Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 14 Issue 1(2022), Pages 28-30.
https://doi.org/10.54671/bmaa-2022-1-3

# STABILITY OF SOLUTIONS TO NONLINEAR EVOLUTION PROBLEMS

ALEXANDER G. RAMM

Abstract. Let $u' = F(u,t)$, $u(0) = u_0$, (1), $u \in H$, $H$ is a Hilbert space, $F(u,t)$ is a nonlinear operator in $H$. If $F(u,t) = A(t)u + B(u,t)$, where $A(t)$ is a linear operator, $B(u,t) = O(\|u\|^2)$ for $\|u\| \to 0$, then problem (1) has a solution $u = 0$ if $u_0 = 0$. If $\|u_0\|$ is small then the stability problem is: will the solution to (1) exist for all $t > 0$ and be small for all $t > 0$? A.M. Lyapunov gave in 1892 sufficient conditions for this to happen. In our paper a new technical tool is given for answering the above question. This tool (a nonlinear inequality) allows one to give old and new results on Lyapunov stability. One of such results is proved in this paper.

## 1. Introduction

Let

$$u' = F(u,t), \quad u(0) = u_0, \quad u' := \frac{du}{dt}, \tag{1.1}$$

where $u \in H$, $H$ is a Hilbert space, and $F(u,t)$ is a nonlinear operator in $H$. The first important question is the existence of a solution to (1.1) for all $t > 0$. The second one is the stability of the solution to (1): if a solution to (1) exists for all $t > 0$, will a solution to (1) with an initial data $u_1$ close to $u_0$ exist for all $t > 0$ and will it be close to the solution $u(t)$ of (1) for all $t > 0$? Let $\mathbb{R}_+ := [0, \infty)$.

Let us assume that

$$F(u,t) = a(t) + A(t)u + B(u,t), \tag{1.2}$$

where

$$\|a(t)\| \le b(t), \quad (A(t)u, u) \le \gamma(t)\|u\|^2, \quad (B(u,t), u) \le \alpha(t, \|u\|)\|u\|. \tag{1.3}$$

*We assume that $\gamma(t), b(t), \alpha(t,g)$ are continuous with respect to $t$. The $\alpha(t,g) \ge 0$ is Lipschitz-continuous and non-decreasing with respect to $g \ge 0$.*

To give sufficient conditions for a solution to problem (1.1) to exist for all $t \ge 0$, we formulate the following result proved in [2]– [4].

**Theorem 1.** *Let $g(t) \in C^1(\mathbb{R}_+)$, $g \ge 0$,*

$$g' \le -\gamma(t)g + \alpha(t,g) + b(t), \quad t \in \mathbb{R}_+. \tag{1.4}$$

2000 *Mathematics Subject Classification.* 34D20; 34G20; 47J35.

*Key words and phrases.* nonlinear evolution problems; stability.

©2022 Universiteti i Prishtinës, Prishtinë, Kosovë.

Submitted June 2,2021. Published March 8,2022.
Communicated by Naim L. Braha.

*If* $0 \le \mu(t) \in C^1(\mathbb{R}_+$ *and*

$$b(t) + \alpha(t, \frac{1}{\mu(t)}) \le \frac{1}{\mu}\left[\gamma(t) - \frac{\mu'(t)}{\mu(t)}\right], \tag{1.5}$$

$$g(0) < \frac{1}{\mu(0)}, \tag{1.6}$$

*then any solution to* (1.4) *exists on* $\mathbb{R}_+$ *and*

$$0 \le g(t) < \frac{1}{\mu(t)}. \tag{1.7}$$

Using Theorem 1 we show how to answer the two basic questions formulated in the Introduction.

## 2. A new result on stability

We keep assumptions on $F(u,t)$ made in the Introduction. Let us denote $g(t) := \|u(t)\|$, and derive for $g$ inequality (1.4). Multiply (1.1) by $u$ in $H$ and get

$$(u', u) \le b(t)g + \gamma(t)g^2 + \alpha(t, g)g. \tag{2.1}$$

Since $(u,u)' = 2g'g$, it follows from (2.1) that inequality (1.4) holds with $-\gamma$ replaced by $\gamma$. Thus in (1.5) $\gamma$ should be replaced by $-\gamma$. Let us choose

$$\mu = c_0 + c_1(1+t)^{-a}, \quad \frac{\mu'}{\mu} = -\frac{ac_1}{(1+t)^{a+1}[c_0 + c_1(1+t)^{-a}]}, \tag{2.2}$$

where $c_0, c_1, a$, are positive constants, and check conditions (1.5) and (1.6). Condition (1.6) is satisfied if

$$\|u_0\| = g(0) < \frac{1}{c_0 + c_1}. \tag{2.3}$$

Condition (1.5) is satisfied if

$$b(t)+\alpha(t, \frac{1}{c_0 + c_1(1+t)^{-a}}) \le \frac{1}{c_0 + c_1(1+t)^{-a}}\left[-\gamma(t) + \frac{ac_1}{(1+t)^{a+1}[c_0 + c_1(1+t)^{-a}]}\right]. \tag{2.4}$$

Let $\gamma(t) \le 0.5\frac{ac_1}{(1+t)^{a+1}[c_0+c_1(1+t)^{-a}]}$, $b(t) \le 0.25\frac{ac_1}{(1+t)^{a+1}[c_0+c_1(1+t)^{-a}]^2}$, $\alpha(t,g) < \alpha(t)g^2$, with $\alpha(t) \le 0.25\frac{ac_1}{(1+t)^{a+1}}$. Then conditions (1.6) is satisfied if $c_0 + c_1$ is sufficiently large for any fixed $u_0$. Condition (2.4) is satisfied if inequalities for $\gamma(t), b(t)$ and $\alpha(t)$ are satisfied. Then Theorem 1 implies Theorem 2.

**Theorem 2.** *If inequalities* (2.3) *and* (2.4) *hold, then equation* (1.1) *has a solution defined for all* $t \ge 0$ *and*

$$\|u(t)\| < \frac{1}{c_0 + c_1(1+t)^{-a}}, \quad t > 0. \tag{2.5}$$

**Example 1.** Let $u_0 = 0$, and $b(t) = 0$. Then problem (1.1) has a solution $u(t) = 0$. If in (1.4) $\gamma(t) = \gamma > 0$ and $\alpha(t,g) \le \alpha(t)g^2$ for $g \le R$, $R = const > 0$, then one can take $\mu(t) = \mu_0 e^{qt}$, $0 < q < \gamma$. Inequality (1.6) is satisfied if $\|u_0\| < 1/\mu_0$. Inequality (1.5) is satisfied if $\alpha(t) \le \mu(t)(\gamma - q)$. This inequality holds if $\alpha(t) < e^{qt}$ and $\mu_0 \le \gamma - q$. This gives the classical result of Lyapunov type: under our assumptions for any $u_0$ the solution to (1.1) exists for all $t > 0$ and $\|u(t)\| < ce^{-qt}$, where $c > 0$ is a constant and $q < \gamma$ is a constant, cf [1].

One can always reduce (1.1) to the case when $u_0 = 0$ by introducing a new function $v(t) = u(t) - u_0$, [1], Ch. 7. Therefore, our assumption $u_0 = 0$ and the corresponding solution $u(t) = 0$ are not restrictive.

**Example 2.** In this example a new result is given. Suppose that $\gamma(t) \leq 0$ in (1.4). If $\gamma(t) = -\gamma$, $\gamma = const > 0$, and $\alpha(t, g) \leq \alpha(t)g^2$, then the Lyapunov's result says that the zero solution to problem (1.1) with $u_0 = 0$ is unstable: if $\|u_0\| > 0$, then the solution will not stay bounded as $t \to \infty$. Our result is stated in Theorem 3.

**Theorem 3.** *Assume that* $0 < -\gamma(t) \to 0$ *as* $t \to \infty$*, and assumptions of Theorem 2 hold. Then the solution to* (1.1) *exists for all* $t > 0$ *and estimate* (2.5) *holds.*

## References

[1] Yu.Daleckii, M.Krein, **Stability of solutions to differential equations in Banach spaces**, Am.Math.Soc., Providence, RI, 1974.

[2] A.G.Ramm, **Dynamical systems method for solving operator equations**, Elsevier, Amsterdam, 2007.

[3] A.G.Ramm, A nonlinear inequality and evolution problems, Journ, Ineq. and Special Funct., (JIASF), 1, N1, (2010), 1-9.

[4] A.G.Ramm, N. Hoang, **Dynamical Systems Method and Applications. Theoretical Developments and Numerical Examples.** Wiley, Hoboken, 2012.

Department of Mathematics, Kansas State University,
Manhattan, KS 66506, USA.
http://www.math.ksu.edu/ ~ ramm
*E-mail address*: ramm@math.ksu.edu