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# THE GENERALIZED HANKEL-CLIFFORD TRANSFORMATION WITH COMPACT SUPPORT ON CERTAIN RANGE

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Abstract. The Paley-Wiener theorem for the generalized Hankel-Clifford transforms is obtained. The generalized Hankel-Clifford transforms of square integrable functions with compact supports, rapid decreasing functions, infinitely differentiable functions with compact supports, of analytic functions are studied. The range of the generalized Hankel-Clifford transform of compactly supported functions which are either square integrable (Paley-Wiener Theorem) or infinitely differentiable (Paley-Wiener-Schwartz Theorem) is characterized. Such developed transforms are supported by an application to Mathematical Physics at the end of the section of the study.

## 1. Introduction

The generalized Hankel-Clifford transformations defined by

$$f(x) = (h_{1,\alpha,\beta} g)(x) = x^{-(\alpha+\beta)} \int_0^\infty \mathcal{J}_{\alpha,\beta}(xy)\, g(y)\, dy, \tag{1.1}$$

and

$$p(x) = (h_{2,\alpha,\beta} t)(x) = \int_0^\infty y^{-(\alpha+\beta)} \mathcal{J}_{\alpha,\beta}(xy)\, t(y)\, dy \tag{1.2}$$

if the integral converges in some sense (absolutely, improper, or mean convergence). Here $\mathcal{J}_{\alpha,\beta}(z) = z^{(\alpha+\beta)/2} J_{\alpha-\beta}(2\sqrt{z})$, $J_{\alpha-\beta}(z)$ being the Bessel function of the first kind and order $(\alpha-\beta) \geq -1/2$ were extended by Malgonde [1] to certain generalized functions [6]. It is analogous from [5] and as represented in [2] that if $Re(\alpha-\beta) \geq -1/2$, then the generalized Hankel-Clifford transformations is an automorphism of $L_2(R_+)$ and the inverse generalized Hankel-Clifford transformations on $L_2(R_+)$ has the symmetric form

$$g(x) = (h_{1,\alpha,\beta} f)(x) = x^{-(\alpha+\beta)} \int_0^\infty \mathcal{J}_{\alpha,\beta}(xy)\, f(y)\, dy, \tag{1.3}$$

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$$t\left(x\right)=\left(h_{2,\alpha,\beta}p\right)\left(x\right)=\int\limits_{0}^{\infty}y^{-(\alpha+\beta)}\mathcal{J}_{\alpha,\beta}\left(xy\right)p\left(y\right)dy. \tag{1.4}$$

Let us take note here of some properties of Bessel functions that we shall use quite a few times in this work (see [4]).

**Definition 1.1.** *The behaviors of $J_{\alpha-\beta}$ near the origin and the infinity are from [8] as follows:*

$$J_{\alpha-\beta}(2x^{1/2})=O\Big(x^{1/2}\Big)^{\alpha-\beta} \tag{1.5}$$

*as $x\to 0+$.*

$$J_{\alpha-\beta}\left(2x^{1/2}\right)\approx\left(2\pi\right)x^{-1/4}\cos\left(2x^{1/2}-\frac{1}{2}\left(\alpha-\beta\right)\pi-\frac{1}{4}\pi\right)\sum_{m=0}^{\infty}\frac{\left(-1\right)^{m}\left(\alpha-\beta,2m\right)}{\left(4x^{1/2}\right)^{2m}}$$

$$-\sin\left(2x^{1/2}-\frac{1}{2}\left(\alpha-\beta\right)\pi-\frac{1}{4}\pi\right)\sum_{m=0}^{\infty}\frac{\left(-1\right)^{m}\left(\alpha-\beta,2m+1\right)}{\left(4x^{1/2}\right)^{2m+1}} \tag{1.6}$$

as, $x\to\infty$ where $(\alpha-\beta,k)$ is understood as in [4].

**Definition 1.2.** *The main differentiation formulas for $J_{\alpha-\beta}$ in [1] are:*

$$\frac{d}{dx}\left[x^{(\alpha-\beta)/2}J_{\alpha-\beta}\left(2\sqrt{x}\right)\right]=x^{(\alpha-\beta-1)/2}J_{\alpha-\beta-1}\left(2\sqrt{x}\right). \tag{1.7}$$

$$\frac{d}{dx}\left[x^{-(\alpha-\beta)/2}J_{\alpha-\beta}\left(2\sqrt{x}\right)\right]=-x^{-(\alpha-\beta+1)/2}J_{\alpha-\beta+1}\left(2\sqrt{x}\right). \tag{1.8}$$

$$x^{\alpha+\beta+1}\frac{d}{dx}\left[x^{-(\alpha-\beta)/2}J_{\alpha-\beta}\left(2\sqrt{x}\right)\right]=-x^{\alpha+\beta+1/2}J_{\alpha-\beta+1}\left(2\sqrt{x}\right) \tag{1.9}$$

for $x,y>0$.

**Definition 1.3.** *The generalized Kepinski type differential operator from [1] is defined as*

$$\Delta_{\alpha,\beta}=\Delta_{\alpha,\beta,x}=x^{-\alpha}Dx^{\alpha-\beta+1}Dx^{\beta}=xD^{2}+(\alpha-\beta+1)D+\alpha\beta x^{-1} \tag{1.10}$$

*where $\alpha-\beta\geq -1/2$ and $D=D_x=\frac{d}{dx}$.*

**Property 1.1.** *By combining* (1.7) *and* (1.8) *and* (1.10)*, it can be easily inferred*

$$\Delta_{\alpha,\beta}\mathcal{J}_{\alpha,\beta}(x)=-\mathcal{J}_{\alpha,\beta}(x) \tag{1.11}$$

**Property 1.2.** *The generalized Hankel-Clifford transforms can be extended to*

$$f\left(x\right)=\left(h_{1,\alpha,\beta}g\right)\left(x\right)=x^{-(\alpha+\beta)}\int\limits_{0}^{\infty}\mathcal{J}_{\alpha,\beta,m}\left(xy\right)g\left(y\right)dy, \tag{1.12}$$

*where $\mathcal{J}_{\alpha,\beta,m}\left(x\right)=x^{(\alpha+\beta)/2}J_{\alpha-\beta,m}\left(2\sqrt{x}\right)$ and $J_{\alpha-\beta,m}\left(2\sqrt{x}\right)$ being the truncated Bessel function of the first kind analogous to [2] and is represented as*

$$J_{\alpha-\beta,m}\left(2\sqrt{x}\right)=J_{\alpha-\beta}\left(2\sqrt{x}\right)-\sum_{k=0}^{m-1}\frac{(-1)^{k}(\sqrt{x})^{(\alpha-\beta+2k)}}{\Gamma(\alpha-\beta+k+1)k!}$$

*and the integral is taken in sense of* $L_2$.
*The generalized Hankel-Clifford transforms and its inverse will have a bounded operator in* $L_2(R_+)$ *from [9] and has been extended from [2] as:*

$$g(x) = x^{(-3\alpha+\beta-1)/2} \frac{d}{dx} x^{(3\alpha-\beta+1)/2} \int_0^\infty x^{(\alpha-\beta+1)} J_{\alpha-\beta+1,m+1}\left(2\sqrt{xy}\right) f(y)\, dy \tag{1.13}$$

*for* $x \in R_+; 1/2 - m < \operatorname{Re}(\alpha-\beta) < m + 1/2,\ m > 0$.

$$\mathcal{J}_{\alpha,\beta-1,m+1}(x) = x^{-1/2} J_{\alpha-\beta+1,m+1}\left(2\sqrt{x}\right). \tag{1.14}$$

**Property 1.3.** *Using the equivalent form of the [equation (7); 2], we get*

$$\frac{d}{dx}\left[x^{(\alpha-\beta+1)} J_{\alpha-\beta+1,m+1}\left(2\sqrt{x}\right)\right] = x^{(\alpha-\beta)+1/2} J_{\alpha-\beta,m}\left(2\sqrt{x}\right), \tag{1.15}$$

where $\operatorname{Re}(\alpha-\beta) < m + 1/2, m > 0$.
Then symmetric to formula [(8); 2] can be extended to

$$g_N(x) = x^{(-3\alpha+\beta-1)/2} \frac{d}{dx} x^{(3\alpha-\beta+1)/2} \int_{1/N}^{N} x^{(\alpha-\beta+1)} J_{\alpha-\beta+1,m+1}\left(2\sqrt{xy}\right) f(y)\, dy$$

$$= x^{-(\alpha+\beta)} \int_{1/N}^{N} \mathcal{J}_{\alpha,\beta,m}(xy)\, g(y)\, dy \tag{1.16}$$

In this paper, the range of the generalized Hankel-Clifford transformations on some spaces of functions has been described. The range of the generalized Hankel-Clifford transforms of compactly supported functions which are either square integrable (Paley-Wiener Theorem) or infinitely differentiable (Paley-Wiener-Schwartz Theorem) is also characterized.
One of the main tools of our next two theorems is the Plancherel's theorem for the generalized Hankel-Clifford transformations as proved in [10] can be represented as

$$\|h_{1,\alpha,\beta} g\|_2 = \|g\|_2 \tag{1.17}$$

where $\|g\|_p = \|g\|_{L_p(R_+)}, 1 \le p < \infty$ , that is valid only when $(\alpha-\beta) \ge -1/2$.
For complex $(\alpha-\beta)$, the Plancherel's equation (1.16) is replaced by the inequalities

$$C^{-1}\|g\|_2 \le \|h_{1,\alpha,\beta} g\|_2 \le C\|g\|_2, (\alpha-\beta) \ge -1/2 \tag{1.18}$$

where $C \in [1, \infty)$ is a constant independent of $g$.

## 2. Range of the generalized Hankel-Clifford transforms of rapid decreasing and square integrable functions

The range of the generalized Hankel-Clifford transforms of rapid decreasing and square integrable functions is described by the following:

**Theorem 2.1.** *Let* $y^m g(y) \in L_2(R_+)$ *for all* $m = 0, 1, 2, 3...$ .
*A function* $f(x)$ *be the generalized Hankel-Clifford transform* $\hbar_{1,\alpha,\beta}$ *of* $g(y)$*order* $\operatorname{Re}(\alpha-\beta) \ge -1/2$ *if and only if:*
*i)* $f(x)$ *is infinitely differentiable on* $R_+$.
*ii)* $\Delta^m_{\alpha,\beta,x} f(x)$, $m = 0, 1, 2, 3...$, *belong to* $L_2(R_+)$;
*iii)* $\Delta^m_{\alpha,\beta,x} f(x)$, $m = 0, 1, 2, 3...$, *tends to* 0 *as* $x$ *tends to* 0 *and to infinity;*

*iv)* $\Delta^m_{\alpha,\beta,x} f(x)$, $m = 0, 1, 2, 3...$ *tends to* 0 *as* $x$ *tends to infinity and are bounded at* 0.

Proof. Necessary:

i) Let $y^m g(y) \in L_2(R_+)$ for all $m = 0, 1, 2, 3...$ then $y^m g(y) \in L_1(R_+)$ for all $m = 0, 1, 2, 3....$

Let $f(x)$ be the generalized Hankel-Clifford transform $\hbar_{1,\alpha,\beta}$ of $g(y)$. Indeed, it is easily verified that ([2, 4]).

$$\frac{\partial^m}{\partial y^m}\left(y^{-\alpha-\beta}(xy)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left(2\sqrt{xy}\right)\right) = \sum_{j=0}^{m} a_j(\alpha)\, y^{-\left(\frac{\alpha+\beta+j}{2}\right)} y^{j-m} x^{\left(\frac{\alpha+\beta+j}{2}\right)} J_{\alpha-\beta_{-j}}\left(2\sqrt{xy}\right) \tag{2.1}$$

where the $a_j(\alpha)$ are constants depending on $\alpha$ only.

Considering

$$\begin{aligned} &D^k\left[x^{-\alpha}(xy)^{\left(\frac{\alpha+\beta}{2}\right)+j/2} J_{\alpha-\beta-j}\left(2\sqrt{xy}\right)\right] \\ &= y^{\alpha} D^k\left[(xy)^{-\left(\frac{\alpha-\beta}{2}\right)+j/2} J_{\alpha-\beta-j}\left(2\sqrt{xy}\right)\right] \\ &= (-1)^k y^{(\alpha+\beta+k)/2}\left[x^{-(\alpha-\beta-j+k)/2} J_{\alpha-\beta-j+k}\left(2\sqrt{xy}\right)\right] \end{aligned}$$

$$\left[(xy)^{-(\alpha-\beta-j+k)/2} J_{\alpha-\beta-j+k}\left(2\sqrt{xy}\right)\right] \sim \frac{1}{2^{(\alpha-\beta-j+k)/2}\Gamma\left(\left(\frac{\alpha-\beta-j+k}{2}\right)+1\right)} \quad \text{as } x \to 0^+$$

$$= O\left[(xy)^{-\left(\frac{\alpha-\beta-j+k}{2}\right)-1/4} e^{2\sqrt{x}\left|\text{Im } \sqrt{y}\right|}\right] \quad \text{as } x \to \infty.$$

It follows that

$$\gamma^{a,\alpha}_{m,k}\left(y^{(-\alpha-\beta)/2} x^{(\alpha-\beta-j)/2} J_{\alpha-\beta-j}\left(2\sqrt{xy}\right)\right) < \infty.$$

Therefore

$$\begin{aligned} &\gamma^{a,\alpha}_{m,k}\left[\frac{\partial^m}{\partial y^m}\left\{y^{(-\alpha-\beta)}(xy)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left(2\sqrt{xy}\right)\right\}\right] \\ &\quad \le \sum_{j=0}^{m} |a_j(\alpha)|\, |y|^{\frac{j}{2}-m}\ \gamma^{a,\alpha}_{m,k}\left(y^{(-\alpha-\beta)/2} x^{(\alpha-\beta-j)/2} J_{\alpha-\beta-j}\left(2\sqrt{xy}\right)\right) < \infty. \end{aligned}$$

for a fixed $y \in \Omega$.

ii) Since $x^{\left(\frac{\alpha+\beta}{2}\right)} J_{\alpha-\beta}\left(2\sqrt{x}\right)$ is the solution of differential equation by Malgonde and Lakshmi Gorty in [8]

$$f''(x) + (1-\alpha-\beta)\, x^{-1} f'(x) + \left(\alpha\beta x^{-2} + 1\right) f(x) = 0. \tag{2.2}$$

Therefore

$$\Delta^m_{\alpha,\beta,x}\left\{x^{-\alpha-\beta}\mathcal{J}_{\alpha,\beta}(xy)\right\} = (-y)^m\left\{x^{-\alpha-\beta}\mathcal{J}_{\alpha,\beta}\left(xy\right)\right\}. \tag{2.3}$$

Consequently

$$\Delta^m_{\alpha,\beta,x}\left\{x^{-\alpha-\beta}\mathcal{J}_{\alpha,\beta}(xy)\right\} = (-1)^m\int\limits_0^\infty x^{-\alpha-\beta}\mathcal{J}_{\alpha,\beta}\left(xy\right)y^m g\left(y\right)dy, \tag{2.4}$$

with $(\alpha-\beta) > -1/2$. Plancherel's inequality gives $y^m g\left(y\right)\in L_2\left(R_+\right)$, and $\Delta^m_{\alpha,\beta,x}\left\{x^{-\alpha-\beta}\mathcal{J}_{\alpha,\beta}\left(xy\right)\right\},\ (\alpha-\beta)\geq -1/2,\ m = 0,1,2,3,...\in L_2\left(R_+\right).$

iii) For the kernel $x^{-\alpha-\beta}J_{\alpha,\beta}\left(xy\right)$ has asymptotes $x^{(\alpha-\beta+1)/2}$ as $x$ tends to 0, is uniformly bounded on $(0,\infty)$ if $(\alpha-\beta)\geq -1/2$ and $y^m g\left(y\right)\in L_1\left(0,\infty\right)$,
then applying dominated convergence theorem,

$$\lim_{x\to\infty}\left[\Delta^m_{\alpha,\beta,x}\left\{f\left(x\right)\right\}\right] = (-1)^m\int\limits_0^\infty x^{-\alpha-\beta}\mathcal{J}_{\alpha,\beta}\left(xy\right)y^m g\left(y\right)dy = 0. \tag{2.5}$$

$(\alpha-\beta)\geq -1/2$.
For every $\varepsilon > 0$ one can choose large enough so that

$$\left|\int\limits_N^\infty x^{-\alpha-\beta}\mathcal{J}_{\alpha,\beta}\left(xy\right)y^m g\left(y\right)dy\right| < \varepsilon. \tag{2.6}$$

uniformly with respect to $x\in R_+$.
By applying the generalized Riemann-Lebesgue theorem,

$$\lim_{x\to\infty}\int\limits_0^N x^{-\alpha-\beta}\mathcal{J}_{\alpha,\beta}\left(xy\right)y^m g\left(y\right)dy = 0, \tag{2.7}$$

$0 < N < \infty, (\alpha-\beta)\geq -1/2$.

Because $\varepsilon$ can be taken arbitrarily small,

$$\lim_{x\to\infty}\int\limits_0^\infty x^{-\alpha-\beta}\mathcal{J}_{\alpha,\beta}\left(xy\right)y^m g\left(y\right)dy = 0, \tag{2.8}$$

$0\leq N\leq\infty, (\alpha-\beta)\geq -1/2$.
Hence

$$\lim_{x\to\infty}\left[\Delta^m_{\alpha,\beta,x}\left\{f\left(x\right)\right\}\right] = 0,\ m = 0,1,2,...,\ (\alpha-\beta)\geq -1/2. \tag{2.9}$$

iv) Using (1.6), we get

$$(-1)^m\frac{d}{dx}\left[\Delta^m_{\alpha,\beta,x}f\left(x\right)\right] = \int\limits_0^\infty x^{(\alpha-\beta-1)/2}J_{\alpha-\beta-1}\left(2\sqrt{x}\right)y^m g\left(y\right)dy. \tag{2.10}$$

From (2.5) and (2.8) of (iii), we can state that the right hand side of (2.10) tends to zero as $x$ tends to infinity. Since $x^{(\alpha-\beta-1)/2}J_{\alpha-\beta-1}$ is uniformly bounded on $R_+$, therefore the right hand side of (2.10) is also uniformly bounded.
Sufficiency:
If $f\left(x\right)$ satisfies the conditions i) to iv) of the theorem 2.1.

Then $\Delta^m_{\alpha,\beta,x} f(x)$, $m = 0, 1, 2, 3...$, belong to $L_2(R_+)$.
Let $g_m(y)$ be its generalized Hankel-Clifford transform:

$$g_m(y) = \int_0^\infty x^{-\alpha-\beta} \mathcal{J}_{\alpha,\beta}(xy)\, \Delta^m_{\alpha,\beta,x} f(x)\, dx;\ m = 0, 1, 2, 3, ... \tag{2.11}$$

$(\alpha - \beta) \geq -1/2$.
Since

$$g^N_m(y) = \int_{1/N}^N x^{-\alpha-\beta} \mathcal{J}_{\alpha,\beta}(xy)\, \Delta^m_{\alpha,\beta,x} f(x)\, dx;\ m = 0, 1, 2, 3, ... \tag{2.12}$$

$(\alpha - \beta) \geq -1/2$.
Here $g^N_m(y) \to g_m(y)$ in $L_2$ norm as $N \to \infty$.

Integrating (2.12) by parts twice,

$$\begin{aligned} g^N_m(y) &= x^{-\alpha-\beta} \mathcal{J}_{\alpha,\beta}(xy) \frac{d}{dx} \Delta^{m-1}_{\alpha,\beta,x}\ f(x)\Big|^N_{1/N} \\ &\quad - \frac{\partial}{\partial x} \left\{ x^{-\alpha-\beta} \mathcal{J}_{\alpha,\beta}(xy) \right\} \Delta^{m-1}_{\alpha,\beta,x}\ f(x)\Big|^N_{1/N} \\ &\quad + \int_{1/N}^N \Delta_{\alpha,\beta,x} \left\{ x^{-\alpha-\beta} \mathcal{J}_{\alpha,\beta}(xy) \right\} \Delta^{m-1}_{\alpha,\beta,x} f(x)\, dx. \end{aligned} \tag{2.13}$$

$$\begin{aligned} &g^N_m(y) \\ &= N^{-\alpha-\beta} \mathcal{J}_{\alpha,\beta}(Ny) \frac{d}{dx} \Delta^{m-1}_{\alpha,\beta,x} f(N) - N^{-\alpha-\beta-1} \mathcal{J}_{\alpha,\beta}\left(N^{-1} y\right) \frac{d}{dx} \Delta^{m-1}_{\alpha,\beta,x} f\left(N^{-1}\right) \\ &\quad + (\alpha+\beta) N^{-\alpha-\beta-1} \left\{ N^{-\alpha-\beta} \mathcal{J}_{\alpha,\beta}(Ny) \right\} \Delta^{m-1}_{\alpha,\beta,x} f(N) \\ &\quad - N^{-\alpha-\beta} N^{-\alpha-\beta-1} \left\{ \mathcal{J}_{\alpha,\beta-1}(Ny) \right\} \Delta^{m-1}_{\alpha,\beta,x} f(N) \\ &\quad - (\alpha+\beta) N^{-\alpha-\beta-2} \left\{ N^{-\alpha-\beta-1} \mathcal{J}_{\alpha,\beta}\left(N^{-1} y\right) \right\} \Delta^{m-1}_{\alpha,\beta,x} f\left(N^{-1}\right) \\ &\quad + N^{-\alpha-\beta-1} N^{-\alpha-\beta-2} \left\{ \mathcal{J}_{\alpha,\beta-1}\left(N^{-1} y\right) \right\} \Delta^{m-1}_{\alpha,\beta,x} f\left(N^{-1}\right) \\ &\quad + \int_{1/N}^N \Delta_{\alpha,\beta,x} \left\{ x^{-\alpha-\beta} \mathcal{J}_{\alpha,\beta}(xy) \right\} \Delta^{m-1}_{\alpha,\beta,x} f(x)\, dx. \end{aligned} \tag{2.14}$$

The following can be concluded:
a) $x^{-\alpha-\beta} \mathcal{J}_{\alpha,\beta}(xy)$ is uniformly bounded and $\frac{d}{dx} \Delta^{m-1}_{\alpha,\beta,x} f(N) \to 0$ as $N \to \infty$.
b) $\frac{d}{dx} \Delta^{m-1}_{\alpha,\beta,x} f\left(N^{-1}\right)$ is bounded, whereas $N^{-\alpha-\beta-1} \mathcal{J}_{\alpha,\beta}\left(N^{-1} y\right)$ has an order $O\left(N^{(-\alpha-\beta-1)/2}\right)$ is $\infty$.
c) $(\alpha+\beta) N^{-\alpha-\beta-1} \left\{ N^{-\alpha-\beta} \mathcal{J}_{\alpha,\beta}(Ny) \right\}$ and $\Delta^{m-1}_{\alpha,\beta,x} f(N)$ is of $O(1)$ .
d) $N^{-\alpha-\beta} N^{-\alpha-\beta-1} \left\{ \mathcal{J}_{\alpha,\beta-1}(Ny) \right\}$ and $\Delta^{m-1}_{\alpha,\beta,x} f(N)$ is of $O(1)$, tends to zero as $N \to \infty$.
e) $(\alpha+\beta) N^{-\alpha-\beta-2} \left\{ N^{-\alpha-\beta-1} \mathcal{J}_{\alpha,\beta}\left(N^{-1} y\right) \right\}$ and $\Delta^{m-1}_{\alpha,\beta,x} f\left(N^{-1}\right)$ is of $O(1)$, tends to zero as $N \to \infty$.
f) $\int_{1/N}^N \Delta_{\alpha,\beta,x} \left\{ x^{-\alpha-\beta} \mathcal{J}_{\alpha,\beta}(xy) \right\} \Delta^{m-1}_{\alpha,\beta,x} f(x)\, dx$ converges to $(-y)\, g_{m-1}(y)$ as $N \to \infty$.

Hence $g_m(y) = (-y)\, g_{m-1}(y)$, therefore $g_m(y) = (-y)^m g_0(y)$, $m = 0, 1, 2, \ldots$. But $f$ is the generalized Hankel-Clifford transforms of $g$. Thus $f(x)$is the generalized Hankel-Clifford transforms of the function $g(y) = g_0(y)$ such that $y^m g(y) \in L_2(R_+)$, $n = 0, 1, 2, \ldots$ and theorem 2.1 is thus proved.

## 3. Generalized Hankel-Clifford transform of infinitely differentiable functions with compact supports

**Theorem 3.1.** *(Paley-Wiener theorem for the generalized Hankel-Clifford transforms of square integrable functions with compact supports) A function $f$ is the generalized Hankel-Clifford transforms of a square integrable function $g$ with compact support on $[0, \infty)$ if and only if $f$ satisfies conditions i)-iv) of Theorem 2.1 and*

$$\lim_{n\to\infty} \left\|\Delta^m_{\alpha,\beta,x} f(x)\right\|_2^{1/2m} = \sigma_g < \infty, \tag{3.1}$$

*where $\sigma_g = \sup\{y : y \in supp\ \ g\}$ and the support of a function is the smallest closed set, outside it the function vanishes almost everywhere.*

Proof. Necessary: Let $f(x)$ be the generalized Hankel-Clifford transforms of $g(y) \in L_2(R_+)$ and assuming $\sigma_g > 0$ and $\sigma_g < \infty$:

$$f(x) = x^{-(\alpha+\beta)} \int_0^{\sigma_g} \mathcal{J}_{\alpha,\beta}(xy)\, g(y)\, dy \tag{3.2}$$

$y^m g(y) \in L_2(R_+)$, $\forall m = 0, 1, 2, \ldots, f$ satisfies conditions i)-iv) of theorem 2.1.
Invoking the right side of the inequality (1.17) in (3.2), we get:

$$\left\|\Delta^m_{\alpha,\beta,x} f(x)\right\|_2^2 \le C \int_0^{\sigma_g} y^{2m} |g(y)|^2 dy \ \le\ C \int_0^{\sigma_g} \sigma_g^{2m} |g(y)|^2 dy. \tag{3.3}$$

Hence

$$\overline{\lim_{m\to\infty}} \left\|\Delta^m_{\alpha,\beta,x} f(x)\right\|_2^{1/2m} \le \overline{\lim_{m\to\infty}} C^{1/2m} \sigma_g \left\{ \int_0^{\sigma_g} |g(y)|^2 dy \right\}^{1/2m} = \sigma_g. \tag{3.4}$$

Since $\sigma_g$ is the least upper bound of the support of $g$, for every $\varepsilon$, $0 < \varepsilon < \sigma_g$, gives $\int\limits_{\sigma_g-\varepsilon}^{\sigma_g} |g(y)|^2 dy > 0$.
Consequently left side of the inequality in (1.17), gives

$$\underline{\lim_{m\to\infty}} \left\|\Delta^m_{\alpha,\beta,x} f(x)\right\|_2^{1/2m} \ge \underline{\lim_{m\to\infty}} C^{-1/2m} (\sigma_g - \varepsilon) \left\{ \int_{\sigma_g-\varepsilon}^{\sigma_g} |g(y)|^2 dy \right\}^{1/2m} = \sigma_g - \varepsilon. \tag{3.5}$$

Sufficient:
Suppose now that $f$ satisfies the conditions i)-iv) of theorem 2.1, and the limit in (3.1) exists and equals $\sigma < \infty$.
Using theorem 2.1, $f$ is the generalized Hankel-Clifford transforms of a function $g$ such that $y^m g(y) \in L_2(R_+)$, $\forall m = 0, 1, 2, \ldots$. It is to be proved that $\sigma < \infty$ and

$\sigma = \sigma_g$. From theorem 2.1 it is observed that (2.4) is valid. Therefore, applying the inequalities (1.17) it is obtained as:

$$C^{-1}\|y^m g(y)\|_2 \le \left\|\Delta^m_{\alpha,\beta,x} f(x)\right\|_2 \le C\|y^m g(y)\|_2. \tag{3.6}$$

Hence

$$\lim_{m\to\infty} C^{-1}\|y^m g(y)\|_2 \le \lim_{m\to\infty} \left\|\Delta^m_{\alpha,\beta,x} f(x)\right\|_2 \le \lim_{m\to\infty} C\|y^m g(y)\|_2 \le \lim_{m\to\infty} C\|\sigma^m g(y)\|_2. \tag{3.7}$$

Consequently

$$\lim_{m\to\infty} \|y^m g(y)\|_2^{1/2m} = \sigma. \tag{3.8}$$

Suppose that $\sigma_g > \sigma$. Then there exists a positive $\varepsilon$ such that

$$\int_{\sigma_g+\varepsilon}^{\infty} |g(y)|^2 dy > 0. \tag{3.9}$$

Then

$$\sigma = \lim_{m\to\infty} \|y^m g(y)\|_2^{1/2m} \ge \lim_{m\to\infty} \left\{ \int_{\sigma+\varepsilon}^{\infty} y^{2m} |g(y)|^2 dy \right\}^{1/2m}$$

$$\ge \lim_{m\to\infty} (\sigma+\varepsilon) \left\{ \int_{\sigma+\varepsilon}^{\infty} |g(y)|^2 dy \right\}^{1/2m} = \sigma+\varepsilon. \tag{3.10}$$

which is impossible. Hence $\sigma_g \le \sigma$ and $g$ has a compact support.

Suppose that $\sigma_g < \sigma$ . Then there exists a positive $\varepsilon$ such that $\int_0^{\sigma-\varepsilon} |g(y)|^2 dy > 0$. Thus

$$\sigma = \overline{\lim_{m\to\infty}} \|y^m g(y)\|_2^{1/2m} \le \overline{\lim_{m\to\infty}} \left\{ \int_0^{\sigma-\varepsilon} y^{2m} |g(y)|^2 dy \right\}^{1/2m}$$

$$\le \overline{\lim_{m\to\infty}} (\sigma-\varepsilon) \left\{ \int_0^{\sigma-\varepsilon} |g(y)|^2 dy \right\}^{1/2m} = \sigma-\varepsilon. \tag{3.11}$$

which is impossible. Hence $\sigma_g \ge \sigma$ and thus $\sigma = \sigma_g$. Thus the theorem is proved.

## 4. Generalized Erdelyi-Kober fractional integral operator

Let the generalized Erdelyi-Kober fractional integral operator as defined by [7]

$$h(x) = (K_{\alpha,\beta} g_1)(x) = \frac{2^{(\alpha+\beta)/2}}{\Gamma\left(\frac{\alpha-\beta+1}{2}\right)} \int_x^{\infty} \left(y^2 - x^2\right)^{\left(\frac{\alpha+\beta-1}{2}\right)} y\, g_1(y)\, dy; \tag{4.1}$$

where $\mathrm{Re}(\alpha-\beta) > 0; x \in R$.

**Theorem 4.1.** *(Paley-Wiener-Schwartz theorem for generalized Hankel-Clifford transform of infinitely differentiable functions with compact supports) A function $f \in h_{1,\alpha,\beta}$ is the generalized Hankel-Clifford transform for $(\alpha-\beta) \geq -1/2$ of a function $g \in h_{1,\alpha,\beta}$ with compact support if and only if*

$$\lim_{m\to\infty} \left\| \frac{d^m}{dx^m} x^{\alpha} f(x) \right\|_p^{1/m} = \sigma_g, 1 \leq p \leq \infty. \tag{4.2}$$

Proof. The integral representation of generalized Hankel-Clifford function $J_{\alpha,\beta}(xy)$ analogously can be written as [3],

$$J_{\alpha,\beta}(x) = \frac{2^{(1+\alpha+\beta)/2} x^{-(\alpha+\beta)/2} y^{(\alpha+\beta)/2}}{\sqrt{\pi}\Gamma\left(\frac{\alpha-\beta+1}{2}\right)} \int_0^1 \left(1-t^2\right)^{\left(\frac{\alpha+\beta-1}{2}\right)} \cos\left(2t\sqrt{x}\right) dt, \tag{4.3}$$

$\mathrm{Re}(\alpha-\beta) \geq -1/2$. Substituting $x$ by $xy^2$ and $t$ by $t/y$, it gives

$$J_{\alpha,\beta}(xy) = \frac{2^{(1+\alpha+\beta)/2} x^{-(\alpha+\beta)/2} y^{3(\alpha+\beta)/2}}{\sqrt{\pi}\Gamma\left(\frac{\alpha-\beta+1}{2}\right)} \int_0^y \left(y^2-t^2\right)^{\left(\frac{\alpha+\beta-1}{2}\right)} y^{-(\alpha+\beta-1)} \cos\left(2t\sqrt{x}\right) dt. \tag{4.4}$$

The generalized Hankel-Clifford transform can be rewritten as

$$f(x) = \frac{2^{(1+\alpha+\beta)/2} x^{-(\alpha+\beta)/2} y^{3(\alpha+\beta)/2}}{\sqrt{\pi}\Gamma\left(\frac{\alpha-\beta+1}{2}\right)} \int_0^\infty y^{-(\alpha+\beta-1)} g(y) \int_0^y \left(y^2-t^2\right)^{\left(\frac{\alpha+\beta-1}{2}\right)} \cos\left(2t\sqrt{x}\right) dtdy. \tag{4.5}$$

If $y^{-(\alpha+\beta)/2} g(y) \in L_1(R_+)$, then the repeated integral (4.5) converges absolutely. Therefore, Fubini-Tonelli theorem [5 ] is applied to interchange the order of integration in (4.5);

$$f(x) = \frac{2^{(1+\alpha+\beta)/2} x^{-(\alpha+\beta)/2} y^{(\alpha+\beta)/2}}{\sqrt{\pi}\Gamma\left(\frac{\alpha-\beta+1}{2}\right)} \int_0^\infty \cos\left(2t\sqrt{x}\right) dt \int_t^\infty \left(y^2-t^2\right)^{\left(\frac{\alpha+\beta-1}{2}\right)} y\, g(y)\, dy. \tag{4.6}$$

Considering $f_1(x) = x^{(\alpha+\beta)/2} f(x)$ and $g_1(y) = y^{-(\alpha+\beta)/2} g(y)$,

$$f_1(x) = \frac{2^{(1+\alpha+\beta)/2}}{\sqrt{\pi}\Gamma\left(\frac{\alpha-\beta+1}{2}\right)} \int_0^\infty \cos\left(2t\sqrt{x}\right) dt \int_t^\infty \left(y^2-t^2\right)^{\left(\frac{\alpha+\beta-1}{2}\right)} y\, g_1(y)\, dy. \tag{4.7}$$

Therefore $f_1(x)$ can be viewed as composition of the Fourier cosine transform

$$f_1(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \cos\left(2t\sqrt{x}\right) h(t)\, dt,\ 0 \leq x < \infty, \tag{4.8}$$

where

$$h(t) = \frac{2^{(\alpha+\beta)/2}}{\Gamma\left(\frac{\alpha-\beta+1}{2}\right)} \int_t^\infty \left(y^2-t^2\right)^{\left(\frac{\alpha+\beta-1}{2}\right)} y\, g_1(y)\, dy. \tag{4.9}$$

and the generalized Erdelyi-Kober fractional integral operator (4.1) $K^{(\alpha-\beta+1)/2}$ of order $(\alpha-\beta+1)/2$ multiplied by a constant.

It is from the definition that $\hat{f} \in S(R)$ is the Fourier transform of an infinitely differentiable function $f$ on $R$ with compact support if and only if

$$\sigma_{|f|} = \lim_{m\to\infty} \left\| \frac{d^m}{dx^m} \hat{f}(x) \right\|_{L_p(R)}^{1/m}, 1 \le p < \infty, \tag{4.10}$$

where $\sigma_{|f|} = \sup\{|y| : y \in \operatorname{supp} \ f\}$.
Restricting the Fourier transform only on even functions it is observed that a function $\hat{f} \in S_e(R)$ is the Fourier cosine transform (4.8) of a function $h \in S_e$ with compact support if and only if

$$\sigma_h = \lim_{m\to\infty} \left\| \frac{d^m}{dx^m} f_1(x) \right\|_p^{1/m}. \tag{4.11}$$

On the other hand, the Erdelyi-Kober fractional integral operator $K^{(\alpha-\beta+1)/2}$is a bijection in the space of infinitely differentiable functions on $\overline{R_+}$ with compact supports and $\sigma_h = \sigma_{g_1}$. From $g_1(y) = y^{-(\alpha+\beta)/2} g(y)$it is obtained that $\sigma_g = \sigma_{g_1}$, theorem 4.1 follows now from formula (4.7).

## 5. Conclusion

1. The Paley-Wiener theorem for the generalized Hankel-Clifford is obtained.
2. The generalized Hankel-Clifford of square integrable functions with compact supports, rapid decreasing functions, infinitely differentiable functions with compact supports, of analytic functions are studied.
3. The range of the generalized Hankel-Clifford transform of compactly supported functions which are either square integrable or infinitely differentiable is characterized.
4. The study leads to application in Mathematical Physics.

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## References

[1] S. P. Malgonde, *Generalized Hankel-Clifford transformation of certain spaces of distributions*, Rev. Acad. Canaria Cienc. **XII (1-2)**, (2000), 51–73.
[2] V. U. Kim Tuan, *On the range of the Hankel and extended Hankel transforms*, Journal of Mathematical Analysis and Applications, **209 (2)**, (1997), 460–478.
[3] I. N. Sneddon, *The use of integral transforms*, Tata McGraw-Hill, New Delhi, 1979.
[4] G. N. Watson, *A treatise on the theory of Bessel functions*, Cambridge Univ. Press, London, 1958.
[5] E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals*, Chelsea Publishing Company, New York, 1986.
[6] A. H. Zemanian, *Generalized integral transformations*, Interscience Publishers, New York, 1968.
[7] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications*, Gordon and Breach, Switzerland, Australia, Belgium, France, Germany, Great Britain, 1993.
[8] S. P. Malgonde and V. R. Lakshmi Gorty, *A Parseval equation and the distributional finite generalized Hankel-Clifford transformation*, August (2010), ICM 2010, Hyderabad India.
[9] P. Heywood and P. G. Rooney, *On the inversion of the extended Hankel transformation*, J. Math. Anal. Appl. **160 (1)**, (1991), 284–302.

[10] S. P. Malgonde and V. R. Lakshmi Gorty, *Two complex variants of generalized Hankel-Clifford transformation*, The Journal of the Indian Academy of Mathematics, **34 (2)**, (2012), 599–614.

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