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DEFERRED INVARIANT STATISTICAL CONVERGENT TRIPLE SEQUENCES VIA ORLICZ FUNCTION

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Abstract. In this paper, we shall study deferred invariant statistical convergence and strongly deferred invariant convergence with triple sequences via Orlicz function. We also introduce some inclusion relations.

1. Introduction

The idea of statistical convergence was given by Zygmund [26]. Later, this concept was introduced by Fast [7] and Steinhaus [22]. Under different names statistical convergence has been studied many different areas [5, 9, 10]. The deferred Cesàro mean $D_{p,q}$ of sequences of real numbers was introduced by Agnew [1] and it was investigated by Dağadur and Sezgek [4], Et and Yılmazer [6], Küçükaslan and Yılmaztürk [11], Temizsu [23] and many authors. Several authors including Schaefer [21], Mursaleen [13, 14], Savaş [19] and others have studied invariant convergent sequences [17]. Mursaleen defined strongly σ-convergence [15]. In [3], Bromwich firstly studied double sequences and many others was investigated this notion [8, 25]. Recently, the concept of statistical convergence for triple sequences was presented by Şahiner et al. [24]. Also, the readers should refer to the monographs [2] and [16] for the background on the sequence spaces and related topics.

In this paper, we introduce deferred invariant statistical $\widetilde{\phi}$-convergence, strongly deferred invariant $\widetilde{\phi}$-convergence and σ-statistical $\widetilde{\phi}$-convergence which are some combinations of the definitions for deferrred statistical convergence, strongly deferred invariant convergence, Orlicz function, σ-statistical convergence and triple sequences. In addition to these definitions, natural inclusion theorems will also be presented.

2. Definitions and notations

Before we enter the motivation for this paper and the presentation of the main results we give some preliminaries.

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The idea of the statistical convergence of sequence of real numbers is based on the notion of natural density of subsets of $\mathbb{N}$, the set of all positive integers which is defined as follows:

Definition 2.1. *A sequence $x = (x_k)$ is said to be statistically convergent to the number $\xi \in \mathbb{R}$ if for every $\varepsilon > 0$,*

$$\lim_{k\to\infty} \frac{1}{n} \left|\{k \le n : |x_k - \xi| \ge \varepsilon\}\right| = 0,$$

where the vertical bars indicate the number of elements in the enclosed set.

In several literary works, statistical convergence of any real sequence is identified relatively to absolute value. While we have known that the absolute value of real numbers is special of an Orlicz function [18], that is, a function $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ in such a way that it is even, non-decreasing on $\mathbb{R}^+$, continuous on $\mathbb{R}$, and satisfying

$$\widetilde{\phi}(x) = 0 \text{ if and only if } x = 0 \text{ and } \widetilde{\phi}(x) \to \infty \text{ as } x \to \infty.$$

Further, an Orlicz function $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ is said to satisfy the $\triangle_2$ condition, if there exists an positive real number M such that $\widetilde{\phi}(2x) \le M.\widetilde{\phi}(x)$ for every $x \in \mathbb{R}^+$.

Deferred Cesàro mean as a generalization of real (or complex) valued sequence $x = (x_k)$ by

$$(D_{p,q}x)_n := \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} x_k, \ n = 1, 2, 3, ...$$

where $\{p(n)\}$ and $\{q(n)\}$ are sequences of nonnegative integers satisfying the conditions

$$p(n) < q(n) \ \text{ and } \ \lim_{n\to\infty} q(n) = \infty.$$

Deferred density of $K \subset \mathbb{N}$ defined by

$$\delta_p^q(K) = \lim_{n\to\infty} \frac{|\{k : p(n) < k \le q(n), \ k \in K\}|}{q(n) - p(n)}.$$

Definition 2.2. *([11]) A real valued sequence $x = (x_k)$ is said to be deferred statistically convergent to ξ provided that*

$$\lim_{n\to\infty} \frac{|\{p(n) < k \le q(n), \ |x_k - \xi| \ge \varepsilon\}|}{q(n) - p(n)} = 0$$

for each $\varepsilon > 0$ and it is written by $x_k \to \xi\,(DS_{p,q})$.

Remark. *If $p(n) = 0$ and $q(n) = n$, then the above definition is coincide with the definition of statistical convergence.*

Also, Dağadur and Sezgek [4] introduced deferred statistical convergence of double sequences. Let $x = (x_{nk})$ be a double sequence and $\beta(t) = b(t) - a(t)$, $\gamma(u) = d(u) - c(u)$. Then the double sequence x is said to be deferred statistically convergent to ξ if for every $\varepsilon > 0$,

$$\lim_{t,u\to\infty} \frac{|\{(n,k) : a(t) < n \le b(t), \ c(u) < k \le d(u); \ |x_{nk} - \xi| \ge \varepsilon\}|}{\beta(t)\gamma(u)} = 0.$$

Recently, the concept of statistical convergence for triple sequences was presented by Şahiner et al. [24], as follows: A function $x : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ (or $\mathbb{C}$) is called a real (complex) triple sequence. A triple sequence (x_{nkl}) is said to be convergent to L in Pringsheim's sense if for every $\varepsilon > 0$, there exists $n_0(\varepsilon) \in \mathbb{N}$ such that $|x_{nkl} - L| < \varepsilon$ whenever $n, k, l \geq n_0$.

A subset K of $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ is said to have natural density $\delta_3(K)$ if

$$\delta_3(K) = \lim_{p,q,r \to \infty} \frac{|K(p,q,r)|}{pqr} \text{ exists,}$$

where the vertical bars denotes the number of (n,k,l)'s in K such that $n \leq p$, $k \leq q$ and $l \leq r$.

Definition 2.3. ([24]) *A real triple sequence (x_{nkl}) is said to be statistically convergent to the number ξ if for each $\varepsilon > 0$,*

$$\delta_3(\{(n,k,l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : |x_{nkl} - \xi| \geq \varepsilon\}) = 0.$$

A continuous linear functional φ on ℓ_∞, the space of real bounded sequences, is said to be a Banach limit if

(i) $\varphi(x) \geq 0$, for the sequence $x = (x_n)$ with $x_n \geq 0$ for all $n \in \mathbb{N}$,
(ii) $\varphi(e) = 1$, where $e = (1,1,1,...)$ and
(iii) $\varphi(x_{\sigma(n)}) = \varphi(x_n)$ for all $x \in \ell_\infty$.

The mapping σ is assumed to be one-to-one and such that $\sigma^m(n) \neq n$ for all $n, m \in \mathbb{Z}^+$, where $\sigma^m(n)$ denotes the m th iterate of the mapping σ at n. Thus, φ extends the limit functional on c, the space of convergent sequences, in the sense that $\varphi(x_n) = \lim x_n$ for all $x \in c$. In the case σ is translation mappings $\sigma(n) = n + 1$, the σ-mean is often called a Banach limit. The space V_σ of the bounded sequences whose invariant means are equal may be defined, as follows;

$$V_\sigma = \left\{ x \in \ell_\infty : \lim_{m \to \infty} \frac{1}{m} \sum_{k=1}^{m} x_{\sigma^k(n)} = \xi, \text{ uniformly in } n \right\}.$$

In [21], Schaefer proved that a bounded sequence $x = (x_k)$ of real numbers is σ-convergent to ξ if and only if

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} x_{\sigma^k(m)} = \xi,$$

uniformly in m. A sequence $x \in \ell_\infty$ is said to be almost convergent to the value ξ if all of its Banach limits equal to ξ.

The concept of almost convergence of sequences of real numbers $x = (x_n)$ was firstly introduced by Lorentz [12]. A bounded sequence $x = (x_n)$ is almost convergent to ξ if and only if

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} x_{m+k} = \xi$$

uniformly in m.

Let X and Y be two sequence spaces and $A = (a_{nk})$ be an infinite matrix. If for each $x \in X$ the series $A_n(x) = \sum_{k=1}^{\infty} a_{nk} x_k$ converges for each n and the sequence $Ax = (A_n x) \in Y$, we say that A maps X into Y. By (X, Y) we denote the set of all matrices which maps X into Y. A matrix A is called regular if $A \in (c, c)$ and $\lim_{k \to \infty} A_k(x) = \lim_{k \to \infty} x_k$ for all $x = (x_k)_{k \in \mathbb{N}} \in c$, as usual, stands for the set of all convergent sequences. It is well known that the necessary and sufficient conditions, in order to be A regular, are

(a) $\|A\| = \sup_n \sum_k |a_{nk}| < \infty$;
(b) $\lim_n a_{nk} = 0$, for each k;

(c) $\lim_n \sum_k a_{nk} = 1$.

3. Main results

Following the above definitions and results, we aim in this section to introduce some new notions of deferred invariant statistical $\widetilde{\phi}$-convergence, strongly deferred invariant $\widetilde{\phi}$-convergence and σ-statistical $\widetilde{\phi}$-convergence for the triple sequences. We also introduce some inclusion relations.

Definition 3.1. *Let $x = (x_{nkl})$ be a triple sequence and $\beta(t) = b(t) - a(t)$, $\gamma(u) = d(u) - c(u)$, $\eta(v) = f(v) - e(v)$. Then deferred Cesàro mean $D_{\beta\gamma\eta}$ of the triple sequence x is defined by*

$$(D_{\beta\gamma\eta}x)_{nkl} = \frac{1}{\beta(t)\gamma(u)\eta(v)} \sum_{n=a(t)+1}^{b(t)} \sum_{k=c(u)+1}^{d(u)} \sum_{l=e(v)+1}^{f(v)} x_{nkl},$$

where $\{a(t)\}$, $\{b(t)\}$, $\{c(u)\}$, $\{d(u)\}$, $\{e(v)\}$, $\{f(v)\}$ are sequences of nonnegative integers satisfying the conditions

$$a(t) < b(t), \ c(u) < d(u), \ e(v) < f(v)$$

and

$$\lim_{t\to\infty} b(t) = \infty, \quad \lim_{u\to\infty} d(u) = \infty, \quad \lim_{v\to\infty} f(v) = \infty.$$

Definition 3.2. *Let $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ be an Orlicz function. Then, the sequence $x = (x_{nkl})$ is said to be deferred statistically $\widetilde{\phi}$-convergent to the number $\xi \in \mathbb{R}$ if for every $\varepsilon > 0$,*

$$\lim_{t,u,v\to\infty} \frac{1}{\beta(t)\gamma(u)\eta(v)} \left|\left\{a(t) < n \leq b(t), \ c(u) < k \leq d(u), \ e(v) < l \leq f(v) : \widetilde{\phi}(x_{nkl} - \xi) \geq \varepsilon\right\}\right|.$$

Definition 3.3. *Let $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ be an Orlicz function. Then, the sequence $x = (x_{nkl})$ is said to be strongly $(D_{\beta\gamma\eta})^{\widetilde{\phi}}$-convergent to $\xi \in \mathbb{R}$ if*

$$\lim_{t,u,v\to\infty} \frac{1}{\beta(t)\gamma(u)\eta(v)} \sum_{n=a(t)+1}^{b(t)} \sum_{k=c(u)+1}^{d(u)} \sum_{l=e(v)+1}^{f(v)} \widetilde{\phi}(x_{nkl} - \xi) = 0.$$

Definition 3.4. *Let $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ be an Orlicz function. Then, the sequence $x = (x_{tuv})$ of real numbers is σ-statistically $\widetilde{\phi}$-convergent to ξ if for every $\varepsilon > 0$,*

$$\lim_{t,u,v\to\infty} \frac{1}{tuv} \left|\left\{n \leq t, \ k \leq u, \ l \leq v : \widetilde{\phi}\left(x_{\sigma^n(p),\sigma^k(q),\sigma^l(r)} - \xi\right) \geq \varepsilon\right\}\right| = 0$$

uniformly in p, q and r, and is written as $x_{tuv} \to \xi\, (S_\sigma)^{\widetilde{\phi}}$.

Definition 3.5. *A sequence $x = (x_{tuv})$ is called to be deferred invariant convergent to $\xi \in \mathbb{R}$ if*

$$\lim_{t,u,v} \frac{1}{\beta(t)\gamma(u)\eta(v)} \sum_{n=a(t)+1}^{b(t)} \sum_{k=c(u)+1}^{d(u)} \sum_{l=e(v)+1}^{f(v)} x_{\sigma^n(p),\sigma^k(q),\sigma^l(r)} = \xi$$

uniformly in p, q, r.

Definition 3.6. *Let $\widetilde{\phi}:\mathbb{R}\rightarrow\mathbb{R}$ be an Orlicz function. Then, the sequence $x=(x_{tuv})$ is called to be strongly deferred invariant $\widetilde{\phi}$-convergent to $\xi\in\mathbb{R}$ if*

$$\lim_{t,u,v}\frac{1}{\beta(t)\gamma(u)\eta(v)}\sum_{n=a(t)+1}^{b(t)}\sum_{k=c(u)+1}^{d(u)}\sum_{l=e(v)+1}^{f(v)}\widetilde{\phi}\left(x_{\sigma^{n}(p),}x_{\sigma^{k}(q),}x_{\sigma^{l}(r)}-\xi\right)=0$$

uniformly in p,q and r, and is written as $x_{tuv}\rightarrow\xi\left(D_{\beta\gamma\eta}\right)_{\sigma}^{\widetilde{\phi}}$.

Remark. *If we take $b(t)=t$, $d(u)=u$, $f(v)=v$, $a(t)=0$, $c(u)=0$, $e(v)=0$, $\widetilde{\phi}=|x|$ and a real valued sequence $x=(x_t)$ instead of $x=(x_{tuv})$, then above definition of strongly deferred invariant $\widetilde{\phi}$-convergence coincides with strongly σ-convergence in* [15].

Definition 3.7. *Let $\widetilde{\phi}:\mathbb{R}\rightarrow\mathbb{R}$ be an Orlicz function and $\widetilde{\phi}_s=|x|^s$. Then, the sequence $x=(x_{tuv})$ is called to be strongly s-deferred invariant $\widetilde{\phi}$-convergent to $\xi\in\mathbb{R}$ if*

$$\lim_{t,u,v}\frac{1}{\beta(t)\gamma(u)\eta(v)}\sum_{n=a(t)+1}^{b(t)}\sum_{k=c(u)+1}^{d(u)}\sum_{l=e(v)+1}^{f(v)}\widetilde{\phi}_s\left(x_{\sigma^{n}(p),}x_{\sigma^{k}(q),}x_{\sigma^{l}(r)}-\xi\right)=0$$

uniformly in p, q, r and is shown as $x_{tuv}\rightarrow\xi\left(D_{\beta\gamma\eta}\right)_{\sigma}^{\widetilde{\phi},s}$, where $0<s<\infty$.

Remark. *If we take $b(t)=t$, $d(u)=u$, $f(v)=v$, $a(t)=0$, $c(u)=0$, $e(v)=0$, $\widetilde{\phi}_s=|x|^s$ and a real valued sequence $x=(x_t)$ instead of $x=(x_{tuv})$, then above definition of strongly s-deferred invariant $\widetilde{\phi}$-convergence coincides with the definition of strongly s-invariant convergence in* [19].

Definition 3.8. *Let $\widetilde{\phi}:\mathbb{R}\rightarrow\mathbb{R}$ be an Orlicz function. Then, the sequence $x=(x_{tuv})$ is called to be deferred invariant statistically $\widetilde{\phi}$-convergent to $\xi\in\mathbb{R}$ if for every $\varepsilon>0$,*

$$\lim_{t,u,v}\tfrac{1}{\beta(t)\gamma(u)\eta(v)}\cdot\quad\Big|\Big\{a(t)<n\leq b(t),c(u)<k\leq d(u),e(v)<l\leq f(v) \\ :\widetilde{\phi}\left(x_{\sigma^{n}(p),}x_{\sigma^{k}(q),}x_{\sigma^{l}(r)}-\xi\right)\geq\varepsilon\Big\}\Big|=0$$

uniformly in p,q and r, and is written as $x_{tuv}\rightarrow\xi\left(DS_{\beta\gamma\eta}\right)_{\sigma}^{\widetilde{\phi}}$.

Remark. *If we take $b(t)=t$, $d(u)=u$, $f(v)=v$, $a(t)=0$, $c(u)=0$, $e(v)=0$, $\widetilde{\phi}=|x|$ and a real valued sequence $x=(x_t)$ instead of $x=(x_{tuv})$, then above definition of deferred invariant statistically $\widetilde{\phi}$-convergence coincides with the definition of invariant statistically convergence in* [20].

We use $\beta'(t)=b'(t)-a'(t)$, $\gamma'(u)=d'(u)-c'(u)$, $\eta'(v)=f'(v)-e'(v)$ where $\{a'(t)\}$, $\{b'(t)\}$, $\{c'(u)\}$, $\{d'(u)\}$, $\{e'(v)\}$, $\{f'(v)\}$ are sequences of nonnegative integers satisfying the conditions

$$a'(t)<b'(t),\ c'(u)<d'(u),\ e'(v)<f'(v)$$

and

$$\lim_{t\rightarrow\infty}b'(t)=\infty,\quad\lim_{u\rightarrow\infty}d'(u)=\infty,\quad\lim_{v\rightarrow\infty}f'(v)=\infty$$

in the next theorem.

Theorem 3.9. *Let $\widetilde{\phi}:\mathbb{R}\to\mathbb{R}$ be an Orlicz function and $\{a(t)\}, \{b(t)\}, \{c(u)\}, \{d(u)\}, \{e(v)\}, \{f(v)\}, \{a'(t)\}, \{b'(t)\}, \{c'(u)\}, \{d'(u)\}, \{e'(v)\}, \{f'(v)\}$ be sequences of non-negative integers satisfying $a(t)\le a'(t)<b'(t)\le b(t)$, $c(u)\le c'(u)<d'(u)\le d(u)$, $e(v)\le e'(v)<f'(v)\le f(v)$ for all $t,u,v\in\mathbb{N}$ and*

$$\limsup_{t,u,v\to\infty}\frac{\beta(t)\gamma(u)\eta(v)}{\beta'(t)\gamma'(u)\eta'(v)}<\infty. \tag{3.1}$$

Then, $x_{tuv}\to\xi\left(DS_{\beta\gamma\eta}\right)_{\sigma}^{\widetilde{\phi}}$ implies $x_{tuv}\to\xi\left(DS_{\beta'\gamma'\Gamma'}\right)_{\sigma}^{\widetilde{\phi}}$.

Proof. From the inclusion

$$\begin{aligned}&\Big\{(n,k,l): a'(t)<n\le b'(t), c'(u)<k\le d'(u), e'(v)<l\le f'(v)\\ &\qquad ;\ \widetilde{\phi}\left(x_{\sigma^n(p)},x_{\sigma^k(q)},x_{\sigma^l(r)}-\xi\right)\ge\varepsilon\Big\}\\ &\subset\Big\{(n,k,l): a(t)<n\le b(t), c(u)<k\le d(u), e(v)<l\le f(v)\\ &\qquad ;\ \widetilde{\phi}\left(x_{\sigma^n(p)},x_{\sigma^k(q)},x_{\sigma^l(r)}-\xi\right)\ge\varepsilon\Big\}\end{aligned}$$

now, we can write

$$\begin{aligned}&\lim_{t,u,v\to\infty}\tfrac{1}{\beta'(t)\gamma'(u)\eta'(v)}.\Big|\Big\{(n,k,l): a'(t)<n\le b'(t),\ c'(u)<k\le d'(u),\\ &\qquad e'(v)<l\le f'(v);\ \widetilde{\phi}\left(x_{\sigma^n(p)},x_{\sigma^k(q)},x_{\sigma^l(r)}-\xi\right)\ge\varepsilon\Big\}\Big|\\ &\le\lim_{t,u,v\to\infty}\frac{\beta(t)\gamma(u)\eta(v)}{\beta'(t)\gamma'(u)\eta'(v)}.\frac{1}{\beta(t)\gamma(u)\eta(v)}\Big|\Big\{(n,k,l): a(t)<n\le b(t),\\ &\quad c(u)<k\le d(u),\ e(v)<l\le f(v);\ \widetilde{\phi}\left(x_{\sigma^n(p)},x_{\sigma^k(q)},x_{\sigma^l(r)}-\xi\right)\ge\varepsilon\Big\}\Big|.\end{aligned} \tag{3.2}$$

From (3.1) and (3.2), we have $x_{tuv}\to\xi\left(DS_{\beta'\gamma'\Gamma'}\right)_{\sigma}^{\widetilde{\phi}}$. □

Theorem 3.10. *Let $\widetilde{\phi}:\mathbb{R}\to\mathbb{R}$ be an Orlicz function and $\{a(t)\}, \{b(t)\}, \{c(u)\}, \{d(u)\}, \{e(v)\}, \{f(v)\}, \{a'(t)\}, \{b'(t)\}, \{c'(u)\}, \{d'(u)\}, \{e'(v)\}, \{f'(v)\}$ be sequences of non-negative integers satisfying $a(t)\le a'(t)<b'(t)\le b(t)$, $c(u)\le c'(u)<d'(u)\le d(u)$, $e(v)\le e'(v)<f'(v)\le f(v)$ for all $t,u,v\in\mathbb{N}$ and*

$$\limsup_{t,u,v\to\infty}\frac{\beta(t)\gamma(u)\eta(v)}{\beta'(t)\gamma'(u)\eta'(v)}<\infty.$$

Then, $x_{tuv}\to\xi\left(D_{\beta\gamma\eta}\right)_{\sigma}^{\widetilde{\phi}}$ implies $x_{tuv}\to\xi\left(D_{\beta'\gamma'\eta'}\right)_{\sigma}^{\widetilde{\phi}}$.

Proof. In the light of hypothesis, one can see that

$$\begin{aligned}&\frac{1}{\beta(t)\gamma(u)\eta(v)}\sum_{n=a(t)+1}^{b(t)}\sum_{n=c(u)+1}^{d(u)}\sum_{l=e(v)+1}^{f(v)}\widetilde{\phi}\left(x_{\sigma^n(p)},x_{\sigma^k(q)},x_{\sigma^l(r)}-\xi\right)\\ &\ge\frac{1}{\beta(t)\gamma(u)\eta(v)}\sum_{n=a'(t)+1}^{b'(t)}\sum_{n=c'(u)+1}^{d'(u)}\sum_{l=e'(v)+1}^{f'(v)}\widetilde{\phi}\left(x_{\sigma^n(p)},x_{\sigma^k(q)},x_{\sigma^l(r)}-\xi\right)\\ &\ge\frac{\beta'(t)\gamma'(u)\eta'(v)}{\beta(t)\gamma(u)\eta(v)}\frac{1}{\beta'(t)\gamma'(u)\eta'(v)}.\\ &\sum_{n=a'(t)+1}^{b'(t)}\sum_{n=c'(u)+1}^{d'(u)}\sum_{l=e'(v)+1}^{f'(v)}\widetilde{\phi}\left(x_{\sigma^n(p)},x_{\sigma^k(q)},x_{\sigma^l(r)}-\xi\right).\end{aligned}$$

From the definition, taking limits as $t, u, v \to \infty$ the desired result is obtained. $\square$

Definition 3.11. *Let $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ be an Orlicz function. A triple sequence $\{x_{nkl}\}$ is said to be $\widetilde{\phi}$-bounded if there exists $M > 0$ such that $\widetilde{\phi}(x_{nkl}) \leq M$ for all $n, k, l \in \mathbb{N}$. We denote the space of all bounded triple sequences by $\ell^3_\infty\left(\widetilde{\phi}\right)$.*

Theorem 3.12. *Let $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ be an Orlicz function. Then, the following statements hold:*

(i) If $x_{tuv} \to \xi \left(D_{\beta\gamma\eta}\right)^{\widetilde{\phi}}_{\sigma}$, then $x_{tuv} \to \xi \left(DS_{\beta\gamma\eta}\right)^{\widetilde{\phi}}_{\sigma}$.

(ii) If $x = (x_{tuv}) \in \ell^3_\infty\left(\widetilde{\phi}\right)$ and $x_{tuv} \to \xi \left(DS_{\beta\gamma\eta}\right)^{\widetilde{\phi}}_{\sigma}$, then $x_{tuv} \to \xi \left(D_{\beta\gamma\eta}\right)^{\widetilde{\phi}}_{\sigma}$.

Proof. (i) Let $x_{tuv} \to \xi \left(D_{\beta\gamma\eta}\right)^{\widetilde{\phi}}_{\sigma}$. For an arbitrary $\varepsilon > 0$, we get

$$\frac{1}{\beta(t)\gamma(u)\eta(v)} \sum_{n=a(t)+1}^{b(t)} \sum_{k=c(u)+1}^{d(u)} \sum_{l=e(v)+1}^{f(v)} \widetilde{\phi}\left(x_{\sigma^n(p)}, x_{\sigma^k(q)}, x_{\sigma^l(r)} - \xi\right)$$

$$= \frac{1}{\beta(t)\gamma(u)\eta(v)} \sum_{n=a(t)+1}^{b(t)} \sum_{k=c(u)+1}^{d(u)} \sum_{\substack{l=e(v)+1 \\ \widetilde{\phi}\left(x_{\sigma^n(p)}, x_{\sigma^k(q)}, x_{\sigma^l(r)} - \xi\right) \geq \varepsilon}}^{f(v)} \widetilde{\phi}\left(x_{\sigma^n(p)}, x_{\sigma^k(q)}, x_{\sigma^l(r)} - \xi\right)$$

$$+ \frac{1}{\beta(t)\gamma(u)\eta(v)} \sum_{n=a(t)+1}^{b(t)} \sum_{k=c(u)+1}^{d(u)} \sum_{\substack{l=e(v)+1 \\ \widetilde{\phi}\left(x_{\sigma^n(p)}, x_{\sigma^k(q)}, x_{\sigma^l(r)} - \xi\right) < \varepsilon}}^{f(v)} \widetilde{\phi}\left(x_{\sigma^n(p)}, x_{\sigma^k(q)}, x_{\sigma^l(r)} - \xi\right)$$

$$\geq \frac{1}{\beta(t)\gamma(u)\eta(v)} \sum_{n=a(t)+1}^{b(t)} \sum_{k=c(u)+1}^{d(u)} \sum_{\substack{l=e(v)+1 \\ \widetilde{\phi}\left(x_{\sigma^n(p)}, x_{\sigma^k(q)}, x_{\sigma^l(r)} - \xi\right) \geq \varepsilon}}^{f(v)} \widetilde{\phi}\left(x_{\sigma^n(p)}, x_{\sigma^k(q)}, x_{\sigma^l(r)} - \xi\right)$$

$$\geq \frac{\varepsilon}{\beta(t)\gamma(u)\eta(v)}. \quad \left|\{a(t) < n \leq b(t),\ c(u) < k \leq d(u),\ e(v) < l \leq f(v) : \widetilde{\phi}\left(x_{\sigma^n(p)}, x_{\sigma^k(q)}, x_{\sigma^l(r)} - \xi\right) \geq \varepsilon\}\right|$$

for each p, q, r. Hence, we have

$$\lim_{t,u,v\to\infty} \frac{1}{\beta(t)\gamma(u)\eta(v)}. \quad \left|\{a(t) < n \leq b(t),\ c(u) < k \leq d(u),\ e(v) < l \leq f(v) : \widetilde{\phi}\left(x_{\sigma^n(p)}, x_{\sigma^k(q)}, x_{\sigma^l(r)} - \xi\right) \geq \varepsilon\}\right| = 0.$$

uniformly in p, q, r, that is $x_{tuv} \to \xi \left(DS_{\beta\gamma\eta}\right)^{\widetilde{\phi}}_{\sigma}$.

(ii) Suppose that $(x_{tuv}) \to \xi \left(DS_{\beta\gamma\eta}\right)^{\widetilde{\phi}}_{\sigma}$ and $x = (x_{tuv})$ is bounded, say

$$\widetilde{\phi}\left(x_{\sigma^n(p)}, x_{\sigma^k(q)}, x_{\sigma^l(r)} - \xi\right) \leq M$$

for all n, k, l and p, q, r. Given $\varepsilon > 0$, we get

$$\frac{1}{\beta(t)\gamma(u)\eta(v)} \sum_{n=a(t)+1}^{b(t)} \sum_{k=c(u)+1}^{d(u)} \sum_{l=e(v)+1}^{f(v)} \widetilde{\phi}\left(x_{\sigma^n(p)}, x_{\sigma^k(q)}, x_{\sigma^l(r)} - \xi\right)$$

$$= \frac{1}{\beta(t)\gamma(u)\eta(v)} \sum_{\substack{n=a(t)+1}}^{b(t)} \sum_{k=c(u)+1}^{d(u)} \sum_{\substack{l=e(v)+1 \\ \widetilde{\phi}\left(x_{\sigma^n(p)},x_{\sigma^k(q)},x_{\sigma^l(r)}-\xi\right)\geq\varepsilon}}^{f(v)} \widetilde{\phi}\left(x_{\sigma^n(p)},x_{\sigma^k(q)},x_{\sigma^l(r)}-\xi\right)$$

$$+ \frac{1}{\beta(t)\gamma(u)\eta(v)} \sum_{\substack{n=a(t)+1}}^{b(t)} \sum_{k=c(u)+1}^{d(u)} \sum_{\substack{l=e(v)+1 \\ \widetilde{\phi}\left(x_{\sigma^n(p)},x_{\sigma^k(q)},x_{\sigma^l(r)}-\xi\right)<\varepsilon}}^{f(v)} \widetilde{\phi}\left(x_{\sigma^n(p)},x_{\sigma^k(q)},x_{\sigma^l(r)}-\xi\right)$$

$$\leq \frac{M}{\beta(t)\gamma(u)\eta(v)} \sum_{\substack{n=a(t)+1 \\ \widetilde{\phi}\left(x_{\sigma^n(p)},x_{\sigma^k(q)},x_{\sigma^l(r)}-\xi\right)\geq\varepsilon}}^{b(t)} \sum_{k=c(u)+1}^{d(u)} \sum_{l=e(v)+1}^{f(v)} 1$$

$$+ \frac{\varepsilon}{\beta(t)\gamma(u)\eta(v)} \sum_{\substack{n=a(t)+1 \\ \widetilde{\phi}\left(x_{\sigma^n(p)},x_{\sigma^k(q)},x_{\sigma^l(r)}-\xi\right)<\varepsilon}}^{b(t)} \sum_{k=c(u)+1}^{d(u)} \sum_{l=e(v)+1}^{f(v)} 1$$

$$\leq \frac{M}{\beta(t)\gamma(u)\eta(v)} \left|\left\{a(t)<n\leq b(t),\ c(u)<k\leq d(u),\ e(v)<l\leq f(v) \right.\right.$$
$$\left.\left. : \widetilde{\phi}\left(x_{\sigma^n(p)},x_{\sigma^k(q)},x_{\sigma^l(r)}-\xi\right)\geq\varepsilon\right\}\right|$$
$$+ \frac{\varepsilon}{\beta(t)\gamma(u)\eta(v)} \left|\left\{a(t)<n\leq b(t),\ c(u)<k\leq d(u),\ e(v)<l\leq f(v) \right.\right.$$
$$\left.\left. \widetilde{\phi}\left(x_{\sigma^n(p)},x_{\sigma^k(q)},x_{\sigma^l(r)}-\xi\right)<\varepsilon\right\}\right|$$

for each p,q,r, hence we have

$$\lim_{t,u,v\to\infty} \frac{1}{\beta(t)\gamma(u)\eta(v)} \sum_{n=a(t)+1}^{b(t)} \sum_{k=c(u)+1}^{d(u)} \sum_{l=e(v)+1}^{f(v)} \widetilde{\phi}\left(x_{\sigma^n(p)},x_{\sigma^k(q)},x_{\sigma^l(r)}-\xi\right)=0$$

uniformly in p,q,r, and the proof is completed. □

Theorem 3.13. *Let $\widetilde{\phi}:\mathbb{R}\to\mathbb{R}$ be an Orlicz function. If the sequence $\left\{\frac{a(t)}{\beta(t)}\frac{c(u)}{\gamma(u)}\frac{e(v)}{\eta(v)}\right\}$ is bounded, then $(x_{tuv})\to\xi\,(S)_{\sigma}^{\widetilde{\phi}}$ implies $(x_{tuv})\to\xi\,(DS_{\beta\gamma\eta})_{\sigma}^{\widetilde{\phi}}$.*

Proof. Let $(x_{tuv})\to\xi\,(S)_{\sigma}^{\widetilde{\phi}}$ then for every $\varepsilon>0$,

$$\lim_{t,u,v\to\infty}\frac{1}{tuv}\left|\left\{(n,k,l)\leq(t,u,v):\widetilde{\phi}\left(x_{\sigma^n(p)},x_{\sigma^k(q)},x_{\sigma^l(r)}-\xi\right)\geq\varepsilon\right\}\right|=0$$

uniformly p,q,r. Hence, for every $\varepsilon>0$

$$\lim_{t,u,v\to\infty}\frac{1}{b(t)d(u)f(v)}\left|\left\{n\leq b(t),\ k\leq d(u),\ l\leq f(v) : \widetilde{\phi}\left(x_{\sigma^n(p)},x_{\sigma^k(q)},x_{\sigma^l(r)}-\xi\right)\geq\varepsilon\right\}\right|=0 \tag{3.3}$$

uniformly in p,q,r. From the inequality

$$\left|\left\{a(t)<n\leq b(t),\ c(u)<k\leq d(u),\ e(v)<l\leq f(v) : \widetilde{\phi}\left(x_{\sigma^n(p)},x_{\sigma^k(q)},x_{\sigma^l(r)}-\xi\right)\geq\varepsilon\right\}\right|$$
$$\leq\left|\left\{n\leq b(t),\ k\leq d(u),\ l\leq f(v):\widetilde{\phi}\left(x_{\sigma^n(p)},x_{\sigma^k(q)},x_{\sigma^l(r)}-\xi\right)\geq\varepsilon\right\}\right|,$$

we have

$$\frac{1}{\beta(t)\gamma(u)\eta(v)}\left|\left\{a(t)<n\leq b(t),\ c(u)<k\leq d(u),\ e(v)<l\leq f(v)\right.\right.$$
$$\left.\left.:\widetilde{\phi}\left(x_{\sigma^n(p)},x_{\sigma^k(q)},x_{\sigma^l(r)}-\xi\right)\geq\varepsilon\right\}\right|$$
$$=\frac{b(t)-a(t)+a(t)}{b(t)-a(t)}\frac{d(u)-c(u)+c(u)}{d(u)-c(u)}\frac{f(v)-e(v)+e(v)}{f(v)-e(v)}\frac{1}{b(t)}\frac{1}{d(u)}\frac{1}{f(v)}$$
$$\times\left|\left\{a(t)<n\leq b(t),\ c(u)<k\leq d(u),\ e(t)<l\leq f(t)\right.\right.$$
$$\left.\left.:\widetilde{\phi}\left(x_{\sigma^n(p)},x_{\sigma^k(q)},x_{\sigma^l(r)}-\xi\right)\geq\varepsilon\right\}\right|$$
$$\leq\left(1+\frac{a(t)}{b(t)-a(t)}\right)\left(1+\frac{c(u)}{d(u)-c(u)}\right)\left(1+\frac{e(v)}{f(v)-e(v)}\right)\frac{1}{b(t)}\frac{1}{d(u)}\frac{1}{f(v)}$$
$$\times\left|\left\{n\leq b(t),\ k\leq d(u),\ l\leq f(v):\widetilde{\phi}\left(x_{\sigma^n(p)},x_{\sigma^k(q)},x_{\sigma^l(r)}-\xi\right)\geq\varepsilon\right\}\right|$$

for each p,q,r. By the (3.3), we obtain $(x_{tuv})\rightarrow\xi\,(DS_{\beta\gamma\eta})_{\sigma}^{\widetilde{\phi}}$. □

Theorem 3.14. *Let $\widetilde{\phi}=\mathbb{R}\rightarrow\mathbb{R}$ be an Orlicz function, $x=(x_{tuv})$ be a triple sequence and $b(t)=t,\ d(u)=u,\ f(v)=v$ for all $t,u,v\in\mathbb{N}$. Let $\{a(t)\},\ \{c(u)\},\ \{e(v)\}$ be arbitrary sequences. Then, $x_{tuv}\rightarrow\xi\,(DS_{\beta\gamma\eta})_{\sigma}^{\widetilde{\phi}}$ if and only if $x_{tuv}\rightarrow\xi\,(S)_{\sigma}^{\widetilde{\phi}}$.*

Proof. Let $b(t)=t,\ d(u)=u,\ f(v)=v$ for all $t,u,v\in\mathbb{N}$ and $\{a(t)\},\ \{c(u)\},\ \{e(v)\}$ be arbitrary three given sequences. We assume that $(DS_{\beta\gamma\eta})_{\sigma}^{\widetilde{\phi}}-\lim_{t,u,v\rightarrow\infty}x_{tuv}=\xi$. We shall apply the same technique given in [1]. We define the sequences as

$$a(t)=t^{(1)}>a\left(t^{(1)}\right)=t^{(2)}>a\left(t^{(2)}\right)=t^{(3)}>...$$
$$c(u)=u^{(1)}>c\left(u^{(1)}\right)=u^{(2)}>c\left(u^{(2)}\right)=u^{(3)}>...$$
$$e(v)=v^{(1)}>e\left(v^{(1)}\right)=v^{(2)}>e\left(v^{(2)}\right)=v^{(3)}>...$$

for all $t,u,v\in\mathbb{N}$.

The set $\left\{1<n\leq t,\ 1<k\leq u,\ 1<l\leq v:\ \widetilde{\phi}\left(x_{\sigma^n(p)},x_{\sigma^k(q)},x_{\sigma^l(r)}-\xi\right)\geq\varepsilon\right\}$ can be written as follows

$$\left\{n\leq t,k\leq u,l\leq v:\widetilde{\phi}\left(x_{\sigma^n(p)},x_{\sigma^k(q)},x_{\sigma^l(r)}-\xi\right)\geq\varepsilon\right\}$$
$$=\left\{n\leq t^{(1)},k\leq u^{(1)},l\leq v^{(1)}:\widetilde{\phi}\left(x_{\sigma^n(p)},x_{\sigma^k(q)},x_{\sigma^l(r)}-\xi\right)\geq\varepsilon\right\}$$
$$\cup\left\{t^{(1)}<n\leq t,u^{(1)}<k\leq u,v^{(1)}<l\leq v:\widetilde{\phi}\left(x_{\sigma^n(p)},x_{\sigma^k(q)},x_{\sigma^l(r)}-\xi\right)\geq\varepsilon\right\}$$

$$\cup\left\{t^{(1)}<n\leq t,\ u^{(1)}<k\leq u,\ l\leq v^{(1)}:\widetilde{\phi}\left(x_{\sigma^n(p)},x_{\sigma^k(q)},x_{\sigma^l(r)}-\xi\right)\geq\varepsilon\right\}$$
$$\cup\left\{t^{(1)}<n\leq t,\ k\leq u^{(1)},\ l\leq v^{(1)}:\widetilde{\phi}\left(x_{\sigma^n(p)},x_{\sigma^k(q)},x_{\sigma^l(r)}-\xi\right)\geq\varepsilon\right\}$$
$$\cup\left\{n\leq t^{(1)},\ u^{(1)}<k\leq u,\ v^{(1)}<l\leq v:\widetilde{\phi}\left(x_{\sigma^n(p)},x_{\sigma^k(q)},x_{\sigma^l(r)}-\xi\right)\geq\varepsilon\right\}$$
$$\cup\left\{n\leq t^{(1)},\ u^{(1)}<k\leq u,\ l\leq v^{(1)}:\widetilde{\phi}\left(x_{\sigma^n(p)},x_{\sigma^k(q)},x_{\sigma^l(r)}-\xi\right)\geq\varepsilon\right\}$$
$$\cup\left\{t^{(1)}<n\leq t,\ k\leq u^{(1)},v^{(1)}<l\leq v:\widetilde{\phi}\left(x_{\sigma^n(p)},x_{\sigma^k(q)},x_{\sigma^l(r)}-\xi\right)\geq\varepsilon\right\}$$
$$\cup\left\{n\leq t^{(1)},k\leq u^{(1)},v^{(1)}<l\leq v:\widetilde{\phi}\left(x_{\sigma^n(p)},x_{\sigma^k(q)},x_{\sigma^l(r)}-\xi\right)\geq\varepsilon\right\}.$$

Accordingly, the sets can be written as:

$$\left\{n\leq t^{(1)},k\leq u^{(1)},l\leq v^{(1)}:\widetilde{\phi}\left(x_{\sigma^n(p)},x_{\sigma^k(q)},x_{\sigma^l(r)}-\xi\right)\geq\varepsilon\right\}$$
$$=\left\{n\leq t^{(2)},k\leq u^{(2)},l\leq v^{(2)}:\widetilde{\phi}\left(x_{\sigma^n(p)},x_{\sigma^k(q)},x_{\sigma^l(r)}-\xi\right)\geq\varepsilon\right\}$$
$$\cup\left\{t^{(2)}<n\leq t^{(1)},u^{(2)}<k\leq u^{(1)},v^{(2)}<l\leq v^{(1)}:\widetilde{\phi}\left(x_{\sigma^n(p)},x_{\sigma^k(q)},x_{\sigma^l(r)}-\xi\right)\geq\varepsilon\right\}$$
$$\cup\left\{t^{(2)}<n\leq t^{(1)},\ u^{(2)}<k\leq u^{(1)},\ l\leq v^{(2)}:\widetilde{\phi}\left(x_{\sigma^n(p)},x_{\sigma^k(q)},x_{\sigma^l(r)}-\xi\right)\geq\varepsilon\right\}$$
$$\cup\left\{t^{(2)}<n\leq t^{(1)},\ k\leq u^{(2)},\ l\leq v^{(2)}:\widetilde{\phi}\left(x_{\sigma^n(p)},x_{\sigma^k(q)},x_{\sigma^l(r)}-\xi\right)\geq\varepsilon\right\}$$
$$\cup\left\{n\leq t^{(2)},\ u^{(2)}<k\leq u^{(1)},\ v^{(2)}<l\leq v^{(1)}:\widetilde{\phi}\left(x_{\sigma^n(p)},x_{\sigma^k(q)},x_{\sigma^l(r)}-\xi\right)\geq\varepsilon\right\}$$
$$\cup\left\{n\leq t^{(2)},\ u^{(2)}<k\leq u^{(1)},\ l\leq v^{(2)}:\widetilde{\phi}\left(x_{\sigma^n(p)},x_{\sigma^k(q)},x_{\sigma^l(r)}-\xi\right)\geq\varepsilon\right\}$$
$$\cup\left\{t^{(2)}<n\leq t^{(1)},\ k\leq u^{(2)},v^{(2)}<l\leq v^{(1)}:\widetilde{\phi}\left(x_{\sigma^n(p)},x_{\sigma^k(q)},x_{\sigma^l(r)}-\xi\right)\geq\varepsilon\right\}$$
$$\cup\left\{n\leq t^{(2)},k\leq u^{(2)},v^{(2)}<l\leq v^{(1)}:\widetilde{\phi}\left(x_{\sigma^n(p)},x_{\sigma^k(q)},x_{\sigma^l(r)}-\xi\right)\geq\varepsilon\right\},$$

$$\left\{t^{(1)}<n\leq t,u^{(1)}<k\leq u,l\leq v^{(1)}:\widetilde{\phi}\left(x_{\sigma^n(p)},x_{\sigma^k(q)},x_{\sigma^l(r)}-\xi\right)\geq\varepsilon\right\}$$
$$=\left\{t^{(1)}<n\leq t,u^{(1)}<k\leq u,l\leq v^{(2)}:\widetilde{\phi}\left(x_{\sigma^n(p)},x_{\sigma^k(q)},x_{\sigma^l(r)}-\xi\right)\geq\varepsilon\right\}$$
$$\cup\left\{t^{(1)}<n\leq t,u^{(1)}<k\leq u,v^{(2)}<l\leq v^{(1)}:\widetilde{\phi}\left(x_{\sigma^n(p)},x_{\sigma^k(q)},x_{\sigma^l(r)}-\xi\right)\geq\varepsilon\right\}$$

and

$$\left\{t^{(1)}<n\leq t,k\leq u^{(1)},v^{(1)}<l\leq v:\widetilde{\phi}\left(x_{\sigma^n(p)},x_{\sigma^k(q)},x_{\sigma^l(r)}-\xi\right)\geq\varepsilon\right\}$$
$$=\left\{t^{(1)}<n\leq t,k\leq u^{(2)},v^{(1)}<l\leq v:\widetilde{\phi}\left(x_{\sigma^n(p)},x_{\sigma^k(q)},x_{\sigma^l(r)}-\xi\right)\geq\varepsilon\right\}$$
$$\cup\left\{t^{(1)}<n\leq t,u^{(2)}<k\leq u^{(1)},v^{(1)}<l\leq v:\widetilde{\phi}\left(x_{\sigma^n(p)},x_{\sigma^k(q)},x_{\sigma^l(r)}-\xi\right)\geq\varepsilon\right\}$$

and

$$\left\{n\leq t^{(1)},u^{(1)}<k\leq u,v^{(1)}<l\leq v:\widetilde{\phi}\left(x_{\sigma^n(p)},x_{\sigma^k(q)},x_{\sigma^l(r)}-\xi\right)\geq\varepsilon\right\}$$
$$=\left\{n\leq t^{(2)},u^{(1)}<k\leq u,v^{(1)}<l\leq v:\widetilde{\phi}\left(x_{\sigma^n(p)},x_{\sigma^k(q)},x_{\sigma^l(r)}-\xi\right)\geq\varepsilon\right\}$$
$$\cup\left\{t^{(2)}<n\leq t^{(1)},u^{(1)}<k\leq u,v^{(1)}<l\leq v:\widetilde{\phi}\left(x_{\sigma^n(p)},x_{\sigma^k(q)},x_{\sigma^l(r)}-\xi\right)\geq\varepsilon\right\}.$$

Now, the set $\left\{1 < n \leq t^{(2)}, 1 < k \leq u^{(2)}, 1 < l \leq v^{(2)} : \widetilde{\phi}\left(x_{\sigma^n(p),} x_{\sigma^k(q),} x_{\sigma^l(r)} - \xi\right) \geq \varepsilon\right\}$ can be written as

$$\begin{aligned}
&\left\{n \leq t^{(2)}, k \leq u^{(2)}, l \leq v^{(2)} : \widetilde{\phi}\left(x_{\sigma^n(p),} x_{\sigma^k(q),} x_{\sigma^l(r)} - \xi\right) \geq \varepsilon\right\} \\
&= \left\{n \leq t^{(3)}, k \leq u^{(3)}, l \leq v^{(3)} : \widetilde{\phi}\left(x_{\sigma^n(p),} x_{\sigma^k(q),} x_{\sigma^l(r)} - \xi\right) \geq \varepsilon\right\} \\
&\cup \left\{t^{(3)} < n \leq t^{(2)}, u^{(3)} < k \leq u^{(2)}, v^{(3)} < l \leq v^{(2)} : \widetilde{\phi}\left(x_{\sigma^n(p),} x_{\sigma^k(q),} x_{\sigma^l(r)} - \xi\right) \geq \varepsilon\right\} \\
&\cup \left\{t^{(3)} < n \leq t^{(2)},\ u^{(3)} < k \leq u^{(2)},\ l \leq v^{(3)} : \widetilde{\phi}\left(x_{\sigma^n(p),} x_{\sigma^k(q),} x_{\sigma^l(r)} - \xi\right) \geq \varepsilon\right\} \\
&\cup \left\{t^{(3)} < n \leq t^{(2)},\ k \leq u^{(3)},\ l \leq v^{(3)} : \widetilde{\phi}\left(x_{\sigma^n(p),} x_{\sigma^k(q),} x_{\sigma^l(r)} - \xi\right) \geq \varepsilon\right\} \\
&\cup \left\{n \leq t^{(3)},\ u^{(3)} < k \leq u^{(2)},\ v^{(3)} < l \leq v^{(2)} : \widetilde{\phi}\left(x_{\sigma^n(p),} x_{\sigma^k(q),} x_{\sigma^l(r)} - \xi\right) \geq \varepsilon\right\} \\
&\cup \left\{n \leq t^{(3)},\ u^{(3)} < k \leq u^{(2)},\ l \leq v^{(3)} : \widetilde{\phi}\left(x_{\sigma^n(p),} x_{\sigma^k(q),} x_{\sigma^l(r)} - \xi\right) \geq \varepsilon\right\} \\
&\cup \left\{t^{(3)} < n \leq t^{(2)},\ k \leq u^{(3)}, v^{(3)} < l \leq v^{(2)} : \widetilde{\phi}\left(x_{\sigma^n(p),} x_{\sigma^k(q),} x_{\sigma^l(r)} - \xi\right) \geq \varepsilon\right\} \\
&\cup \left\{n \leq t^{(3)}, k \leq u^{(3)}, v^{(3)} < l \leq v^{(2)} : \widetilde{\phi}\left(x_{\sigma^n(p),} x_{\sigma^k(q),} x_{\sigma^l(r)} - \xi\right) \geq \varepsilon\right\}.
\end{aligned}$$

For the general form, the set

$$\left\{1 < n \leq t^{(h_1-1)}, 1 < k \leq u^{(h_2-1)}, 1 < l \leq v^{(h_3-1)} : \widetilde{\phi}\left(x_{\sigma^n(p),} x_{\sigma^k(q),} x_{\sigma^l(r)} - \xi\right) \geq \varepsilon\right\}$$

is written

$$\begin{aligned}
&\left\{n \leq t^{(h_1-1)}, k \leq u^{(h_2-1)}, l \leq v^{(h_3-1)} : \widetilde{\phi}\left(x_{\sigma^n(p),} x_{\sigma^k(q),} x_{\sigma^l(r)} - \xi\right) \geq \varepsilon\right\} \\
&= \left\{n \leq t^{(h_3)}, k \leq u^{(h_2)}, l \leq v^{(h_3)} : \widetilde{\phi}\left(x_{\sigma^n(p),} x_{\sigma^k(q),} x_{\sigma^l(r)} - \xi\right) \geq \varepsilon\right\} \\
&\cup \left\{t^{(h_1)} < n \leq t^{(h_1-1)}, u^{(h_2)} < k \leq u^{(h_2-1)}, v^{(h_3)} < l \leq v^{(h_3-1)} : \widetilde{\phi}\left(x_{\sigma^n(p),} x_{\sigma^k(q),} x_{\sigma^l(r)} - \xi\right) \geq \varepsilon\right\} \\
&\cup \left\{t^{(h_1)} < n \leq t^{(h_1-1)},\ u^{(h_2)} < k \leq u^{(h_2-1)},\ l \leq v^{(h_3)} : \widetilde{\phi}\left(x_{\sigma^n(p),} x_{\sigma^k(q),} x_{\sigma^l(r)} - \xi\right) \geq \varepsilon\right\} \\
&\cup \left\{t^{(h_1)} < n \leq t^{(h_1-1)},\ k \leq u^{(h_2)},\ l \leq v^{(h_3)} : \widetilde{\phi}\left(x_{\sigma^n(p),} x_{\sigma^k(q),} x_{\sigma^l(r)} - \xi\right) \geq \varepsilon\right\} \\
&\cup \left\{n \leq t^{(h_1)},\ u^{(h_2)} < k \leq u^{(h_2-1)},\ v^{(h_3)} < l \leq v^{(h_3-1)} : \widetilde{\phi}\left(x_{\sigma^n(p),} x_{\sigma^k(q),} x_{\sigma^l(r)} - \xi\right) \geq \varepsilon\right\} \\
&\cup \left\{n \leq t^{(h_1)},\ u^{(h_2)} < k \leq u^{(h_2-1)},\ l \leq v^{(h_3)} : \widetilde{\phi}\left(x_{\sigma^n(p),} x_{\sigma^k(q),} x_{\sigma^l(r)} - \xi\right) \geq \varepsilon\right\} \\
&\cup \left\{t^{(h_1)} < n \leq t^{(h_1-1)},\ k \leq u^{(h_2)}, v^{(h_3)} < l \leq v^{(h_3-1)} : \widetilde{\phi}\left(x_{\sigma^n(p),} x_{\sigma^k(q),} x_{\sigma^l(r)} - \xi\right) \geq \varepsilon\right\} \\
&\cup \left\{n \leq t^{(h_1)}, k \leq u^{(h_2)}, v^{(h_3)} < l \leq v^{(h_3-1)} : \widetilde{\phi}\left(x_{\sigma^n(p),} x_{\sigma^k(q),} x_{\sigma^l(r)} - \xi\right) \geq \varepsilon\right\}.
\end{aligned}$$

where

$$\begin{aligned}
t^{(h_1)} &\geq 1, t^{(h_1+1)} = 0, \\
u^{(h_2)} &\geq 1, u^{(h_2+1)} = 0, \\
v^{(h_3)} &\geq 1, v^{(h_3+1)} = 0
\end{aligned}$$

are satisfied for the fixed positive integers h_1, h_2 and h_3. Consequently,

$$\frac{1}{tuv}\left|\left\{n \leq t, k \leq u, l \leq v : \widetilde{\phi}\left(x_{\sigma^n(p)}, x_{\sigma^k(q)}, x_{\sigma^l(r)} - \xi\right) \geq \varepsilon\right\}\right|$$
$$= \sum_{g=0}^{(h_1+1)} \sum_{i=0}^{(h_2+1)} \sum_{j=0}^{(h_3+1)} \frac{\triangle t^{(g)} \triangle u^{(i)} \triangle v^{(j)}}{tuv} . K_{tuvnkl}$$

is obtained here. Here,

$$\triangle t^{(g)} : t^{(g)} - t^{(g+1)}$$
$$\triangle u^{(i)} : u^{(i)} - u^{(i+1)}$$
$$\triangle v^{(j)} : v^{(j)} - v^{(j+1)}.$$

Moreover, since $x_{tuv} \to \xi \left(DS_{\beta\gamma\eta}\right)_{\sigma}^{\widetilde{\phi}}$, the sequence

$$K_{tuvnkl} = \left\{\frac{1}{\triangle t^{(g)} \triangle u^{(i)} \triangle v^{(j)}} \left\{\left| t^{(g+1)} < n \leq t^{(g)}, u^{(i+1)} < k \leq u^{(i)},\right.\right.\right.$$
$$\left.\left.\left. v^{(j+1)} < l \leq v^{(j)} : \widetilde{\phi}\left(x_{\sigma^n(p)}, x_{\sigma^k(q)}, x_{\sigma^l(r)} - \xi\right) \geq \varepsilon \right\}\right|\right\} \tag{3.4}$$

is convergent to zero for all $g, i, j \in \mathbb{N}$. If we consider the matrix $Z = (z_{tuvnkl})$ defined by

$$z_{tuvnkl} := \begin{cases} \frac{\triangle t^{(g)} \triangle u^{(i)} \triangle v^{(j)}}{tuv}, & t^{(g+1)} < n \leq t^{(g)}, u^{(i+1)} < k \leq u^{(i)}, \\ & v^{(j+1)} < l \leq v^{(j)},\ g, i, j = 1, 2, 3, ... \\ 0, & \text{otherwise} \end{cases},$$

then the statistical convergence of the sequence $x = (x_{tuv})$ is equivalent to the convergence of Z-transform of the sequence given by (3.4). Since the matrix (z_{tuvnkl}) is regular,

$$\lim_{t,u,v \to \infty} \frac{1}{tuv}\left|\left\{n \leq t, k \leq u, l \leq v : \widetilde{\phi}\left(x_{\sigma^n(p)}, x_{\sigma^k(q)}, x_{\sigma^l(r)} - \xi\right) \geq \varepsilon\right\}\right| = 0$$

is obtained. This step completes the proof.

Conversely, since $\left\{\frac{a(t)}{\beta(t)}, \frac{c(u)}{\gamma(u)}, \frac{d(v)}{\eta(v)}\right\}$ is bounded for $b(t) = t$, $d(u) = u$, $f(v) = v$, by Theorem 3.13, we have $x_{tuv} \to \xi\,(S)_{\sigma}^{\widetilde{\phi}}$ implies $x_{tuv} \to \xi \left(DS_{\beta\gamma\eta}\right)_{\sigma}^{\widetilde{\phi}}$. □

When $\sigma(p) = p + 1$, $\sigma(q) = q + 1$, $\sigma(r) = r + 1$, from Definitions 3.5 − 3.8 we have the following definitions of deferred almost convergence, strongly deferred almost $\widetilde{\phi}$-convergence, strongly s-deferred almost $\widetilde{\phi}$-convergence, deferred almost statistically $\widetilde{\phi}$-convergence for a sequence $x = (x_{nkl})$.

Definition 3.15. *A sequence $x = (x_{tuv})$ is called to be deferred almost convergent to $\xi \in \mathbb{R}$ if*

$$\lim_{t,u,v \to \infty} \frac{1}{\beta(t)\gamma(u)\eta(v)} \sum_{n=a(t)+1}^{b(t)} \sum_{k=c(u)+1}^{d(u)} \sum_{l=e(v)+1}^{f(v)} x_{n+p,k+q,l+r} = \xi$$

uniformly in p, q, r.

Definition 3.16. *Let $\widetilde{\phi}:\mathbb{R}\to\mathbb{R}$ be an Orlicz function. Then, the sequence $x=(x_{tuv})$ is called to be strongly deferred almost $\widetilde{\phi}$-convergent to $\xi\in\mathbb{R}$ if*

$$\lim_{t,u,v\to\infty}\frac{1}{\beta(t)\gamma(u)\eta(v)}\sum_{n=a(t)+1}^{b(t)}\sum_{k=c(u)+1}^{d(u)}\sum_{l=e(v)+1}^{f(v)}\widetilde{\phi}\left(x_{n+p,k+q,l+r}-\xi\right)=0$$

uniformly in p,q,r.

Definition 3.17. *Let $\widetilde{\phi}:\mathbb{R}\to\mathbb{R}$ be an Orlicz function and $\widetilde{\phi}_s=|x|^s$. Then, the sequence $x=(x_{tuv})$ is called to be strongly s-deferred almost $\widetilde{\phi}$-convergent to $\xi\in\mathbb{R}$ if*

$$\lim_{t,u,v\to\infty}\frac{1}{\beta(t)\gamma(u)\eta(v)}\sum_{n=a(t)+1}^{b(t)}\sum_{k=c(u)+1}^{d(u)}\sum_{l=e(v)+1}^{f(v)}\widetilde{\phi}_s\left(x_{n+p,k+q,l+r}-\xi\right)=0$$

uniformly in p,q,r where $0<s<\infty$.

Definition 3.18. *Let $\widetilde{\phi}:\mathbb{R}\to\mathbb{R}$ be an Orlicz function. Then, the sequence $x=(x_{tuv})$ is called to be deferred almost statistically $\widetilde{\phi}$-convergent to $\xi\in\mathbb{R}$ if for every $\varepsilon>0$,*

$$\lim_{t,u,v\to\infty}\frac{1}{\beta(t)\gamma(u)\eta(v)}\left|\left\{a(t)<n\le b(t),\ c(u)<k\le d(u),\ e(v)<l\le f(v)\right.\right.$$
$$\left.\left. :\widetilde{\phi}\left(x_{n+p,k+q,l+r}-\xi\right)\ge\varepsilon\right\}\right|=0$$

uniformly in p,q,r.

In case $\sigma(p)=p+1$, $\sigma(q)=q+1$, $\sigma(r)=r+1$, we have the following remark.

Remark. *So, the similar inclusions given by Theorems $3.12-3.14$ hold between strongly deferred almost $\widetilde{\phi}$-convergent sequences, deferred almost statistical $\widetilde{\phi}$-convergent sequences and σ-statistical $\widetilde{\phi}$-convergent sequences.*

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