Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://bmathaa.org
Volume 10 Issue 2(2018), Pages 19-30.

STABILITY OF A LINEAR INTEGRO-DIFFERENTIAL EQUATION OF FIRST ORDER WITH VARIABLE DELAYS

CEMİL TUNÇ, İREM AKBULUT

Abstract. We consider a linear integro-differential equation (IDE) of first order with two variable delays. We construct new conditions guaranteeing the trivial solution of this IDE is stable. The technique of the proof based on the use of the fixed-point theory. Our findings generalize and improve some results can be found in the literature.

1. Introduction

It is well known from the related literature that the IDEs are often used to model some practical problems in mechanics, physics, biology, ecology and the other scientific fields (see, e.g. Burton [5], Levin [10], Rahman [14], Wazwaz [24] and their references). This paper deals with the following linear IDE of the first order with two variable delays

$$\frac{dx}{dt} = -b(t)x - \sum_{i=1}^{2} \int_{t-r_i(t)}^{t} a(t,s)x(s)ds, \tag{1.1}$$

where $t \geq 0, t \in \Re,\ x \in \Re,\ \Re = (-\infty, \infty),\ r_i = [0,\infty) \to [0,\infty), (i = 1,2)$, are continuous differentiable functions such that $t - r_i(t) \to \infty$ as $t \to \infty$ with $r_0 = \max_{t\geq 0}\{t - r_1(t), t - r_2(t)\}$, $a : [-r_0,\infty) \times [-r_0,\infty) \to \Re$ and $b : (0,\infty) \to \Re$ are continuous functions. In the past decades, a number of researches have dealt with qualitative behaviors of linear and non-linear IDEs by means of the fixed point theory, the perturbation methods, the variations of parameters formulas, the Lyapunov's function or functional method, etc., (see Ardjouni and Djoudi [1], Becker and Burton [2], Burton ([3], [4]), Gabsi et al. [6], Gözen and Tunç [7], Graef and Tunç [8], Jin and Luo [9], Levin [10], Pi ([11], [12]), Raffoul [13], Tunç ([15], [16],[17],[18], [19]), Tunç and Mohammed [20], Tunç and Tunç ([21], [22],[23]) and their references). Among these investigations, the stability analysis of solutions has been an important topic for IDEs with constant or variable delay and without delay. For various models and kinds of IDEs, many significant results have been presented, see, for example, the references of this article and those registered therein. Here,

---

2000 *Mathematics Subject Classification.* 34K20, 45J05.
Key words and phrases. Contraction mapping, stability, fixed point, integro-differential equation, variable delays, first order.

Submitted April 4, 2018. Published May 15, 2018.
Communicate by Tongxing Li.

we would not give the details of the works and their applications. However, we would like to summarize here a few related results on the topic.

First, in 1963, Levin [10] considered the linear IDE

$$\frac{dx}{dt} = -\int_0^t a(t-s)g(x(s))ds$$

and the author gave sufficient conditions guaranteeing that if any solution of the above IDE exists on $[0, \infty)$, then the solution of this IDE, its first and second order derivatives tends to zero when $t \to \infty$. The proof of the main result of [10] involves the use of a suitably chosen Lyapunov function.

Later, in 2004, Burton [4] took into consideration the following non-linear IDE with constant delay of the form

$$\frac{dx}{dt} = -\int_{t-r}^t a(t,s)g(x(s))ds.$$

In [4], instead of using a Lyapunov functional, the author studied asymptotic stability of the above IDE by using the concept of a contraction mapping in line with the fixed point theory for a class of equations which has been comprehensively studied in the last fifty years. The results of [4] look very interesting.

Finally, Jin and Luo [9] studied a scalar integro-differential equation (in both the linear and nonlinear cases) establishing sufficient conditions for the existence, stability and asymptotic stability for the null solutions. The IDEs examined by the authors are given by

$$\frac{dx}{dt} = -\int_{t-r(t)}^t a(t,s)x(s)ds$$

and

$$\frac{dx}{dt} = -\int_{t-r(t)}^t a(t,s)g(x(s))ds.$$

In [9], in order to have the possibility of applying Banach's fixed point theorem, first IDE is written in an equivalent form and if $x(t) = \psi(t)$ on $[-r_0, 0)$, it is shown that the solution $x(t)$ of first IDE is bounded on $[-r_0, \infty)$ and the trivial solution of the same IDE is also stable. Under an additional condition, it is proved for the second IDE that $x(t) \to 0$ as $t \to \infty$. The nonlinear case (the second IDE) looks somewhat more complicated but it admits a treatment similar to that of the former one.

The motivation to consider IDE (1.1) and investigation its some qualitative properties come from the papers of Levin [10], Burton [4], Jin and Luo [9] and the sources that found in the references of this article. Here, we give new results on the boundedness, stability, asymptotic stability and some other properties of solutions of IDE (1.1). The considered IDE, the results and assumptions to be given here are different from that can be found in the literature and complete that ones. These are the contributions of this paper to the literature and its novelty and originality.

The following definition may be useful for readers.

Definition 1. The zero solution of IDE (1.1) is said to be stable at $t = 0$ if, for every, $\varepsilon > 0$, there exists a $\delta > 0$ such that $\psi : [-r_0, 0] \to (-\delta, \delta)$ implies that $|x(t)| < \varepsilon$ for $t \geq -r_0$.

We can write IDE (1.1) as

$$\frac{dx}{dt}+b(t)x=\sum_{i=1}^{2}G(t,t-r_i(t))(1-r_i'(t))x(t-r_i(t)) + \sum_{i=1}^{2}\frac{d}{dt}\int_{t-r_i(t)}^{t}G(t,s)x(s)ds, \tag{1.2}$$

where

$$G(t,s)=\int_t^s a(u,s)du \ with \ G(t,t-r_i(t))=\int_t^{t-r_i(t)} a(u,t-r_i(t))du. \tag{1.3}$$

Theorem 1. If $x(t)$ is a solution of IDE (1.1) on an interval $[0,T)$ and satisfies the initial condition $x(t)=\psi(t)$ for $t\in[-r_0,0]$, then $x(t)$ is a solution of IE

$$\begin{aligned}x(t) =&exp(-\int_0^t b(s)ds)\psi(0)-\exp(-\int_0^t b(u)du)\sum_{i=1}^{2}\int_{-r_i(0)}^{0}[b(u)+G(0,u)]\psi(u)du\\&+\sum_{i=1}^{2}\int_{t-r_i(t)}^{t}[b(u)+G(t,u)]x(u)du\\&-\int_0^t exp(-\int_s^t b(u)du)b(s)\sum_{i=1}^{2}\int_{s-r_i(s)}^{s}[b(u)+G(s,u)]x(u)duds\\&+\sum_{i=1}^{2}exp(-\int_s^t b(u)du)[b(s-r_i(s)+G(s,s-r_i(s))](1-r_i'(s))x(s-r_i(s))ds\\&-2\int_0^t exp(-\int_s^t b(u)du)b(s)x(s)ds\end{aligned} \tag{1.4}$$

on $[0,T)$, where $b:[-r_0,\infty)\to\Re$ is an arbitrary continuous function. Conversely, if a continuous function $x(t)$ is equal to $\psi(t)$ for $t\in[-r_0,0]$ and is a solution of IE (1.4) on an interval $[0,\tau)$, then $x(t)$ is a solution of IDE (1.1) on $[0,\tau)$.

Proof. Multiplying both sides of IDE (1.2) by the factor $exp(\int_0^t b(u)du)$ and integrating from 0 to any t, $t\in[0,\tau)$, then we get

$$\begin{aligned}&x'(t)exp(\int_0^t b(u)du)+exp(\int_0^t b(u)du)b(t)x(t)\\&=exp(\int_0^t b(u)du)\sum_{i=1}^{2}G(t,t-r_i(t))(1-r_i'(t))x(t-r_i(t))\\&+exp(\int_0^t b(u)du)\sum_{i=1}^{2}\frac{d}{dt}\int_{t-r_i(t)}^{t}G(t,s)x(s)ds\end{aligned}$$

so that

$$\begin{aligned}\frac{d}{dt}\int_0^t\left(x(s)exp(\int_0^s b(u)du)\right)=&\int_0^t[\sum_{i=1}^{2}G(s,s-r_i(s))(1-r_i'(s))x(s-r_i(s))\\&+\sum_{i=1}^{2}\frac{d}{ds}\int_{s-r_i(s)}^{s}G(s,u)x(u)du]exp(\int_0^s b(u)du).\end{aligned}$$

Hence

$$\begin{aligned} x(t)exp(\int_0^t b(u)du) - \psi(0)exp(\int_0^t b(u)du)& \\ &= \int_0^t [\sum_{i=1}^{2} G(s, s-r_i(s))(1-r_i'(s))x(s-r_i(s)) \\ &+ \sum_{i=1}^{2} \frac{d}{ds}\int_{s-r_i(s)}^{s} G(s,u)x(u)du]exp(\int_0^s b(u)du) \end{aligned}$$

so that

$$\begin{aligned} x(t) =&exp(-\int_0^t b(u)du)\psi(0) + \int_0^t [\sum_{i=1}^{2} G(s, s-r_i(s))(1-r_i'(s))x(s-r_i(s)) \\ &+ \sum_{i=1}^{2} \frac{d}{ds}\int_{s-r_i(s)}^{s} G(s,u)x(u)du]exp(\int_0^s b(u)du). \end{aligned} \tag{1.5}$$

Hence, we can write that

$$\begin{aligned} x(t) =&exp(-\int_0^t b(s)ds)\psi(0) \\ &+ \sum_{i=1}^{2}\int_0^t exp(-\int_s^t b(u)du)\frac{d}{ds}\int_{s-r_i(s)}^{s}[b(u)+G(s,u)]x(u)duds \\ &+ \sum_{i=1}^{2}\int_0^t exp(-\int_s^t b(u)du)[b(s-r_i(s))+G(s,s-r_i(s))](1-r_i'(s))x(s-r_i(s))ds \\ &- 2\int_0^t exp(-\int_s^t b(u)du)b(s)x(s)ds. \end{aligned} \tag{1.6}$$

We will now show that estimate (1.6) is equal to estimate (1.5).
In fact, it follows from (1.6) that

$$\begin{aligned} x(t) =&exp(-\int_0^t b(s)ds)\psi(0) + \sum_{i=1}^{2}\int_0^t exp(-\int_s^t b(u)du)\frac{d}{ds}\int_{s-r_i(s)}^{s} b(u)x(u)duds \\ &+ \sum_{i=1}^{2}\int_0^t exp(-\int_s^t b(u)du)\frac{d}{ds}\int_{s-r_i(s)}^{s} G(s,u)x(u)duds \\ &+ \sum_{i=1}^{2}\int_0^t exp(-\int_s^t b(u)du)b(s-r_i(s))(1-r_i'(s))x(s-r_i(s))ds \\ &+ \sum_{i=1}^{2}\int_0^t exp(-\int_s^t b(u)du)G(s-r_i(s))(1-r_i'(s))x(s-r_i(s))ds \\ &- 2\int_0^t exp(-\int_s^t b(u)du)b(s)x(s)ds. \end{aligned}$$

Applying integration by part to the second and third terms of (1.6), we get

$$\begin{aligned}
x(t) =&exp(-\int_0^t b(s)ds)\psi(0)\\
&-\sum_{i=1}^{2} exp(-\int_s^t b(s)ds)\int_{s-r_i(s)}^{s}[b(u)+G(s,u)]x(u)du|_0^t\\
&-\sum_{i=1}^{2} exp(-\int_s^t b(u)du)b(s)\int_{s-r_i(s)}^{s}[b(u)+G(s,u)]x(u)duds\\
&+\sum_{i=1}^{2} exp(-\int_s^t b(u)du)[b(s-r_i(s))+G(s,s-r_i(s))](1-r_i'(s))x(s-r_i(s))ds\\
&-2\int_0^t exp(-\int_s^t b(u)du)b(s)x(s)ds
\end{aligned}$$

$$\begin{aligned}
=&exp(-\int_0^t b(s)ds)\psi(0)\\
&-\sum_{i=1}^{2} exp(-\int_s^t b(s)ds)\int_{-r_i(0)}^{0}[b(u)+G(0,u)]x(u)du\\
&+\sum_{i=1}^{2}\int_{t-r_i(t)}^{t}[b(u)+G(t,u)]x(u)du\\
&-\sum_{i=1}^{2} exp(-\int_s^t b(u)du)b(s)\int_{s-r_i(s)}^{s}[b(u)+G(s,u)]x(u)duds\\
&+\sum_{i=1}^{2} exp(-\int_s^t b(u)du)[b(s-r_i(s))+G(s,s-r_i(s))](1-r_i'(s))x(s-r_i(s))ds\\
&-2\int_0^t exp(-\int_s^t b(u)du)b(s)x(s)ds,
\end{aligned}$$

which leads to (1.4).

Conversely, suppose that there exists a continuous function $x(t)$ which is equal to $\psi(t)$ on $[-r_0,0]$ and satisfies IE (1.4) on an interval $[0,\tau)$. Then, the function $x(t)$ is differentiable on $[0,\tau)$ and if we calculate the time derivative of IE (1.4) with the aid of Leibniz's rule, then we obtain IDE (1.2).

Next, we will define a mapping directly from (1.4). By Theorem 1, a fixed point of that map will be a solution of IE (1.4) and IDE (1.1). To obtain stability of the zero solution of IDE (1.1), we need the mapping defined by IE (1.4) to map bounded functions into bounded functions.

Let $(C,\|.\|)$ be the set of real-valued and bounded continuous functions on $[-r_0,\infty)$ with the supremum norm $\|.\|$ that is, for $\phi\in C$,

$$\|\phi\|=\sup\{|\phi(t)|:t\in[-r_0,\infty)\}.$$

In other words, we carry out our investigations in the complete metric space (C,ρ), where ρ denotes the supremum (uniform) metric for $\phi_1,\phi_2\in C$, and it is defined by

$$\rho(\phi_1, \phi_2) = \|\phi_1 - \phi_2\|.$$

For a given continuous initial function $\psi : [-r_0, 0) \to \Re$, define the set $C_\psi \subset C$ by

$$C_\psi = \{\phi : [-r_0, \infty) \to \Re | \phi \in C, \phi(t) = \psi(t) \ for \ t \in [-r_0, 0]\}.$$

Let $\|.\|$ denote the supremum on $[-r_0, 0]$ or on $[-r_0, \infty)$. Finally, note that $(C_\psi, \|.\|)$ is itself a complete metric space since C_ψ is a closed subset of C.

Theorem 2. Let $b : [-r_0, \infty) \to \Re$ be a continuous function and T be a mapping on C_ψ defined for $\phi \in C_\psi$ by

$$(T\phi)(t) = \psi(t) \ if \ t \in [-r_0, 0]$$

and

$$\begin{aligned}
(T\phi)(t) = {} & exp(-\int_0^t b(s)ds)\psi(0) \\
& - exp(-\int_0^t b(u)du) \sum_{i=1}^{2} \int_{-r_i(0)}^{0} [b(u) + G(0, u)]\psi(u)du \\
& + \sum_{i=1}^{2} \int_{t-r_i(t)}^{t} [b(u) + G(t, u)]\phi(u)du \\
& - exp(-\int_s^t b(u)du) b(s) \sum_{i=1}^{2} \int_{s-r_i(s)}^{s} [b(u) + G(s, u)]\phi(u)duds \\
& + \sum_{i=1}^{2} exp(-\int_s^t b(u)du)[b(s - r_i(s)) + G(s, s - r_i(s))](1 - r_i'(s))\phi(s - r_i(s))ds \\
& - 2\int_0^t exp(-\int_s^t b(u)du) b(s)\phi(s)ds. \qquad (1.7)
\end{aligned}$$

Assume that there exist constants $k \geq 0$ and $\alpha > 0$ such that

$$-\int_0^t b(s)ds \leq k \qquad (1.8)$$

and

$$\begin{aligned}
& \sum_{i=1}^{2} \int_{t-r_i(t)}^{t} |b(u) + G(t, u)|du \\
& + \sum_{i=1}^{2} exp(-\int_s^t b(u)du)|b(s)| \int_{s-r_i(s)}^{s} |b(u) + G(s, u)|duds \\
& + \sum_{i=1}^{2} exp(-\int_s^t b(u)du)|b(s - r_i(s)) + G(s, s - r_i(s))||(1 - r_i'(s)|ds \\
& + 2\int_0^t exp(-\int_s^t b(u)du)|b(s)|ds \leq a \qquad (1.9)
\end{aligned}$$

for $t \geq 0$, then $T : C_\psi \to C_\psi$.

Proof. For $\phi \in C_\psi$, $T\phi$ is continuous and agrees with ψ on $[-r_0, 0]$ by virtue of the definition of T. In view of (1.8) and (1.9), for $t > 0$, we have

$$|(T\phi)(t)| \le exp(k)|\psi(0)| + \exp(k)\sum_{i=1}^{2}\int_{-r_i(0)}^{0}|b(u) + G(0,u)||\phi(u)|du + \alpha\|\phi\|.$$

Hence, subject to the hypotheses of Theorem 2, it is clear that

$$|(T\phi)(t)| \le exp(k)|\psi|(1 + \sum_{i=1}^{2}\int_{-r_i(0)}^{0}|b(u) + G(0,u)|du) + \alpha\|\phi\| < \infty. \tag{1.10}$$

Thus, we can conclude that $T\phi \in C_\psi$.

Theorem 3. Let $k \ge 0$, $\alpha \in (0,1)$ and $b : [-r_0, \infty) \to \Re$ be a continuous function such that (1.8) and (1.9) hold for $t \ge 0$. Then for each continuous function $\psi : [-r_0, 0] \to \Re$, there is an unique continuous function $x : [-r_0, \infty) \to \Re$ with $x(t) = \psi(t)$ on $[-r_0, 0)$. In addition, $x(t)$ is bounded on $[-r_0, \infty)$ and the zero solution of IDE (1.1) is stable at $t = 0$. Finally, if

$$\int_0^t b(s)ds \to \infty \tag{1.11}$$

holds as $t \to \infty$, then $x(t) \to 0$ as $t \to \infty$.

Proof. We now take into consideration the space C_ψ defined by the continuous initial function $\psi : [-r_0, 0] \to \Re$. For $\phi, \eta \in C_\psi$ in the light of the hypotheses of Theorem 3, it can be seen that

$$\begin{aligned}
|(T\phi)(t) - (T\eta)(t)| \le & \sum_{i=1}^{2}\int_{t-r_i(t)}^{t}|b(u) + G(t,u)||\phi(u) - \eta(u)|du \\
& + \int_0^t \exp(-\int_s^t b(u)du)|b(s)|\sum_{i=1}^{2}\int_{s-r_i(s)}^{s}|b(u) + G(s,u)|||\phi(u) - \eta(u)|duds \\
& + \sum_{i=1}^{2}\exp(-\int_s^t b(u)du)|b(s - r_i(s) + G(s, s - r_i(s)))| \\
& \times |1 - r_i'(s)||\phi(s - r_i(s)) - \eta(s - r_i(s))|ds \\
& + 2\int_0^t \exp(-\int_s^t b(u)du)|b(s)||\phi(s) - \eta(s)|ds \\
\le & (\sum_{i=1}^{2}\int_{t-r_i(t)}^{t}|b(u) + G(t,u)|du \\
& + \int_0^t \exp(-\int_s^t b(u)du)|b(s)|\sum_{i=1}^{2}\int_{s-r_i(s)}^{s}|b(u) + G(s,u)|duds \\
& + \sum_{i=1}^{2}\exp(-\int_s^t b(u)du)|b(s - r_i(s) + G(s, s - r_i(s)))||1 - r_i'(s)|ds \\
& + 2\int_0^t \exp(-\int_s^t b(u)du)|b(s)|\|\phi - \eta\|ds.
\end{aligned}$$

By the definition of T and (1.9), T is a contraction mapping with contraction constant α. By Banach's contraction mapping principle, T has a unique fixed point x in C_ψ which is a bounded and continuous function. By Theorem 3, it is a solution of IDE (1.1) on $[0,\infty)$. It follows that x is the only bounded and continuous function satisfying IDE (1.1) on $[0,\infty)$ and the initial condition.

It is clear that the zero solution of IDE (1.1) is stable. If $x(t)$ is a solution with the initial function ψ on $[-r_0, 0]$, then, by (1.10), we have

$$(1-\alpha)\|x\| \leq \exp(k)\|\psi\|(1+\sum_{i=1}^{2}\int_{-r_i(0)}^{0}|b(u)+G(0,u)|du).$$

Then, for each $\varepsilon > 0$, there exists a $\delta > 0$ such that $|x(t)| < \varepsilon$ for all $t \geq -r_0$ if $\|\psi\| < \delta$.

Next we prove that the solution of IDE (1.1) tends to zero when (1.11) holds. First, we define a subset C_ψ^0 of C_ψ by

$$C_\psi^0 = \{\phi : [-r_0,\infty) \to \Re | \phi \in C, \phi(t) = \psi(t) \ for\ t \in [-r_0, 0], \phi(t) \to 0 \ as\ t \to \infty\}.$$

Since C_ψ^0 is a closed subset of C_ψ and (C_ψ^0, ρ) is complete, then the metric space (C_ψ^0, ρ) is also complete. Now we show that $(T\phi)(t) \to 0$ as $t \to \infty$ when $\phi \in C_\psi^0$.

By (1.7) and (1.9), we have

$$\begin{aligned}|(T\phi)(t)| \leq &\exp(-\int_0^t b(s)ds)(|\psi(0)| + \sum_{i=1}^{2}\int_{-r_i(0)}^{0}|b(u)+G(0,u)|)du \\ &+ \alpha\|\phi\|_{[t-r_i(t),t]} + |I_4| + |I_5|,\end{aligned}$$

where $t > 0$, I_4 and I_5 denote fourth and fifth, sixth terms of (1.7), respectively.

We can prove that each of the above terms tend to zero as $t \to \infty$. In fact, it is easy to see that the first term tends to 0 by (1.11) and the second term approaches zero as $t \to \infty$ since $t - r_i(t) \to \infty$, $i = 1, 2$. Hence, for each $\varepsilon > 0$, there exists a $M > 0$ such that

$$\|\phi\|_{[M-r_i(M),\infty)} \leq \frac{\varepsilon}{2\alpha}$$

since $t - r_i(t) \to \infty$ as $t \to \infty$. Then, for $t \geq M$ we can see that

$$\begin{aligned}|I_4| \leq &\int_0^M \left|b(s)\exp(-\int_s^M b(u)du)\sum_{i=1}^{2}\int_{s-r_i(s)}^{s}|b(u)+G(s,u)|duds\|\phi\|\exp(-\int_T^t b(u)du)\right. \\ &+ \int_T^t |b(s)\exp(-\int_s^t b(u)du)\sum_{i=1}^{2}\int_{s-r_i(s)}^{s}|b(u)+G(s,u)|duds\|\phi\|_{[M-r_i(M),\infty)}.\end{aligned}$$

By (1.11), there exists a $\tau \geq M$ such that $\|\phi\|\exp(-\int_T^t b(u)du) < \frac{\varepsilon}{2\alpha}$ for $t > \tau$. Thus, for every $\varepsilon > 0$ there exists a $\tau > 0$ such that $t > \tau$ implies $I_4 < \varepsilon$, that is, $I_4 \to 0$ as $t \to \infty$.

Similarly, we can show that I_5 tends to zero $t \to \infty$. This yields $(T\phi)(t) \to 0$ as $t \to \infty$, and hence $T : C_\psi^0 \to C_\psi$. Therefore, T is a contraction on C_ψ^0 with a unique fixed point x. By Theorem 1, x is a solution of IDE (1.1) on $[0,\infty)$. Hence, $x(t)$ is the only continuous solution of IDE (1.1) agreeing with the initial function ψ. Since $x \in C_\psi^0$, then $x(t) \to 0$ as $t \to \infty$.

We now give an example to show the applicability of Theorem 3.

Example. Let us consider the following integro -differential equation of first order with two variable delays, which is a special case of IDE (1.1):

$$x'(t) = -\frac{0.2t}{t^2+1}x(t) - \sum_{i=1}^{2}\int_{t-r_i(t)}^{t}\frac{0.2}{s^2+1}x(s)ds, \tag{1.12}$$

where $r_1(t) = 0.385t$ and $r_2(t) = 0.476t$.

When we compare IDE (1.12) with IDE (1.1) and take into consideration the hypotheses of Theorem 3, it can easily be seen that

$$b(t) = \frac{0.2t}{t^2+1}$$

and

$$a(t,s) = \frac{0.2}{s^2+1}.$$

Hence, we have

$$G(t,s) = \int_t^s \frac{0.2}{s^2+1}du = \frac{0.2(s-t)}{s^2+1}.$$

For the function $b(t)$, we obtain

$$\int_0^t b(s)ds = \frac{1}{5}\int_0^t \frac{s}{s^2+1}ds = \frac{1}{10}ln(t^2+1).$$

Hence, clearly, it follows that

$$\frac{1}{10}ln(t^2+1) \to \infty \; when \; t \to \infty.$$

Thus, the hypothesis (1.11) holds.

We also have

$$\begin{aligned}\sum_{i=1}^{2}\int_{t-r_i(t)}^{t}|b(u)+G(t,u)|du &= \sum_{i=1}^{2}\int_{t-r_i(t)}^{t}\left|\frac{0.2u}{u^2+1}+\frac{0.2(u-t)}{u^2+1}\right|du \\ &= \int_{0.615t}^{t}\frac{0.4u-0.2t}{u^2+1}du + \int_{0.524t}^{t}\frac{0.4u-0.2t}{u^2+1}du \\ &= \mu_1(t)+\mu_2(t).\end{aligned}$$

By an easy calculation, we can get

$$\mu_1(t) = \int_{0.615t}^{t}\frac{0.4u}{u^2+1} - \int_{0.615t}^{t}\frac{0.2u}{u^2+1}$$

$$\begin{aligned}&= 0.2ln(u^2+1)|_{0.615t}^{t} - 0.2t\arctan u|_{0.615t}^{t} \\ &= 0.2ln(t^2+1) - 0.2ln(0.615^2t^2+1) - 0.2t\arctan t + 0.2t\arctan 0.615t \\ &= 0.2t[\arctan 0.615t - \arctan t] + 0.2ln(t^2+1) - 0.2ln(0.615^2t^2+1).\end{aligned}$$

By the same way, we can calculate μ_2 and obtain

$$\mu_2(t) = 0.2t[\arctan 0.524t - \arctan t] + 0.2ln(t^2+1) - 0.2ln(0.524^2t^2+1).$$

It is obvious that both of these functions, that is, μ_1 and μ_2 are increasing in $[0,\infty)$.

We now need to find that

$$\lim_{t\to\infty}\mu_1(t) + \lim_{t\to\infty}\mu_2(t).$$

If we do the necessary calculations, then

$$\begin{aligned}\lim_{t\to\infty}\mu_1(t) = \lim_{t\to\infty}[&\frac{0.2[(0.615/0.615t^2+1)-(1+1/t^2)]}{-1/t^2}\\ &+0.2\ln(\frac{t^2(1+1/t^2)}{t^2(0.615^2+1/t^2)})]\\ =&\lim_{t\to\infty}0.2[\frac{1}{1+1/t^2}-\frac{0.615}{0.615^2+\frac{1}{t^2}}+\ln(1/0.615^2)]\\ =&0.2[1-1/0.615-2\ln(0.615)]\\ \cong&0.0692499524\end{aligned}$$

and

$$\begin{aligned}\lim_{t\to\infty}\mu_2(t) = \lim_{t\to\infty}[&\frac{0.2[(0.524/0.524t^2+1)-(1+1/t^2)]}{-1/t^2}\\ &+0.2\ln(\frac{t^2(1+1/t^2)}{t^2(0.524^2+1/t^2)})]\\ =&\lim_{t\to\infty}0.2[\frac{1}{1+1/t^2}-\frac{0.524}{0.524^2+\frac{1}{t^2}}+\ln(1/0.524^2)]\\ =&0.2[1-1/0.524-2\ln(0.524)]\\ \cong&0.0768260486.\end{aligned}$$

Hence, we have

$$\sum_{i=1}^{2}\int_{t-r_i(t)}^{t}|b(u)+G(t,u)|<0.07+0.08=0.15,$$

$$\sum_{i=1}^{2}\int_0^t\exp(-\int_s^t b(u)du)|b(s)|\int_{s-r_i(s)}^{s}|b(u)+G(s,u)|duds<0.15,$$

$$\sum_{i=1}^{2}\int_0^t\exp(-\int_s^t b(s)ds)|b(s-r_i(s)+G(s,s-r_i(s)))||1-r_i'(s)|ds$$

$$\begin{aligned}&=0.2(2-1/0.615)\int_0^t\exp(-\int_s^t\frac{u}{u^2+1})\frac{s}{s^2+1/0.615t^2}ds\\ &+0.2(2-1/0.524)\int_0^t\exp(-\int_s^t\frac{u}{u^2+1})\frac{s}{s^2+1/0.524t^2}ds\\ &<0.2[(2-1/0.615)]+0.2[(2-1/0.524)]\\ &\cong 0.09331173587<0.1\end{aligned}$$

and

$$2\int_0^t\exp(-\int_s^t b(u)du)|b(s)|ds<0.3,$$

respectively.

Let $\alpha=0.15+0.15+0.1+0.3=0.7<1$. Hence, we can conclude that $x(t)$ of IDE (1.12) is bounded on $[-r_0,\infty)$ and the zero solution of IDE (1.12) is asymptotically stable.

References

[1] Ardjouni, A., Djoudi, A., Stability in nonlinear neutral integro-differential equations with variable delay using fixed point theory. J. Appl. Math. Comput. 44 (2014) ,no.1-2, 317-336.
[2] Becker, L.C., Burton, T. A., Stability, fixed points and inverses of delays. Proc. Roy. Soc. Edinburgh Sect. A 136 (2006), no. 2, 245-275.
[3] Burton, T. A., Stability by fixed point theory or Liapunov theory: a comparison. Fixed Point Theory 4 (2003), no. 1, 15-32.
[4] Burton, T. A., Fixed points and stability of a non-convolution equation. Proc. Amer. Math. Soc. 132 (2004), no. 12, 3679-3687.
[5] Burton, T.A., Stability by fixed point theory for functional differential equations. Dover Publications, Mineola, New York, 2006.
[6] Gabsi, H., Ardjouni, A., Djoudi, A., Fixed points and stability of a class of nonlinear delay integro-differential equations with variable delays. Facta Univ. Ser. Math. Inform. 32 (2017), no. 1, 31-57.
[7] Gözen, M., Tunç, C., Stability in functional integro-differential equations of second order with variable delay. J. Math. Fundam. Sci. 49 (2017), no. 1, 66-89.
[8] Graef, J. R., Tunç, C., Continuability and boundedness of multi-delay functional integro-differential equations of the second order. Rev. R. Acad. Cienc. Exactas Fs. Nat. Ser. A Math. RACSAM 109 (2015), no. 1, 169-173.
[9] Jin, C., Luo, J., Stability of an integro-differential equation. Comput. Math. Appl. 57 (2009), no. 7, 1080-1088.
[10] Levin, J. J., The asymptotic behavior of the solution of a Volterra equation. Proc. Amer. Math. Soc. 14 (1963) 534-541.
[11] Pi, D., Study the stability of solutions of functional differential equations via fixed points. Nonlinear Anal.74 (2011), no.2, 639-651.
[12] Pi, D., Stability conditions of second order integro-differential equations with variable delay. Abstr. Appl. Anal. 2014, Art. ID 371639, 11 pp.
[13] Raffoul, Y. N., Stability in neutral nonlinear differential equations with functional delays using fixed-point theory. Math. Comput. Modelling, 40 (2004) no.7-8, 691-700.
[14] Rahman, M., Integral equations and their applications. WIT Press, Southampton, 2017.
[15] Tunç, C., A note on the qualitative behaviors of non-linear Volterra integro-differential equation. J. Egyptian Math. Soc. 24 (2016), no. 2, 187-192.
[16] Tunç, C., New stability and boundedness results to Volterra integro-differential equations with delay. J. Egyptian Math. Soc. 24 (2016), no. 2, 210-213.
[17] Tunç, C., Stability and boundedness in Volterra-integro differential equations with delays. Dynam. Systems Appl.26 (2017), no.1, 121-130.
[18] Tunç, C., Qualitative properties in nonlinear Volterra integro-differential equations with delay. Journal of Taibah University for Science 11 (2017), no.2, 309-314.
[19] Tunç, C., On the qualitative behaviors of a functional differential equation of second order. Appl. Appl. Math. 12 (2017), no. 2, 813-842.
[20] Tunç, C., Mohammed, S. A., On the stability and instability of functional Volterra integro-differential equations of first order. Bull. Math. Anal. Appl. 9 (2017), no. 1, 151-160.
[21] Tunç, C., Tunç, O., On the exponential study of solutions of Volterra integro-differential equations with time lag. Electron. J. Math. Anal. Appl., 6 (1), (2018), 253-265.
[22] Tunç, C., Tunç, O., New results on the stability, integrability and boundedness in Volterra integro-differential equations. Bull. Comput. Appl. Math., 6 (1),(2018), 41-58.
[23] Tunç, C., Tunç, O., On behaviors of functional Volterra integro-differential equations with multiple time-lags. Journal of Taibah University for Science, 12 (2), (2018), 173-179.
[24] Wazwaz, A. M., Linear and nonlinear integral equations. Methods and applications. Higher Education Press, Beijing, Springer, Heidelberg, 2011.

Cemİl Tunç
Department of Mathematics Faculty of Sciences Van Yuzuncu Yil University 65080, Van-Turkey
E-mail address: cemtunc@yahoo.com

İrem Akbulut
Department of Mathematics Faculty of Education Siirt University 56100, Siirt-Turkey
E-mail address: iremmmatematik@gmail.com