) = T(f) * g$, $\forall f, g \in L^1(G,A)^\natural$.
(ii) There exists a unique measure $\mu \in M_b\left(G, \widetilde{A}\right)^\natural$ such that $\widehat{Tf} = \widehat{\mu}\widehat{f}$ for any $f \in L^1\left(G, \widetilde{A}\right)^\natural$.
(iii) There exists a unique continuous bounded function $\varphi : \widehat{G} \longrightarrow \widetilde{A}$ such that $\varphi\widehat{f} \in L^1\left(G, \widetilde{A}\right)^\natural$ and $\widehat{Tf} = \varphi\widehat{f}$, for any $f \in L^1\left(G, \widetilde{A}\right)^\natural$.

Proof. Let us consider the mapping $\overline{\overline{T}}$ defined on $L^1(G//K, A)$ by

$$\overline{\overline{T}}\left(\widetilde{f}\right) = \widetilde{Tf}, \forall f \in L^1(G, A)^\natural.$$

$\overline{\overline{T}}$ is a linear bounded operator on $L^1(G//K, A)$. In fact, for $f,\ g \in L^1(G, A)^\natural$ and $\lambda \in \mathbb{C}$, we have

$$\begin{aligned}\overline{\overline{T}}\left(\lambda\widetilde{f} + \widetilde{g}\right) &= T\,\widetilde{(\lambda f + g)} \\ &= \lambda\widetilde{Tf + Tg} \\ &= \lambda\widetilde{Tf} + \widetilde{Tg} \\ &= \lambda\overline{\overline{T}}\left(\widetilde{f}\right) + \overline{\overline{T}}(\widetilde{g}),\end{aligned}$$

and

$$\left\|\overline{\overline{T}}\left(\widetilde{f}\right)\right\|_1 = \left\|\widetilde{Tf}\right\|_1 = \|Tf\|_1 \text{ (see Proposition 3).}$$

Since (G, K) is a Gelfand pair, then $G//K$ is a commutative hypergroup. So by [17] (Theorem 6.1), the following statements are equivalent.

a) $\overline{\overline{T}}\left(\widetilde{f} * \widetilde{g}\right) = \overline{\overline{T}}\left(\widetilde{f}\right) * \widetilde{g}$, $\forall \widetilde{f},\ \widetilde{g} \in L^1(G//K, A)$

b) There exsits a unique measure $\widetilde{\mu} \in M_b\left(G//K, \widetilde{A}\right)$ such that $\widehat{\overline{\overline{T}}\left(\widetilde{f}\right)} = \widehat{\widetilde{\mu}}\widehat{\widetilde{f}}$, $\forall\widetilde{f} \in L^1\left(G//K, \widetilde{A}\right)$.

c) There exists a unique function $\widetilde{\varphi} \in C_b\left(\widehat{G//K}, \widetilde{A}\right)$ such that $\widetilde{\varphi}\widehat{\widetilde{f}} \in L^1\left(\widehat{G//K}, \widetilde{A}\right)$ and $\widetilde{\varphi}\widehat{\widetilde{f}} = \widehat{\overline{\overline{T}}\left(\widetilde{f}\right)}$ for any $\widetilde{f} \in L^1\left(G//K, \widetilde{A}\right)$.

To prove our theorem, it will suffice to show that (i) $\Longleftrightarrow$ a), (ii) $\Longleftrightarrow$ b) and (iii) $\Longleftrightarrow$ c).

(i)$\Longleftrightarrow$a) Let $f, g \in L^1(G, A)^\natural$, then

$$\begin{aligned}T(f * g) = T(f) * g &\Longleftrightarrow \widetilde{T(f * g)} = \widetilde{T(f) * g} \\ &\Longleftrightarrow \overline{\overline{T}}\left(\widetilde{f * g}\right) = \widetilde{T(f)} * \widetilde{g} \\ &\Longleftrightarrow \overline{\overline{T}}\left(\widetilde{f} * \widetilde{g}\right) = \overline{\overline{T}}\left(\widetilde{f}\right) * \widetilde{g}.\end{aligned}$$

(ii)$\Longleftrightarrow$b) We can see from the proof of theorem 3.11 in [10] that $\widehat{\widetilde{f}} = \widetilde{\widehat{f}}$ for any $f \in L^1\left(G, \widetilde{A}\right)^\natural$. In the same way one can see that, for any $\nu \in M_b\left(G, \widetilde{A}\right)^\natural$, the unique measure $\widetilde{\nu}$ defined by $\widetilde{\nu}\left(\widetilde{f}\right) = \nu(f)$ belongs to $M_b\left(G//A, \widetilde{A}\right)$ and $\widehat{\widetilde{\nu}} = \widetilde{\widehat{\nu}}$.

Now let us suppose that there exists a unique measure $\mu \in M_b\left(G, \widetilde{A}\right)^\natural$ such that $\widehat{Tf} = \widehat{\mu}\widehat{f}\ \forall\ f \in L^1\left(G, \widetilde{A}\right)^\natural$. We have

$$\begin{aligned}\widehat{Tf} = \widehat{\mu}\widehat{f} &\Longleftrightarrow \widetilde{\widehat{Tf}} = \widetilde{\widehat{\mu}\widehat{f}} \\ &\Longleftrightarrow \widehat{\widetilde{Tf}} = \widehat{\widetilde{\mu}}\widehat{\widetilde{f}} \\ &\Longleftrightarrow \widehat{\overline{\overline{T}}\left(\widetilde{f}\right)} = \widehat{\widetilde{\mu}}\widehat{\widetilde{f}}.\end{aligned}$$

Since the mapping: $M_b\left(G,\widetilde{A}\right)^\natural \longrightarrow M_b\left(G//K,\widetilde{A}\right)$, $\mu \longmapsto \widehat{\mu}$ is bijective, the uniqueness of μ is equivalent to the uniqueness of $\widetilde{\mu}$, and (ii)$\Longleftrightarrow$b).
(iii)$\Longleftrightarrow$c) Let us remember that $\widetilde{\phi} \in \widehat{G//K}$ if and only if $\phi = \widetilde{\phi} \circ p_K \in \widehat{G}$, and that there exists a one to one correspondence between $C_b(\widehat{G//K})$ and $C_b(\widehat{G})$ by the mapping: $\widetilde{\varphi} \longmapsto \varphi$, where $\varphi(\phi) = \widetilde{\varphi}\left(\widetilde{\phi}\right)$ $\forall \phi \in \widehat{G}$.
Let us suppose that there exists $\widetilde{\varphi} \in C_b\left(\widehat{G//K},\widetilde{A}\right)$ such that
$\widetilde{\varphi}\widehat{\widetilde{f}} \in L^1\left(\widehat{G//K},\widetilde{A}\right)$ and $\widetilde{\varphi}\widehat{\widetilde{f}} = \widehat{\overline{\overline{T}}\left(\widetilde{f}\right)}$ for any $\widetilde{f} \in L^1\left(G//K,\widetilde{A}\right)$. We have

$$\begin{aligned}\widehat{\overline{\overline{T}}\left(\widetilde{f}\right)} = \widetilde{\varphi}\widehat{\widetilde{f}} \quad &\Longleftrightarrow \widehat{\widetilde{T(f)}} = \widetilde{\varphi}\widehat{\widetilde{f}} \\ &\Longleftrightarrow \widehat{T(f)} = \varphi\widehat{f},\end{aligned}$$

and $\widetilde{\varphi}\widehat{\widetilde{f}} \in L^1\left(\widehat{G//K},\widetilde{A}\right)$ $\forall \widetilde{f} \in L^1\left(G//K,\widetilde{A}\right) \Longleftrightarrow \varphi\widehat{f} \in L^1\left(\widehat{G},\widetilde{A}\right)^\natural$ for any $f \in L^1\left(G,\widetilde{A}\right)^\natural$. Moreover $\widetilde{\varphi}$ is unique if and only if φ is unique.
Thus, (iii)$\Longleftrightarrow$c) and the theorem is proved. $\square$

Acknowledgments. The authors would like to thank the anonymous referee for his/her comments that helped us improve this article.

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