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FIXED POINT RESULTS FOR MULTIVALUED MAPPINGS INVOLVING Q-FUNCTION IN QUASI-METRIC SPACES

ABDUL LATIF, NADIAH ZAFER ALSHEHRI, MONAIRAH OMAR ALANSARI

Abstract. In this paper, we present some new results on the existence of fixed points for multivalued mappings endowed with Q-function in the setting of quasi-metric space. Examples are also provided in support of our main results. We conclude that our results in fact either improve or generalize some classical metric fixed point results as well.

1. Introduction And Preliminaries

Let (Z, d) be a metric space. We denote 2^Z a collection of nonempty subsets of Z, $Cl(Z)$ a collection of nonempty closed subsets of Z, $CB(Z)$ a collection of nonempty closed bounded subsets of Z, H the Hausdorff-Pompeiu metric on $CB(Z)$ induced by the metric d, that is; for any $A, B \in CB(Z)$,

$$H(A, B) = \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\right\},$$

where $d(a, B) = \inf_{b\in B} d(a, b)$ and $\mathbb{R}^+ = [0, \infty)$. An element $z \in Z$ is called a fixed point of a multivalued mapping $\Gamma : Z \to 2^Z$ if $z \in \Gamma(z)$. We denote $Fix(\Gamma) = \{z \in Z : z \in \Gamma(z)\}$. A sequence $\{z_n\}$ in Z is called an orbit of Γ at $z_0 \in Z$ if $z_n \in \Gamma(z_{n-1})$ for all $n \geq 1$. A real-valued function β on Z is called a lower semicontinuous if for any sequence $\{z_n\} \subset Z$ with $z_n \to z \in Z$ imply that $\beta(z) \leq \liminf_{n\to\infty} \beta(z_n)$. A function $\chi : \mathbb{R}^+ \to [0, 1)$ is called MT-function (Mizoguchi-Takahashi function) if $\limsup_{r\to t^+} \chi(r) < 1$ for all $t \in \mathbb{R}^+$. It is has been observed that a function χ is MT-function if and only if for any strictly decreasing sequence $\{z_n\}$ in $\mathbb{R}^+$, we have $0 \leq \sup_n \chi(z_n) < 1$. For more characterizations of MT-function, see; [13].
In the sequel till Theorem 1.5, we consider (Z, d) a complete metric space.

Using the concept of Hausdorff-Pompeiu metric, Nadler [28] established the following multivalued version of the well known Banach contraction principle [4].

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Theorem 1.1. ([28]) *Let $\Gamma : Z \to CB(Z)$ be a multivalued contraction mapping (that is; for a fixed constant $h \in (0,1)$ and for every $z_1, z_2 \in Z$, $H(\Gamma(z_1), \Gamma(z_2)) \leq h\ d(z_1, z_2)$). Then $Fix(\Gamma) \neq \emptyset$.*

Using MT-function, Mizoguchi and Takahashi [27] established a real generalization of Theorem 1.1.

Theorem 1.2. ([27]) *Let $\Gamma : Z \to CB(Z)$ be a multivalued mapping such that for every $z_1, z_2 \in Z$,*

$$H(\Gamma(z_1), \Gamma(z_2)) \leq \chi(d(z_1, z_2))d(z_1, z_2),$$

where χ is an MT-function. Then, $Fix(\Gamma) \neq \emptyset$.

A number of fruitful various generalizations of these classical results have been appeared in the literature. For example, see; [10, 12, 27] and references therein. Some interesting fixed point results have been appeared without using the Hausdorff-Pompeiu metric. For example, see; [7, 9, 14] and others. In [14], Feng and Liu proved the following result without using the concept of Hausdorff-Pompeiu metric, which extends Theorem 1.1.

Theorem 1.3. ([14]) *Let $\Gamma : Z \to Cl(Z)$ be a multivalued mapping and let a function β on Z with $\beta(z) = d(z, \Gamma(z))$ be a lower semicontinuous. Then $Fix(\Gamma) \neq \emptyset$ provided that there are some constants $c, h \in (0,1)$, $h < c$ such that for every $z_1 \in Z$, there is $z_2 \in I_c^{z_1}$ satisfying*

$$d(z_2, \Gamma(z_2)) \leq hd(z_1, z_2),$$

where $I_c^{z_1} = \{z_2 \in \Gamma(z_1) : cd(z_1, z_2) \leq d(z_1, \Gamma(z_1))\}$.

Later, Klim and Wardowski [19] established a generalization of Theorem 1.3 as follows.

Theorem 1.4. ([19]) *Let $\Gamma : Z \to Cl(Z)$ be a multivalued mapping and let a function β on Z with $\beta(z) = d(z, \Gamma(z))$ be a lower semicontinuous. Then, $Fix(\Gamma) \neq \emptyset$ provided that there is some $c \in (0,1)$ such that for every $z_1 \in Z$, there is $z_2 \in \Gamma(z_1)$ satisfying*

$$cd(z_1, z_2) \leq d(z_1, \Gamma(z_1)) \quad \text{and} \quad d(z_2, \Gamma(z_2)) \leq \chi(d(z_1, z_2))d(z_1, z_2)$$

where χ is a function from $\mathbb{R}^+$ to $[0, c)$ with $\limsup_{r \to t^+} \chi(r) < c$, for every $t \in \mathbb{R}^+$.

It has been observed in [19] that Theorem 1.4 do not generalize fixed point result of Mizoguchi and Takahashi [27, Theorem 5] (Theorem 1.2). *Ćirić* [8, Theorem 6] established a fixed point result for multivalued nonlinear contractions, which generalizes Theorem 1.2, Theorem 1.3 and Theorem 1.4.

Kada et al. [17] introduced a concept of w-distance on metric spaces as follows.

Let (Z, d) be a metric space. A function $w : Z \times Z \to \mathbb{R}^+$ is called w-distance on Z if the following conditions are satisfied for each $z_1, z_2, z_3 \in Z$:

- (w_1) $w(z_1, z_2) \leq w(z_1, z_3) + w(z_3, z_2)$;
- (w_2) for any $z \in Z$, a function $w(z, \cdot) : Z \to \mathbb{R}^+$ is lower semicontinuous;
- (w_3) for any $\epsilon > 0$, there exists $\delta > 0$ such that $w(z_3, z_1) \leq \delta$ and $w(z_3, z_2) \leq \delta$ imply $d(z_1, z_2) \leq \epsilon$.

Using the concept of w-distance, they improved a number of important results of metric fixed point theory. Note that, in general for $z_1, z_2 \in Z, w(z_1, z_2) \neq w(z_2, z_1)$ and not either of the implications $w(z_1, z_2) = 0 \Leftrightarrow z_1 = z_2$ necessarily hold. Clearly, the metric d is a w-distance on Z. Examples and properties of the w-distance, see [17, 31]. Without using the concept of Hausdorff-Pompeiu metric, Susuki and Takahashi [30] generalized some metric fixed point results including Theorem 1.1 for contractive type mappings with respect to w-distance.

Theorem 1.5. ([30]) *Let $\Gamma : Z \to Cl(Z)$ be a multivalued mapping. If there exists a w-distance w on Z and a constant $\lambda \in (0, 1)$, such that for every $z_1, z_2 \in Z$, and $u \in \Gamma(z_1)$, there is $v \in \Gamma(z_2)$ satisfying*

$$w(u, v) \leq \lambda\ w(z_1, z_2).$$

Then, there exists $z_0 \in Z$ such that $z_0 \in \Gamma(z_0)$ and $w(z_0, z_0) = 0$.

Latif and Albar [23] and Latif and Abdou [21] generalized Theorem 1.3 and Theorem 1.4, respectively with respect to w-distance. For further work in this direction, see; [15, 16, 18, 20, 24, 25, 29, 32] and others.

Now, let us recall the well-known generalization of the standard metric, known as quasi-metric, see [33] and others.

Let Z be a nonempty set. A function $D : Z \times Z \to \mathbb{R}^+$ is said to be a quasi-metric on Z if the following conditions are satisfied for all $z_1, z_2, z_3 \in Z$:

(1) $D(z_1, z_2) = 0$ if and only if $z_1 = z_2$,
(2) $D(z_1, z_2) \leqslant D(z_1, z_3) + D(z_3, z_2)$.

In this case, the pair (Z, D) is called a quasi-metric space. Every metric space is a quasi-metric space. The concepts of Cauchy sequences, convergent sequences, and completeness in the frame work of quasi-metric spaces can be defined in a same manner as in the setting of metric spaces, see [3]. Further, a quasi-metric space can be endowed with a topology induced by its convergence and almost all the concepts and results which are valid for metric spaces can be extended to the framework of quasi-metric space. For further examples of quasi-metric space, see; [1, 2, 5, 6].

A subset A of the quasi-metric space (Z, D) is said to be open if and only if for any $a \in A$, there exists $\epsilon > 0$ such that the open ball $B_0(a, \epsilon) \subset A$. The family of all open subsets of Z will be denoted by τ. It has been observed that τ defines a topology on (Z, D). Further, any nonempty subset A of Z is closed if and only if for any sequence $\{z_n\}$ in A which converges to z, we have $z \in A$, see; [11, 26].

In [3], Al-Homidan et al. introduced a concept of Q-function on quasi-metric spaces.

Let (Z, D) be a quasi-metric space. A function $q : Z \times Z \to \mathbb{R}^+$ is called a Q-function on Z if the following conditions are satisfied for any $z_1, z_2, z_3 \in Z$:

(q_1) $q(z_1, z_2) \leqslant q(z_1, z_3) + q(z_3, z_2)$;
(q_2) If $\{z_n\}$ is a sequence in Z such that $z_n \to z \in Z$ and $q(z_1, z_n) \leqslant N$ for some $N = N(z_1) > 0$ then $q(z_1, z) \leqslant N$;
(q_3) for any $\varepsilon > 0$, there exists $\delta > 0$, such that $q(z_3, z_1) \leqslant \delta$ and $q(z_3, z_2) \leqslant \delta$ imply $D(z_1, z_2) \leqslant \varepsilon$.

It has been observed [3] that every w-distance is a Q-function but the converse may not be true, in general. It is also worth to mention that the concepts of a Q-function and a quasi-metric are not comparable, see [3, Example 3.1 and Example 3.2]. Each discrete metric on quasi-metric space (Z, D) is a Q-function and for other examples of Q-functions, see [26].

Using the technique of [25], the following result is obvious.

Lemma 1.6. *Let S be a closed subset of a quasi-metric space (Z, D) and q be a Q-function on Z. Suppose that there exists $z_1 \in Z$ such that $q(z_1, z_1) = 0$. Then we have $q(z_1, S) = 0 \Leftrightarrow z_1 \in S$, where $q(z_1, S) = \inf\{q(z_1, z_2) : z_2 \in S\}$.*

The following result is useful for our results.

Lemma 1.7. ([3, 24]) *Let (Z, D) be a quasi-metric space and q be a Q-function on Z. Let $\{z_n\}$ and $\{z_n^{'}\}$ be sequences in Z, let $\{\alpha_n\}$ and $\{\gamma_n\}$ be sequences in $\mathbb{R}^+$ converging to zero. Then, the following hold for every $z_1, z_2, z_3 \in Z$:*

- (i) *if $q(z_n, z_2) \leq \alpha_n$ and $q(z_n, z_3) \leq \gamma_n$ for any $n \in \mathbb{N}$, then $z_2 = z_3$. In particular, if $q(z_1, z_2) = 0$ and $q(z_1, z_3) = 0$, then $z_2 = z_3$;*
- (ii) *if $q(z_n, z_n^{'}) \leq \alpha_n$ and $q(z_n, z_3) \leq \gamma_n$ for any $n \in \mathbb{N}$, then $D(z_n^{'}, z_3) \to 0$;*
- (iii) *if $q(z_n, z_m) \leq \alpha_n$ for any $n, m \in \mathbb{N}$ with $m > n$, then $\{z_n\}$ is a Cauchy sequence;*
- (iv) *if $q(z_2, z_n) \leq \alpha_n$ for any $n \in \mathbb{N}$, then $\{z_n\}$ is a Cauchy sequence.*

In [3], Al-Homidan et al. generalized Theorem 1.1 with respect to Q-function.

Theorem 1.8. ([3]) *Let (Z, D) be a complete quasi-metric space and let $\Gamma : Z \to Cl(Z)$ be a multivalued mapping. If there exists a Q-function q on Z and a constant $\lambda \in (0, 1)$ such that for every $z_1, z_2 \in Z$ and $u \in \Gamma(z_1)$, there is $v \in \Gamma(z_2)$ satisfying*

$$q(u, v) \leq \lambda\, q(z_1, z_2).$$

Then, there exists $z_0 \in Z$ such that $z_0 \in \Gamma(z_0)$ and $q(z_0, z_0) = 0$.

The aim of this paper is to present new general results on the existence of fixed points for multivalued mappings involving Q-function on quasi-metric spaces. Consequently, our results unify and generalize the corresponding several known metric fixed point results.

2. Results

In this section, we consider (Z, D) is a quasi-metric space with Q-function q and χ is an MT-function.

First, we prove the following key lemma.

Lemma 2.1. *Let $\Gamma : Z \to 2^Z$ be a multivalued mapping such that for any $u_1 \in Z$, there exists $u_2 \in \Gamma(u_1)$ satisfying*

$$\begin{aligned} q(u_1, u_2) &\leq (2 - \chi(q(u_1, u_2)))q(u_1, \Gamma(u_1)), \\ q(u_2, \Gamma(u_2)) &\leq \chi(q(u_1, u_2))q(u_1, u_2). \end{aligned} \tag{2.1}$$

Then, the existence of an orbit $\{z_n\}$ of Γ in Z implies that the sequences of non-negative reals $\{q(z_n, \Gamma(z_n))\}$ and $\{q(z_n, z_{n+1})\}$ converge to zero.

Proof. Let $z_0 \in Z$ be a fixed arbitrary element. Then, we get $z_1 \in \Gamma(z_0)$ satisfying

$$\begin{gathered} q(z_0, z_1) \leq (2 - \chi(q(z_0, z_1)))q(z_0, \Gamma(z_0)), \\ q(z_1, \Gamma(z_1)) \leq \chi(q(z_0, z_1))q(z_0, z_1). \end{gathered} \tag{2.2}$$

From (2.2), we get

$$q(z_1, \Gamma(z_1)) \leq \chi(q(z_0, z_1))(2 - \chi(q(z_0, z_1)))q(z_0, \Gamma(z_0)). \tag{2.3}$$

Define a function $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\rho(t) = \chi(t)(2 - \chi(t)) = 1 - (1 - \chi(t))^2. \tag{2.4}$$

Note that for any $t \in \mathbb{R}^+$, $\rho\ (t) < 1$, and $\limsup_{r \to t^+} \rho(r) < 1$. From (2.3) and (2.4), we have

$$q(z_1, \Gamma(z_1)) \leq \rho(q(z_0, z_1))q(z_0, \Gamma(z_0)). \tag{2.5}$$

Continuing this process we can get an orbit $\{z_n\}$ of Γ in Z satisfying the following for each integer $n \geq 0$,

$$q(z_n, z_{n+1}) \leq (2 - \chi(q(z_n, z_{n+1})))q(z_n, \Gamma(z_n)), \tag{2.6}$$

$$q(z_{n+1}, \Gamma(z_{n+1})) \leq \rho(q(z_n, z_{n+1}))q(z_n, \Gamma(z_n)). \tag{2.7}$$

Thus for all $n \geq 0$, we have

$$q(z_{n+1}, \Gamma(z_{n+1})) < q(z_n, \Gamma(z_n)). \tag{2.8}$$

It follows that the sequence of non-negative real numbers $\{q(z_n, \Gamma(z_n))\}$ is convergent. Therefore, there is some $\delta \geq 0$ such that

$$\lim_{n \to \infty} q(z_n, \Gamma(z_n)) = \delta. \tag{2.9}$$

Note that

$$q(z_n, \Gamma(z_n)) \leq q(z_n, z_{n+1}) < 2q(z_n, \Gamma(z_n)), \tag{2.10}$$

thus, it follows that the sequence $\{q(z_n, z_{n+1})\}$ is bounded. Thus, there is some $\theta \geq 0$ such that

$$\liminf_{n \to \infty} q(z_n, z_{n+1}) = \theta. \tag{2.11}$$

Note that for each $n \geq 0$, $q(z_n, z_{n+1}) \geq q(z_n, \Gamma(z_n))$, and thus we get $\theta \geq \delta$. Now, we show that $\theta = \delta$. Suppose that $\delta = 0$. Then we get

$$\lim_{n \to \infty} q(z_n, z_{n+1}) = 0. \tag{2.12}$$

Now, if $\theta > \delta > 0$, then $\theta - \delta > 0$ From (2.9) and (2.11) there is a positive integer n_0 such that

$$q(z_n, \Gamma(z_n)) < \delta + \frac{\theta - \delta}{4} \quad \forall n \geq n_0, \tag{2.13}$$

$$\theta - \frac{\theta - \delta}{4} < q(z_n, z_{n+1}) \quad \forall n \geq n_0. \tag{2.14}$$

Then from (2.6), (2.13) and (2.14), we get

$$\begin{aligned} \theta - \frac{\theta - \delta}{4} &< q(z_n, z_{n+1}) \\ &\leq (2 - \chi(q(z_n, z_{n+1})))q(z_n, \Gamma(z_n)) \\ &< (2 - \chi(q(z_n, z_{n+1}))) \left[\delta + \frac{\theta - \delta}{4}\right]. \end{aligned} \tag{2.15}$$

Thus for all $n \geq n_0$,

$$(2-\chi(q(z_n, z_{n+1}))) > \frac{3\theta+\delta}{3\delta+\theta}, \tag{2.16}$$

that is,

$$1+(1-\chi(q(z_n, z_{n+1}))) > 1+\frac{2(\theta-\delta)}{3\delta+\theta}, \tag{2.17}$$

and we get

$$-(1-\chi(q(z_n, z_{n+1})))^2 < -\left[\frac{2(\theta-\delta)}{3\delta+\theta}\right]^2. \tag{2.18}$$

Thus for all $n \geq n_0$,

$$\rho(q(z_n, z_{n+1})) = 1-(1-\chi(q(z_n, z_{n+1})))^2 < 1-\left[\frac{2(\theta-\delta)}{3\delta+\theta}\right]^2. \tag{2.19}$$

Put $h = 1-[2(\theta-\delta)/(3\delta+\theta)]^2$. Clearly $h < 1$ as $\theta > \delta$. Thus, from (2.7) and (2.19), we get

$$q(z_{n+1}, \Gamma(z_{n+1})) \leq hq(z_n, \Gamma(z_n)) \quad \forall n \geq n_0. \tag{2.20}$$

From (2.13) and (2.20), for any $k \geq 1$ we have

$$q(z_{n_0+k}, \Gamma(z_{n_0+k})) \leq h^k q(z_{n_0}, \Gamma(z_{n_0})). \tag{2.21}$$

Since $\delta > 0$ and $h < 1$, there is a positive integer k_0 such that $h^{k_0} q(z_{n_0}, \Gamma(z_{n_0})) < \delta$. Now, since $\delta \leq q(z_n, \Gamma(z_n))$ for each $n \geq 0$, by (2.21) we have

$$\delta \leq q(z_{n_0+k_0}, \Gamma(z_{n_0+k_0})) \leq h^{k_0} q(z_{n_0}, \Gamma(z_{n_0})) < \delta, \tag{2.22}$$

a contradiction. Hence, our assumption $\theta > \delta$ is wrong. Thus $\delta = \theta$. Now, we show that $\theta = 0$. Since $\theta = \delta \leq q(z_n, \Gamma(z_n)) \leq q(z_n, z_{n+1})$, then from (2.11) we can read as

$$\liminf_{n\to\infty} q(z_n, z_{n+1}) = \theta+, \tag{2.23}$$

so, there exists a subsequence $\{q(z_{n_k}, z_{n_k+1})\}$ of $\{q(z_n, z_{n+1})\}$ such that

$$\lim_{k\to\infty} q(z_{n_k}, z_{n_k+1}) = \theta+. \tag{2.24}$$

Note that

$$\limsup_{q(z_{n_k}, z_{n_k+1})\to\theta+} \rho(q(z_{n_k}, z_{n_k+1})) < 1, \tag{2.25}$$

and from (2.7), we have

$$q(z_{n_k}, \Gamma(z_{n_k})) \leq \rho(q(z_{n_k}, z_{n_k+1}))q(z_{n_k}, \Gamma(z_{n_k})). \tag{2.26}$$

Using (2.9), we get

$$\begin{aligned} \delta &= \limsup_{k\to\infty} q(z_{n_k+1}, \Gamma(z_{n_k+1})) \\ &\leq \left(\limsup_{k\to\infty} \rho(q(z_{n_k}, z_{n_k+1}))\right)\left(\limsup_{k\to\infty} q(z_{n_k}, \Gamma(z_{n_k}))\right) \\ &= \left(\limsup_{q(z_{n_k}, z_{n_k+1})\to\theta+} \rho(q(z_{n_k}, z_{n_k+1}))\right)\delta. \end{aligned} \tag{2.27}$$

If we suppose that $\delta > 0$, then from last inequality, we have

$$\limsup_{q(z_{n_k}, z_{n_k+1})\to\theta+} \rho(q(z_{n_k}, z_{n_k+1})) \geq 1, \tag{2.28}$$

which contradicts with (2.25). Thus $\delta = 0$. Then from (2.9) and (2.10), we have

$$\lim_{n\to\infty} q(z_n, \Gamma(z_n)) = 0+, \tag{2.29}$$

and thus

$$\lim_{n\to\infty} q(z_n, z_{n+1}) = 0+. \tag{2.30}$$

□

Using Lemma 2.1, we obtain the following fixed point result.

Theorem 2.2. *Assume that all the hypotheses of* Lemma 2.1 *hold. If Z is complete, then there exists an orbit of Γ which converges in Z. Further, if there is a lower semicontinuous function β on Z with $\beta(z) = q(z, \Gamma(z))$, then, there exists $u_0 \in Z$ such that $\beta(u_0) = 0$. Also, if the mapping Γ is closed valued and $q(u_0, u_0) = 0$ then $u_0 \in \Gamma(u_0)$.*

Proof. In the light of Lemma 2.1, we have an orbit $\{z_n\}$ of Γ such that (2.29) and (2.30) hold. Now, let

$$\alpha = \limsup_{q(z_{n_k}, z_{n_k+1})\to\theta+} \rho(q(z_{n_k}, z_{n_k+1})). \tag{2.31}$$

Clearly, $\alpha < 1$. Let a be such that $\alpha < a < 1$. Then there is some $n_0 \in \mathbb{N}$ such that

$$\rho(q(z_n, z_{n+1})) < a \quad \forall n \geq n_0. \tag{2.32}$$

Thus it follows from (2.7),

$$q(z_{n+1}, \Gamma(z_{n+1})) \leq a q(z_n, \Gamma(z_n)) \quad \forall n \geq n_0. \tag{2.33}$$

By induction we get

$$q(z_{n+1}, \Gamma(z_{n+1})) \leq a^{n+1-n_0} q(z_{n_0}, \Gamma(z_{n_0})) \quad \forall n \geq n_0. \tag{2.34}$$

Now, using (2.10) and (2.34), we have

$$q(z_n, z_{n+1}) \leq 2a^{n-n_0} q(z_{n_0}, \Gamma(z_{n_0})) \quad \forall n \geq n_0. \tag{2.35}$$

Now, we show that $\{z_n\}$ is a Cauchy sequence, for all $m > n \geq n_0$, we get

$$\begin{aligned} q\left(z_n, z_m\right) &\leq \sum_{k=n}^{m-1} q\left(z_k, z_{k+1}\right) \\ &\leq 2\sum_{k=n}^{m-1} a^{k-n_0} q\left(z_{n_0}, \Gamma\left(z_{n_0}\right)\right) \\ &\leq 2\left(\frac{a^{n-n_0}}{1-a}\right) q(z_{n_0}, \Gamma(z_{n_0})). \end{aligned} \tag{2.36}$$

Since $a < 1$, an orbit $\{z_n\}$ turned to be a Cauchy sequence in the complete space Z. Thus we have some $u_0 \in Z$ with $\lim_{n\to\infty} z_n = u_0$. Since β is lower semicontinuous and from (2.29) , we have

$$0 \leq \beta(u_0) \leq \liminf_{n\to\infty} \beta(z_n) = q(z_n, \Gamma(z_n)) = 0, \tag{2.37}$$

and thus, $\beta(u_0) = q(u_0, \Gamma(u_0)) = 0$. Since $q(u_0, u_0) = 0$, and $\Gamma(u_0)$ is closed, it follows from Lemma 1.6 that $u_0 \in \Gamma(u_0)$. □

Now, we present another interesting fixed point result by replacing the assumption of the real-valued function β of Theorem 2.2 with another suitable assumption.

Theorem 2.3. *Suppose that all the hypotheses of* Theorem 2.2 *except the assumption of the real-valued function* β *hold. Assume that*

$$\inf\{q(z,u)+q(z,\Gamma(z)) : z \in Z\} > 0, \tag{2.38}$$

for every $u \in Z$ *with* $u \notin \Gamma(u)$. *Then* $Fix(\Gamma) \neq \emptyset$.

Proof. As in the proof of Theorem 2.2, we get a Cauchy sequence $\{z_n\}$ such that $z_n \in \Gamma(z_{n-1})$. Since Z is complete, there exists $u_0 \in Z$ such that the sequence $\{z_n\}$ converges to u_0. From (2.35) and (2.36), we get for all $n \geq n_0$

$$\begin{aligned} q(z_n,u_0) &\leq \left(\frac{2a^{n-n_0}}{1-a}\right) q(z_{n_0},\Gamma(z_{n_0})), \\ q(z_n,\Gamma(z_n)) &\leq q(z_n,z_{n+1}) \leq 2a^{n-n_0} q(z_{n_0},\Gamma(z_{n_0})). \end{aligned} \tag{2.39}$$

Assume that $u_0 \notin \Gamma(u_0)$. Then, we have

$$\begin{aligned} 0 &< \inf\{q(z,u_0)+q(z,\Gamma(z)) : z \in Z\} \\ &\leq \inf\{q(z_n,u_0)+q(z_n,\Gamma(z_n)) : n \geq n_0\} \\ &\leq \inf\left\{\left(\frac{2a^{n-n_0}}{1-a}\right) q(z_{n_0},\Gamma(z_{n_0})) + 2a^{n-n_0} q(z_{n_0},\Gamma(z_{n_0}))\right\} \\ &= \frac{2(2-a)}{(1-a)a^{n_0}} q(z_{n_0},\Gamma(z_{n_0})) \inf\{a^n : n \geq n_0\} = 0, \end{aligned} \tag{2.40}$$

which is impossible and hence $u_0 \in Fix(\Gamma)$. □

Theorem 2.4. *Let* $\Gamma : Z \to Cl(Z)$ *be a multivalued mapping with the space* Z *complete. Assume that the following conditions hold:*

(i) *there exists a function* $\mu : \mathbb{R}^+ \to [b,1)$, *with* $b > 0$, μ *non-decreasing such that for each* $t \in \mathbb{R}^+$

$$\chi(t) < \mu(t), \ \limsup_{r\to t^+} \chi(r) < \limsup_{r\to t^+} \mu(r), \tag{2.41}$$

(ii) *for any* $u_1 \in Z$, *there exists* $u_2 \in \Gamma(u_1)$ *satisfying*

$$\begin{aligned} \mu(q(u_1,u_2))q(u_1,u_2) &\leq q(u_1,\Gamma(u_1)), \\ q(u_2,\Gamma(u_2)) &\leq \chi(q(u_1,u_2))q(u_1,u_2), \end{aligned} \tag{2.42}$$

(iii) *a real-valued function* β *on* Z *defined by* $\beta(z) = q(z,\Gamma(z))$ *is lower semicontinuous.*

Then, there exists $u_0 \in Z$ *such that* $\beta(u_0) = 0$. *Further, if* $q(u_0,u_0) = 0$ *then* $u_0 \in \Gamma(u_0)$.

Proof. Let z_0 be an arbitrary, then there exists $z_1 \in \Gamma(z_0)$ such that

$$\begin{aligned} \mu(q(z_0,z_1))q(z_0,z_1) &\leq q(z_0,\Gamma(z_0)), \\ q(z_1,\Gamma(z_1)) &\leq \chi(q(z_0,z_1))q(z_0,z_1). \end{aligned} \tag{2.43}$$

From (2.43) we have

$$q(z_1,\Gamma(z_1)) \leq \frac{\chi(q(z_0,z_1))}{\mu(q(z_0,z_1))} q(z_0,\Gamma(z_0)). \tag{2.44}$$

Define a function $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\rho(t) = \frac{\chi(t)}{\mu(t)} \quad \forall t \in \mathbb{R}^+. \tag{2.45}$$

Since $\mu(t) > \chi(t)$, we get $\rho(t) < 1$, and $\limsup_{r\to t^+}\rho(r) < 1 \quad \forall\, t \in \mathbb{R}^+$. It follows from (2.44)

$$q(z_1, \Gamma(z_1)) \le \rho(q(z_0, z_1))q(z_0, \Gamma(z_0)). \tag{2.46}$$

Similarly, there exists $z_2 \in \Gamma(z_1)$ such that

$$\begin{aligned} \mu(q(z_1, z_2))q(z_1, z_2) &\le q(z_1, \Gamma(z_1)),\\ q(z_2, \Gamma(z_2)) &\le \chi(q(z_1, z_2))q(z_1, z_2). \end{aligned} \tag{2.47}$$

Then by definition of ρ, we get

$$q(z_2, \Gamma(z_2)) \le \rho(q(z_1, z_2))q(z_1, \Gamma(z_1)). \tag{2.48}$$

Finally, we have an orbit $\{z_n\}$ of Γ at z_0 satisfying

$$\mu(q(z_n, z_{n+1}))q(z_n, z_{n+1}) \le q(z_n, \Gamma(z_n)), \tag{2.49}$$

$$q(z_{n+1}, \Gamma(z_{n+1})) \le \chi(q(z_n, z_{n+1}))q(z_n, z_{n+1}). \tag{2.50}$$

Thus,

$$q(z_{n+1}, \Gamma(z_{n+1})) \le \rho(q(z_n, z_{n+1}))q(z_n, \Gamma(z_n)). \tag{2.51}$$

Since $\rho(t) < 1$ for all $t \in \mathbb{R}^+$, we get

$$q(z_{n+1}, \Gamma(z_{n+1})) < q(z_n, \Gamma(z_n)). \tag{2.52}$$

Thus the sequence of non-negative real numbers $\{q(z_n, \Gamma(z_n))\}$ becomes convergent. Also, we claim that the sequence $\{q(z_n, z_{n+1})\}$ is decreasing. Suppose that $q(z_n, z_{n+1}) \le q(z_{n+1}, z_{n+2})$, then as $\mu(t)$ is non-decreasing, we have

$$\mu(q(z_n, z_{n+1})) \le \mu(q(z_{n+1}, z_{n+2})), \tag{2.53}$$

Now using (2.49), (2.50) and (2.53) with $n = n + 1$, we get

$$\begin{aligned} q(z_{n+1}, z_{n+2}) &\le \frac{\chi(q(z_n, z_{n+1}))}{\mu(q(z_{n+1}, z_{n+2}))}q(z_n, z_{n+1})\\ &\le \frac{\chi(q(z_n, z_{n+1}))}{\mu(q(z_n, z_{n+1}))}q(z_n, z_{n+1})\\ &< \rho(q(z_n, z_{n+1}))q(z_n, z_{n+1})\\ &< q(z_n, z_{n+1}), \end{aligned} \tag{2.54}$$

a contradiction. Thus the sequence $\{q(z_n, z_{n+1})\}$ is decreasing. Now let

$$\limsup_{n\to\infty} \rho(q(z_n, z_{n+1})) = \alpha, \tag{2.55}$$

Note that $\alpha < 1$ and for any $a \in (\alpha, 1)$, there is an $n_0 \in \mathbb{N}$ such that

$$\rho(q(z_n, z_{n+1})) < a \quad \forall n \ge n_0. \tag{2.56}$$

So, from (2.51), for all $n \ge n_0$, we get

$$q(z_{n+1}, \Gamma(z_{n+1})) < aq(z_n, \Gamma(z_n)). \tag{2.57}$$

Thus by induction, we get for all $n \ge n_0$

$$q(z_{n+1}, \Gamma(z_{n+1})) \le a^{n+1-n_0}q(z_{n_0}, \Gamma(z_{n_0})). \tag{2.58}$$

As $\mu(t) \ge b$, using (2.49) and (2.58), we have

$$q(z_n, z_{n+1}) \le \frac{1}{b}q(z_n, \Gamma(z_n)) \le \frac{1}{b}a^{n-n_0}q(z_{n_0}, \Gamma(z_{n_0})), \tag{2.59}$$

for all $n \geq n_0$. Note that $q(z_n, \Gamma(z_n)) \to 0$. Now, for each $m > n \geq n_0$, we have

$$\begin{aligned} q(z_n, z_m) &\leq \sum_{k=n}^{m-1} q(z_k, z_{k+1}) \\ &\leq \frac{1}{b} \sum_{k=n}^{m-1} a^{k-n_0} q(z_{n_0}, \Gamma(z_{n_0})) \\ &\leq \frac{1}{b} \left(\frac{a^{n-n_0}}{1-a} \right) q(z_{n_0}, \Gamma(z_{n_0})). \end{aligned} \tag{2.60}$$

Thus $\{z_n\}$ becomes a Cauchy sequence and hence there is some $u_0 \in Z$ with $\beta(u_0) = q(u_0, \Gamma(u_0)) = 0$ and $u_0 \in \Gamma(u_0)$, as in the proof of Theorem 2.2. □

In the light of Theorem 2.3, we have the following result.

Theorem 2.5. *If all the assumptions of* Theorem 2.4 *without* (*iii*) *hold and*

$$\inf\{q(z, u) + q(z, \Gamma(z)) : z \in Z\} > 0, \tag{2.61}$$

for every $u \in Z$ *with* $u \notin \Gamma(v)$. *Then* $Fix(\Gamma) \neq \emptyset$.

Theorem 2.6. *Let* $\Gamma : Z \to Cl(Z)$ *be a multivalued mapping with* Z *complete and satisfying the conditions as under:*

(*i*) *for any* $u_1 \in Z$, *there exists* $u_2 \in \Gamma(u_1)$ *satisfying*

$$\begin{gathered} q(u_1, u_2) = q(u_1, \Gamma(u_1)), \\ q(u_2, \Gamma(u_2)) \leq \chi(q(u_1, u_2)) q(u_1, u_2), \end{gathered} \tag{2.62}$$

(*ii*) *a real-valued function* β *on* Z, *defined by* $\beta(z) = q(z, \Gamma(z))$ *is lower semicontinuous.*

Then, $\beta(u_0) = 0$, *for some* $u_0 \in Z$. *Moreover,* $u_0 \in \Gamma(u_0)$, *provided* $q(u_0, u_0) = 0$.

Proof. Let $z_0 \in Z$ be any arbitrary point. Then we can choose $z_1 \in \Gamma(z_0)$ such that

$$\begin{gathered} q(z_0, z_1) = q(z_0, \Gamma(z_0)), \\ q(z_1, \Gamma(z_1)) \leq \chi(q(z_0, z_1)) q(z_0, z_1), \quad \chi(q(z_0, z_1)) < 1. \end{gathered} \tag{2.63}$$

Thus, as in the proof of Lemma 2.2 [24], we can get a Cauchy sequence $\{z_n\}$ in Z satisfying $z_n \in \Gamma(z_{n-1})$ and

$$\begin{gathered} q(z_n, z_{n+1}) = q(z_n, \Gamma(z_n)), \\ q(z_{n+1}, \Gamma(z_{n+1})) \leq \chi(q(z_n, z_{n+1})) q(z_n, z_{n+1}), \quad \chi(q(z_n, z_{n+1})) < 1. \end{gathered} \tag{2.64}$$

Consequently, there exists $u_0 \in Z$ such that $\lim_{n\to\infty} z_n = u_0$. Since β is lower semicontinuous, we have

$$0 \leq \beta(u_0) \leq \liminf_{n\to\infty} \beta(z_n) = 0, \tag{2.65}$$

thus, $g(u_0) = q(u_0, \Gamma(u_0)) = 0$. Further by closedness of $\Gamma(u_0)$ and since $q(u_0, u_0) = 0$, it follows from Lemma 1.6 that $u_0 \in \Gamma(u_0)$. □

Remark.

(1) Theorem 2.2 generalizes fixed point results of *Ćirić* [8, Theorem 5] and Latif and Abdou [22, Theorem 2.1].

(2) Theorem 2.4 generalizes fixed point results of *Ćirić* [8, Theorem 6], and Latif and Abdou [22, Theorem 2.3].

(3) Theorem 2.6 improves the results of Ćirić [8, Theorem 7], and Latif and Abdou [22, Theorem 2.5]. Consequently, it contains fixed point result of Klim and Wardowski [19, Theorem 2.2] as a special case.

3. Examples

In support of Theorem 2.2, we present the following example.

Example 1. Consider $Z = [-1, 1]$ with the quasi-metric D defined by

$$D(z_1, z_2) = \begin{cases} 0; & \text{if } z_1 = z_2, \\ |z_2|; & \text{otherwise.} \end{cases}$$

Define a Q-function on Z by

$$q(z_1, z_2) = |z_2|, \quad \text{for all } z_1, z_2 \in Z.$$

Let $\Gamma : Z \to Cl(Z)$ be defined as

$$\Gamma(z) = \begin{cases} \left\{\frac{1}{2}z^2\right\}; & z \in \left[-1, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right], \\ \left\{\frac{1}{7}, \frac{1}{4}\right\}; & z = \frac{1}{2}. \end{cases}$$

Define $\chi : \mathbb{R}^+ \to [0, 1)$ by

$$\chi(t) = \begin{cases} \frac{3}{4}t; & t \in \left[0, \frac{1}{2}\right), \\ \frac{3}{8}; & t \in \left[\frac{1}{2}, \infty\right). \end{cases}$$

Note that

$$\beta(z) = q(z, \Gamma(z)) = \begin{cases} \frac{1}{2}z^2; & z \in \left[-1, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right], \\ \frac{1}{7}; & z = \frac{1}{2}, \end{cases}$$

and β is lower semicontinuous. Moreover, for each $z_1 \in [-1, 1/2) \cup (1/2, 1]$, we have $\Gamma(z_1) = \{(1/2)z_1^2\}$. Take $z_2 = (1/2)z_1^2$, then we have

$$q(z_1, z_2) = q(z_1, \frac{1}{2}z_1^2) = \frac{1}{2}z_1^2 \leq [2 - \chi(q(z_1, z_2))]\frac{1}{2}z_1^2 = [2 - \chi(q(z_1, z_2))]q(z_1, \Gamma(z_1)),$$

also,

$$q(z_1, \Gamma(z_1)) = q\left(\frac{1}{2}z_1^2, \frac{1}{2}\left(\frac{1}{2}z^2\right)^2\right) = \left(\frac{1}{4}z_1^2\right)q(z_1, z_2) < \frac{3}{4}\left(\frac{1}{2}z_1^2\right)q(z_1, z_2) = \chi\left(q(z_1, z_2)\right)q(z_1, z_2).$$

Thus, for all $z_1 \in [-1, 1]$, $z_1 \neq 1/2$, Γ satisfies all the conditions of Theorem 2.2. Now, let $z_1 = 1/2$, then we have $\Gamma(z_1) = \{1/7, 1/4\}$, and

$$q(z_1, \Gamma(z_1)) = q\left(\frac{1}{2}, \left\{\frac{1}{7}, \frac{1}{4}\right\}\right) = \frac{1}{7}.$$

Note that for $z_1 = 1/2$ there is $z_2 = 1/7 \in \Gamma(z_1)$ such that

$$q(z_1, z_2) = \frac{1}{7} < \left[2 - \frac{3}{4}\left(\frac{1}{7}\right)\right]\left(\frac{1}{7}\right) = [2 - \chi(q(z_1, z_2))]q(z_1, \Gamma(z_1)),$$

$$q(z_2, \Gamma(z_2)) = q\left(\frac{1}{7}, \frac{1}{2}\left(\frac{1}{7}\right)^2\right) = \frac{1}{2}\left(\frac{1}{7}\right)^2 < \frac{3}{4}\left(\frac{1}{7}\right)\left(\frac{1}{7}\right) = \chi(q(z_1, z_2))q(z_1, z_2).$$

Thus, for $z_1 = 1/2$ all the conditions of Theorem 2.2 are satisfied and hence $Fix(\Gamma) \neq \emptyset$. Note that $Fix(\Gamma) = \{0\}$. Clearly, Γ fails to satisfy the conditions of [8, Theorem 5] and [22, Theorem 2.1] because (Z, D) is not a metric space.

Further, our result Theorem 2.6 is also a genuine generalization of [19, Theorem 2.2], and [22, Theorem 2.5] as shows under.

Example 2. Let $Z = \mathbb{R}^+$. Define a quasi-metric on Z by

$$D(z_1, z_2) = \begin{cases} 0; & \text{if } z_1 = z_2, \\ z_1; & \text{otherwise.} \end{cases}$$

Define a Q-function on Z by

$$q(z_1, z_2) = z_1 + z_2, \quad \text{for all } z_1, z_2 \in Z.$$

Now, for any real number $a > 1$, define $\Gamma : Z \to Cl(Z)$ by

$$\Gamma(z) = \{\frac{z}{a}\} \cup [(1 + 2z), \infty), \quad \text{for all } z \in Z.$$

Define $\chi : \mathbb{R}^+ \to [0, 1)$ by

$$\chi(t) = \frac{1}{a}, \quad \text{for all } t \in \mathbb{R}^+.$$

Clearly, $\chi(t) < 1$ for all $t \in \mathbb{R}^+$. For any $z \in Z$ we get

$$\beta(z) = q(z, \Gamma(z)) = z + \frac{z}{a} = \left(\frac{a+1}{a}\right) z.$$

Thus, β is continuous. Now for each $z_1 \in Z$, there exists $z_2 = (z_1/a) \in \Gamma(z_1)$ satisfying

$$q(z_1, z_2) = q(z_1, \frac{z_1}{a}) = q(z_1, \Gamma(z_1)),$$

$$q(z_2, \Gamma(z_2)) = \frac{z_1}{a} + \frac{z_1}{a^2} = \frac{1}{a}\left(\frac{a+1}{a}\right) z_1 = \chi(q(z_1, z_2))q(z_1, z_2).$$

Clearly, all the conditions of Theorem 2.6 are true and $Fix(\Gamma) = \{0\}$. Note that $\Gamma(z)$ is not compact for all $z \in Z$ and the Q-function q is not a w-distance on Z, so Γ fails to satisfy assumptions of [19, Theorem 2.2] and [22, Theorem 2.5].

Conclusion. Among others, Feng and Liu [14], Klim and Wardowski [19], and *Ćirić* [8] studied the existence of fixed points for multivalued contractive mappings without using the HausdorffPompeiu metric, and consequently, they generalized some classically known fixed point results, including Theorem 1.1. In this paper, we established some general fixed point results for multivalued generalized contractive mappings on quasi-metric spaces with respect to the Q-function. Our results generalize and improve a number of known fixed point results, including the corresponding fixed point results which are stated in Section 2. In support of our main fixed point theorems, examples are also provided.

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Abdul Latif
Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia
E-mail address: alatif@kau.edu.sa

Nadiah Zafer Alshehri
Department of Mathematics, College of Science, King Khalid University, P.O. Box 960, Abha 61421, Saudi Arabia
E-mail address: nabdullahalshehri0002@stu.kau.edu.sa, nadia@kku.edu.sa

Monairah Omar Alansari
Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia
E-mail address: malansari@kau.edu.sa