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NEW THEOREMS BY MINIMAX INEQUALITIES ON H-SPACE

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Abstract. In this paper, we generalize some theorems by using minimax inequalities from real valued mappings to vector valued mappings in a partial ordering space by a pointed positive cone. An example is given to illustrate our result.

1. Introduction

In the years 1983-1985 C. Horvath obtained minimax inequalities by replacing convexity assumptions with merely topological properties: pseudo-convexity in [6] and contractibility in [7, 8]. Then in 1989 R. Ceppitelli and C. Bardaro in [1] generalized Horvath's generalized minimax inequality in to vector valued in H-spaces.

There are several methods to prove vector valued minimax theorems including F. Ferro in [4] and C. Bardaro-R. Ceppitelli in [1] and some others.

The first two minimax theorem ware obtained by K. Fan [3] in generalizing Sian's minimax theorem [9].

At first, let E be a vector space.We shall denote by 2^E the set of all subsets of E and by $\mathrm{co}(A)$ the convex hull of $A \in 2^E$. Let X be an arbitrary non-empty subset of E. A map $F : X \to 2^E$ is called a KKM map if $\mathrm{co}(\{x_1, \cdots, x_n\}) \subseteq \bigcup_{k=1}^{n} F(x_i)$. for each finite subset $\{x_1, \cdots, x_n\}$ of X, (See[10]).

Let E be a linear space, and C a subset of E, C is called a cone in E if it satisfies:

(i) C is closed, nonempty and $C \neq \{0\}$,
(ii) $a, b \in \mathbb{R}$, $a, b \geq 0$ and $x, y \in C$ imply that $ax + by \in C$,
(iii) $x \in C$ and $-x \in C$ imply that $x = 0$.

The space E can be partially ordered by the cone $C \subset E$; that is, $x \leq y$ if and only if $y - x \in C$. Also we write $x \ll y$ if $y - x \in C^\circ$, where C° denotes the interior of C.
A cone C is called normal if there exists a constant $K > 0$ such that $0 \leq x \leq y$ implies $\|x\| \leq K\|y\|$. The smallest positive such number is called the normal constant of C.

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In the following we always suppose that E is a partially ordered linear space, C is a cone in E with $C^\circ \neq \emptyset$ and $\leq$ is the partial ordering induced on E by C.

Definition 1.1. ([5]) *A partially ordered linear space, is a quadruple $(E, +, \cdot, \leq)$ where $(E, +, \cdot)$ is a linear space over the field $\mathbb{R}$ of real numbers and $\leq$ is a partial ordering on E such that*

(i) *If $x \leq y$, then $x + z \leq y + z$ for every $z \in E$;*
(ii) *If $x \geq 0$ in E, then $\alpha x \geq 0$ whenever $\alpha \geq 0$ in $\mathbb{R}$.*

Definition 1.2. ([5])

(1) *From Definition 1.1 (i), we see that $x \leq y \iff 0 \leq y - x$. So $\leq$ is determined entirely by $E^+ = \{x : x \in E, x \geq 0\}$, the positive cone of E.*
(2) *A lattice is a partially ordered set $(A, \leq)$ such that $\sup\{a, b\}$ and $\inf\{a, b\}$ exist for all elements a and b of A.*
(3) *A Riesz space, or vector lattice, is a partially ordered linear space $(E, +, \cdot, \leq)$ such that $(E, \leq)$ is a lattice.*

Example 1.3. ([5])

(1) *Let E be a partially ordered linear space such that $\sup\{x, 0\}$ exists for every $x \in E$. Then E is a Riesz space.*
(2) *The set of real numbers is a Riesz space.*
(3) *Let K be an arbitrary set. Then $C(K)$, the space of continuous functions on K is Riesz space.*

In this paper let X be a topological space and (E, C) be a topological Riesz space, where C is the positive cone.

Definition 1.4. ([1]) *By H-space we mean a pair $\big(X, \{\Gamma_A\}\big)$, where X is a topological space and $\{\Gamma_A\}$ is a given family of nonempty contractible subsets of X, that is, intuitively it is one that can be continuously shrunk to a point within that space or it is null-homotopic or it is homotopic to some constant map, indexed by the finite subsets of X.*

Let $\big(X, \{\Gamma_A\}\big)$ be an $H-$space. A subset $D \subset X$ is called $H-$convex if, for every finite subset $A \subset D$, it follows that $\Gamma_A \subset D$.

A subset $D \subset X$ is called weakly H-convex if, for every finite subset $A \subset D$, it results that $\Gamma_A \cap D$ is nonempty and contractible.

Finally, a subset $K \subset X$ is called H-compact if, for every finite subset $A \subset X$, there exists a compact, weakly H-convex set $D \subset X$ such that $K \cup A \subset D$.

For every finite subset $A = \{x_1, \ldots, x_n\} \subset X$, we can set $\Gamma_A = \operatorname{co}\{x_1, \ldots, x_n\}$; moreover, any convex subset of X is H-convex and any nonempty compact convex subset is H-compact.

We recall the following remark, since we shall use it in Theorems 1.6 and 1.7 in our main results.

Remark. ([1]) *Every Hausdorff topological vector space is H-space: For every finite subset $A = \{x_1, \ldots, x_n\} \subset X$, we can set $\Gamma_A = \operatorname{co}\{x_1, \ldots, x_n\}$; moreover, any convex subset of X is H-convex and any nonempty compact convex subset is H-compact.*

Every contractible space X is an H-space: at first we may put $\Gamma_A = X$ for every finite subset $A \subset X$ with this structure, the only H-convex subset of X is X itself. For more detail see ([1]).

Definition 1.5. ([1]) *In a given H-space* $\big(X,\{\Gamma_A\}\big)$*, a multifunction* $F : X \longrightarrow 2^X$ *is called* $H-KKM$ *if* $\Gamma_A \subset \bigcup_{x\in A} F(x)$*, for each finite subset* $A \subset X$*.*

We premise same notations: given a multifunction $F : X \longrightarrow 2^X$, we put: $F^{-1}(y) = \big\{x \in X;\ y \in F(x)\big\}$ and $F^*(y) = X - F^{-1}(y)$.

The following theorems are Theorem 1, 2 of Bardaro-Ceppitelli [1].

Theorem 1.6. ([1]) *Let* $\big(X,\{\Gamma_A\}\big)$ *be an H-space and* $F : X \longrightarrow 2^X$ *an* $H-KKM$ *multifunction such that:*

a) *For each* $x \in X$*,* $F(x)$ *is compactly closed, that is,* $B\cap F(x)$ *is closed in* B*, for every compact* $B \subset X$*.*
b) *There is a compact set* $L \subset X$ *and an H-compact* $K \subset X$*, such that, for each weakly H-convex set* D *with* $K \subset D \subset X$*, we have* $\bigcap_{x\in D} \big(F(x)\cap D\big) \subset L$*.*

Then $\bigcap_{x\in X} F(x) \neq \emptyset$*.*

Theorem 1.7. ([1]) *Let* $\big(X,\{\Gamma_A\}\big)$ *be an H-space,* $G, F : X \longrightarrow 2^X$ *two multifunctions such that:*

a) *For every* $x \in X$*,* $G(x)$ *is compactly closed and* $F(x) \subset G(x)$*;*
b) $x \in F(x)$*, for every* $x \in X$*;*
c) *for every* $x \in X$*,* $F^*(x)$ *is H-convex;*
d) *the multifunction* G *verifies property (b) of Theorem 1.6 then* $\bigcap_{x\in X} G(x) \neq \emptyset$*.*

2. Main Results

In this section we will generalize some minimax theorems in Tan [10] and Ding-Tan [2] in to vector valued mappings by Theorem 1.6 and Theorem 1.7.

Remember that, the space E can be partially ordered by the cone $C \subset E$; that is, $x \leq y$ if and only if $y - x \in C$.

Theorem 2.1. *Let* X *be a non-empty convex set in Hausdorff topological vector space. Let* $f, g : X \times X \to (E, C)$ *having the following properties:*

a) $f(x, y) \leq g(x, y)$ *for all* $(x, y) \in X \times X$*, and* $g(x, x) \leq 0$ *for all* $x \in X$*.*
b) *for each fixed* $x \in X$*,* $\big\{y \in X :\ f(x, y) \in -C\big\}$ *is compactly closed.*
c) *For each fixed* $y \in X$*, the set* $\big\{x \in X :\ g(x, y) \notin -C\big\}$ *is convex.*
d) *There exists a non-empty compact convex subset* K *of* X *such that for each* $y \in X \setminus K$ *there exists a point* $x \in K$ *with* $f(x, y) \notin -C$*.*

Then there exists a point $\widehat{y} \in K$ *such that* $f(x, \widehat{y}) \in -C$ *for all* $x \in X$*.*

Proof. For each $x \in X$, define: $K(x) = \big\{y \in X :\ f(x, y) \in -C\big\}$. By (b); $K(x)$ is compactly closed in K for every $x \in X$.

We first prove that the family $\big\{K(x);\ x \in X\big\}$ has the finite intersection property.

Now choose $x_1, \ldots, x_m \in X$. Let $B \equiv \mathrm{co}\big(K \cup \{x_1, \ldots, x_m\}\big)$. Then B is a compact convex subset of X for every $x \in X$ define $F(x) = \big\{y \in B;\ f(x, y) \in -C\big\}$, $G(x) = \big\{y \in B;\ g(x, y) \in -C\big\}$.

By (a), $x \in F(x)$, then for each $x \in B$, F(x) is non-empty.

We shall show that $\bigcap_{x\in B} F(x) \neq \emptyset$. Our next goal is to show that, this theorem satisfy in assumptions of Theorem 1.7. Then we have the following:

Let $y \in G(x)$, there for, $g(x,y) \leq 0$ and by (a) $f(x,y) \leq 0$, and $y \in F(x)$, thus $G(x) \subseteq F(x)$ on the other side $F(x)$ is compactly closed, because B is compact and by (b), $\{y \in X : \ f(x,y) \in -C\}$ is compactly closed.

By (a), for each $x \in X$, $g(x,x) \leq 0$, thus $x \in G(x)$.

We have, for each $y \in X$,

$$\begin{aligned} G^*(y) &= X - G^{-1}(y) = \{x \in X;\ x \notin G^{-1}(y)\} \\ &= \{x \in X;\ y \notin G(x)\} = \{x \in X;\ g(x,y) \notin -C\}, \end{aligned}$$

by (c), $G^*(y)$ is convex.

Suppose that $D \subseteq X$ is weakly H-convex with $L \subseteq D$. We will prove $\bigcap_{x \in D} \big(F(x) \cap D\big) \subseteq L$.

Let $z \in \bigcap_{x \in D} \big(F(x) \cap D\big)$, we would have:

$$(z \in D, \quad z \in F(x)) \Rightarrow (z \in D, \quad f(x,z) \leq 0) \Rightarrow (z \in D, \quad f(x,z) \in -C), \quad (2.1)$$

for all $x \in D$. Suppose z is not in L, therefore by (d), there exists a x_0 in L that $f(x_0,z) \notin -C$. If x_0 be in L by $L \subseteq D$ we have $x_0 \in D$ and this contradicts (2.1) then $z \in L$ and $\bigcap_{x \in D} \big(F(x) \cap D\big) \subseteq L$.

We recover condition (d) of Theorem 1.7.

Therefore,this theorem satisfies in conditions of Theorem 1.7, Hence it follows that $\bigcap_{x \in B} F(x) \neq \emptyset$. In other words, there exists a point $\overline{y} \in B$ such that $f(x,\overline{y}) \leq 0$, so $f(x,\overline{y}) \in -C$ for all $x \in B$.

It follows that $\overline{y} \in L$ by (d) and $\overline{y} \in K(x_1) \cap \cdots \cap K(x_m)$ by definition of $K(x)$. Thus $\{K(x);\ x \in X\}$ has the finite intersection property. By compactness of L, we have $\bigcap_{x \in X} K(x) \neq \emptyset$.

Now, if we choose that $\widehat{y} \in \bigcap_{x \in X} K(x)$, there for $f(x,\widehat{y}) \in -C$ for all $x \in X$, and the proof is complete. □

Example 2.2. *Let f,g be two vector-valued functions on $\mathbb{R}^+ \times \mathbb{R}^+$ and taking values in (E,C) define $f(x,y) = (-3x+y, -4x+y)$, $g(x,y) = (-2x+y, -3x+y)$ and set $C = \{(x,y) \in \mathbb{R}^2; x,y \geq 0\}$. For every $(a,b),(c,d) \in \mathbb{R}^2$ we consider*

$$(a,b) \leq (c,d) \iff a \leq c \text{ and b} \leq \text{d}.$$

Obviously, $f(x,y) \leq g(x,y)$ for all $(x,y) \in \mathbb{R}^+ \times \mathbb{R}^+$ and $g(x,x) \leq 0$ for all $x \in \mathbb{R}^+$.

We have, $\{y \in \mathbb{R}^+; f(x,y) \in -C\} = \{0\}$ therefore, $\{y \in \mathbb{R}^+; f(x,y) \in -C\}$ is compactly closed for each fixed $x \in \mathbb{R}^+$.

For each $y \in \mathbb{R}^+$, $\{x \in \mathbb{R}^+; g(x,y) \notin -C\}$ is convex, because $\{x \in \mathbb{R}^+; g(x,y) \notin -C\} = \emptyset$.

We put $k = \{0\}$, it is seen that $f(x,y) > 0$ for every $y \in \mathbb{R}^+ \backslash \{0\}$ and $x = 0$. Finally we take $\hat{y} = 0$, it is easy to check that $f(x,\hat{y}) \leq 0$ for every $x \in \mathbb{R}^+$.

Theorem 2.3. *Let X be a non-empty convex set in a Hausdorff topological vector space. Assume that $f_1, f_2 : X \times X \to (E,C)$ f_1 and with the following properties:*

a) *$f_1(x,y) \leq f_2(x,y)$ for all $(x,y) \in X \times X$.*
b) *For all $x \in X$, $\{y : f_1(x,y) \leq \alpha\}$ is compactly closed for all $\alpha \in E$.*
c) *For all $x \in X$, $\{x : f_2(x,y) > \alpha\}$ is convex for all $\alpha \in E$.*

d) *There exists a non-empty compact convex subset K of X such that for all $y \in X \setminus K$ there exists a point $x \in X$ with $f_1(x,y) \in C^o + \sup_{z\in X} f_2(z,z)$ if $\sup_{z\in X} f_2(z,z) < \infty$.*

Then the minimax inequality $\min_{y\in K} \sup_{x\in X} f_1(x,y) \leq \sup_{x\in X} f_2(x,y)$ holds.

Proof. Choose $t = \sup_{x\in X} f_2(x,x)$ it exists.

Define $g(x,y) = f_2(x,y) - t$, $f(x,y) = f_1(x,y) - t$. It suffices to show that $g(x,y)$, $f(x,y)$ satisfy in Theorem 2.1.

For all $t < +\infty$, we have $f_1(x,y) - t \leq f_2(x,y) - t$. Therefore $f(x,y) \leq g(x,y)$ and for every $x \in X$, $g(x,x) = f_2(x,x) - t \leq \sup_{x\in X} f_2(x,x) - t = 0$.

For all $x \in X$ and for all $\alpha \in E$, the set $\{y : \; f_1(x,y) \leq \alpha\}$ is compactly closed and for all $t < +\infty$, the set $\{y : \; f(x,y) + t \leq \alpha\}$ is compactly closed if we take $\alpha = t = 0$, we would have $\{y : \; f(x,y) \leq 0\}$ is compactly closed.

For all $x \in X$ and for all $\alpha \in E$, the set $\{x : \; f_2(x,y) > \alpha\}$ is convex and for all $t < +\infty$, the set $\{x : \; g(x,y) + t > \alpha\}$ is convex.

If we take $\alpha = t = 0$, we would have $\{x : \; g(x,y) > 0\}$ is convex therefore $\{x : g(x,y) \notin -C\}$ is convex.

By (d), $f_1(x,y) \in C^o + \sup_{z\in X} f_2(z,z)$, therefore $f_1(x,y) - t \in C^o$ then $f(x,y) \in C^o$.

By Theorem 2.1, $f(x,y) \leq 0$, for all $(x,y) \in X \times X$, therefore $f_1(x,y) \leq t$, so that $f_1(x,y) \leq \sup_{x\in X} f_2(x,x)$ then $\sup_{x\in X} f_1(x,y) \leq \sup_{x\in X} f_2(x,x)$ it follows that $\min_{y\in K} \sup_{x\in X} f_1(x,y) \leq \sup_{x\in X} f_2(x,x)$. □

Theorem 2.4. *Let X be a non-empty convex subset of a topological vector space and (E,C) be an order complete topological Riesz space, and let $f, g : X \times X \to (E,C)$ be such that:*

a) *$f(x,y) \leq g(x,y)$ for all $x, y \in X$ and $g(x,x) \in -C$ for all $x \in X$.*
b) *For each $x \in X$ the set $\{y; \; f(x,y) \in -C\}$ is compactly closed.*
c) *For each $y \in X$, the set $\{x \in X; \; g(x,y) \in C^o\}$ is convex.*
d) *There exists a non-empty compact convex subset X_0 of X and non-empty compact subset K of X such that for each $y \in X \setminus K$, there is an $x_0 \in C^o \big(X_0 \cup \{y\}\big)$ with $f(x,y) \in C^o$.*

Then there exists $\widehat{y} \in K$ such that $f(x,\widehat{y}) \in -C$ for all $x \in X$.

Proof. For each $x \in X$ define $K(x) = \{y \in K : \; f(x,y) \in -C\}$ for every $x \in X$; $K(x)$ is closed in K by (b). Our claim is to prove the family $\{K(x) : \; x \in X\}$ has the finite intersection property.

Choose $x_1, \ldots, x_m \in X$. Let $B \equiv \operatorname{co}\big(K \cup \{x_1, \ldots, x_m\}\big)$ then B is a compact subset of X for every $x \in X$ define:

$$F(x) = \{y \in B; \; f(x,y) \in -C\}, \qquad G(x) = \{y \in B; \; g(x,y) \in -C\}. \tag{2.2}$$

It is clear that $F(x)$ and $G(x)$ is non-empty by (a). We show that $\bigcap_{x\in B} F(x) \neq \emptyset$.

Next, we show, this theorem satisfies in assumptions of Theorem 1.7. Then we have the following:

Let $y \in G(x)$, therefore, $g(x,y) \leq 0$ and by (a) $f(x,y) \leq 0$, and $y \in F(x)$, thus; $G(x) \subseteq F(x)$ on the other hand, $F(x)$ is compactly closed, because B is compact and by (b), $\{y \in X;\ f(x,y) \in -C\}$ is compactly closed.

For each $x \in X$, $g(x,x) \leq 0$, by (a), thus $x \in G(x)$.

For each $y \in X$, we have

$$G^*(y) = X - G^{-1}(y) = \{x \in X;\ x \notin G^{-1}(y)\} = \{x \in X;\ g(x,y) \notin -C\}.$$

By (c), $G^*(y)$ is convex.

Suppose that, $D \subseteq X$ is weakly H-convex. Therefore, $X_0 \subseteq D \subseteq L$. It suffices to show that $\bigcap_{x\in D} \big(F(x) \cap D\big) \subseteq L$.

If $y \in \bigcap_{x\in D} \big(F(x) \cap D\big)$, we would have

$$(y \in D, \quad y \in F(x)) \Rightarrow (y \in D, \quad f(x,y) \in C.) \tag{2.3}$$

for all $x \in D$. Suppose y is not in L, therefore by (d), there exists $x \in \mathrm{co}(X_0 \cup \{y\})$ that $f(x,y) \notin -C$, since $y \in D$. Therefore, $X_0 \cup \{y\} \subseteq D$ and because $X_0 \cup \{y\} \subseteq \mathrm{co}(X_0 \cup \{y\})$. Then $\mathrm{co}(X_0 \cup \{y\}) \subseteq D$ and $x \in D$ and this contradicts (2.3). Thus $y \in L$, $\bigcap_{x\in D} \big(F(x) \cap D\big) \subseteq L$ by Theorem 1.7, we have; $\bigcap_{x\in D} F(x) \neq \emptyset$. On the other hand, there exists a point $\overline{y} \in B$ such that $f(x,\overline{y}) \in -C$ for all $x \in B$.

It follows that $\overline{y} \in L$ by (d) and $\overline{y} \in K(x_1) \cap \ldots \cap K(x_m)$ by definition of $K(x)$. Thus $\{K(x); x \in X\}$ has the finite intersection property by compactness of L, we have $\bigcap_{x\in X} K(x) \neq \emptyset$.

Now, if we choose that $\widehat{y} \in \bigcap_{x\in X} K(x)$, therefore $f(x,\widehat{y}) \in -C$ for all $x \in X$, and the proof is complete. □

The next Corollaries are [2, Corollary 1, 2] which we improved them here.

Corollary 2.5. *Let X be a non-empty compact convex subset of topological vector space and let $f : X \times X \to (E,C)$ be such that for each $x \in X$, $\{y :\ f(x,y) \in -C\}$ is compactly closed. Then for each $t \in E$, one of the following properties holds:*

(1) *There exists $\widehat{y} \in X$ such that $f(x,\widehat{y}) \in t + C$ for all $x \in X$;*
(2) *There exists $A \in \mathcal{F}(X)$ (the family of all non-empty finite subset of X) and $y \in \mathrm{co}(A)$ such that $\min_{x\in A} f(x,y) \in t + C^o$.*

Proof. Define $F(x,y) = f(x,y) - t$ for all $x,y \in X$; therefore for all $x \in X$, $\{y :\ F(x,y) \in t{-}C\}$ is compactly closed. Fix $X_0 = K = X$ therefore condition (iii) of Theorem 2.4 holds. If for every $A \in \mathcal{F}(X)$ and for each $y \in \mathrm{co}(A)$, $\min_{x\in A} F(x,y) \in -C$. Therefore,by Theorem 2.4 exists $\widehat{y} \in X$ such that for all $x \in X$, $F(x,\widehat{y}) \in -C$. It follows that $f(x,\widehat{y}) \in t - C$ for all $x \in X$ and (1) holds.

On the other hand, if there exists $A \in \mathcal{F}(X)$ and $y \in \mathrm{co}(A)$ such that $\min_{x\in A} F(x,y) \in C^o$, then $\min_{x\in A} f(x,y) \in t + C^o$ and finally the condition (2) holds. □

Corollary 2.6. *Let X be a non-empty compact convex subset of a topological vector space and let $f, g : X \times X \to (E,C)$ be such that*

(i) *$f(x,y) \leq g(x,y)$ for all $x,y \in X$;*
(ii) *For each $x \in X$; $\{y :\ f(x,y) \in -C\}$ is compactly closed.*
(iii) *For each $y \in X$ and $t \in E$, the set $\{x \in X;\ g(x,y) \in t + C^o\}$ is convex.*

Then the minimax inequality

$$\min_{y\in X}\sup_{x\in X} f(x,y) \le \sup_{x\in X} g(x,x)$$

holds.

Proof. It suffices to suppose that $t = \sup_{x\in X} g(x,x) < \infty$ we only need to show that the condition (2) of Corollary 2.5 can't occur.

If there exists $A \in F(x)$ and $y \in \mathrm{co}(A)$. Such that $\min_{x\in A} f(x,y) \in t + C^o$. Therefore, by (i), we have, $\min_{x\in A} g(x,y) \in t + C^o$ and by (iii) $g(y,y) \in t + C^o$ contracting $t = \sup_{x\in x} g(x,x)$. Then condition(1) of Corollary 2.5 holds. Then there exists $y \in X$ such that for every $x \in X$, $f(x,y) \in t - C$ and because (E,C) is order complete space then is defined $\sup_{x\in X} f(x,y)$ and we have $\sup_{x\in X} f(x,y) \le t$. Then

$$\min_{y\in X}\sup_{x\in X} f(x,y) \le \sup_{x\in X} g(x,x).$$

□

The next results are [10, Theorems 1, 2] and [2, Theorem 2], that we improved them in this paper.

Corollary 2.7. *Let X be a non-empty convex set in a Hausdorff topological vector space E. Let ϕ and ψ be two real-valued functions on $X \times X$ having the following properties:*

a) *We have $\phi(x,y) \le \psi(x,y)$ for all $(x,y) \in X \times X$, and $\psi(x,x) \le 0$ for all $x \in X$;*
b) *for each fixed $x \in X$, $\phi(x,y)$ is a lower semi-continuous function of y on X;*
c) *for each fixed $y \in X$, the set $\{x \in X : \psi(x,y) > 0\}$ is convex;*
d) *there exists a non-empty compact convex subset K of X such that for each $y \in X \setminus K$ there exists a point $x \in K$ with $\phi(x,y) > 0$.*

Then there exists a point $\hat{y} \in K$ such that $\phi(x,\hat{y}) \le 0$ for all $x \in X$.

Corollary 2.8. *Let X be a non-empty convex set in a Hausdorff topological vector space. Let ϕ_1 and ϕ_2 be two real-valued functions on $X \times X$ having the following properties:*

a) *We have $\phi_1(x,y) \le \phi_2(x,y)$ for all $(x,y) \in X \times X$.*
b) *For each fixed $x \in X$, $\phi_1(x,y)$ is a lower semi-continuous function of y on X.*
c) *For each fixed $y \in X$, $\phi_2(x,y)$ is a quasi-concave function of x on X.*
d) *There exists a non-empty compact convex subset K of X such that for all $y \in X \setminus K$ there exists a point $x \in X$ with $\phi_1(x,y) > \sup_{z\in X} \phi_2(z,z)$ if $\sup_{z\in X} \phi_2(z,z) < \infty$.*

Then the minimax inequality $\min_{y\in K}\sup_{x\in X} \phi_1(x,y) \le \sup_{x\in X} \phi_2(x,x)$ holds.

Corollary 2.9. *Let X be a non-empty convex subset of a topological vector space and let $f, g : X \times X \longrightarrow \mathbb{R} \cup \{-\infty, \infty\}$ be such that*

a) *$f(x,y) \le g(x,y)$ for all $x, y \in X$ and $g(x,x) \le 0$ for all $x \in X$;*

b) *for each fixed* $x \in X$, $f(x, y)$ *is a lower semi-continuous function of* y *on each non-empty compact subset* C *of* X*;*
c) *for each* $y \in X$*, the set* $\{x \in X : g(x, y) > 0\}$ *is convex;*
d) *there exists a non-empty compact convex subset* X_0 *of* X *and a non-empty compact subset* K *of* X *such that for each* $y \in X \setminus K$*, there is an* $x \in \mathrm{co}(X_0 \cup \{y\})$ *with* $f(x, y) > 0$.

Then there exists $\hat{y} \in K$ *such that* $f(x, \hat{y}) \leq 0$ *for all* $x \in X$.

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