ON A UNIQUENESS CONDITION FOR CR FUNCTIONS ON HYPERSURFACES

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Abstract. Let \( f \) be a smooth CR function on a smooth hypersurface \( M \subset \mathbb{C}^n \), such that \( f \) vanishes to infinite order along a \( C^\infty \)-smooth curve \( \gamma \subset M \). Assume that for each \( q \in \gamma \) there exists a truncated double cone \( C \) at \( q \) in \( M \), such that at least one of the following three conditions holds true: (a) There is a constant \( \theta \in \mathbb{R} \), such that \( C \subset \{ |\text{Re}(e^{i\theta}f)| \leq |\text{Im}(e^{i\theta}f)| \} \). (b) \( C \subset \{ \text{Re} f \geq 0 \} \). (c) \( |f(z)||z-q| \to 0 \), \( z \to q \), \( z \in \mathbb{C} \). Then \( f \) vanishes on an \( M \)-open neighborhood of \( \gamma \).

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1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

Our starting point is the following definition of vanishing to infinite order on a submanifold.

Definition 1.1 (See Baouendi & Zachmanoglou [9], p.9). Let \( \Omega \subset \mathbb{R}^N \) be an open set and let \( M \) and \( \gamma \subset M \), be two differentiable submanifolds of \( \Omega \). We say that a continuous complex-valued function \( f \), defined on \( M \), vanishes to infinite order on \( \gamma \), if for every \( \alpha \in \mathbb{R} \), the function,
\[
z \mapsto f(z)(\text{dist}(z, \gamma))^{\alpha},
\]
is bounded in any compact set of \( M \).
Remark 1.2. Let \( f, M, \gamma \) be as in Definition 1.1 and let \( 0 \in \gamma \). We automatically know that for \( \alpha \geq 0 \), \( f(z)(\text{dist}(z, \gamma))^\alpha \), is bounded on any compact subset of \( M \), and that for any pair of \( \alpha < 0 \), \( c > 0 \), it is bounded on the intersection, of any compact subset of \( M \) with \( \{ z \in M : \text{dist}(z, \gamma) \geq c \} \). On the set \( \{ z \in M : \text{dist}(z, \gamma) < 1 \} \) it is obvious that \( f(z)(\text{dist}(z, \gamma))^\alpha \), is bounded on every compact subset if and only if, for every \( k \in \mathbb{N} \), \( f(z)(\text{dist}(z, \gamma))^{-k} \), is bounded locally near each point of \( \gamma \). Hence, vanishing to infinite order on \( \gamma \), is equivalent to the requirement that \( f(z)(\text{dist}(z, \gamma))^{-k} \) is bounded locally near each point of \( \gamma \), for each fix \( k \in \mathbb{N} \). We shall be interested in the version of Definition 1.1 where \( \Omega \subset \mathbb{R}^{2n} \) and where we identify \( \mathbb{R}^{2n} \) with \( \mathbb{C}^n \). Let \( M \subset \mathbb{C}^n \) be a \( CR \) submanifold and let \( \gamma \subset M \) be a submanifold. We say that a continuous \( CR \) function \( f : M \to \mathbb{C} \), \textit{vanishes to infinite order on} \( \gamma \) if for every \( a \in \gamma \), and \( \forall k \in \mathbb{N} \), there exists a constant \( C_k > 0 \), and \( U \subset M \) an open neighborhood of \( a \) satisfying, \[
|f(z)| \leq C_k(\text{dist}(z, \gamma))^k, \quad z \in U. \tag{1.2}
\]
Note that for any \( p \in \gamma \), we have (sufficiently near \( p \)), \( |f(z)|\cdot|z - p|^{-(k+1)} \leq C_{k+1} \Rightarrow |f(z)|\cdot|z - p|^{-k} \leq C_{k+1} |z - p| \), thus letting \( z \to p \), we see that, \[
\lim_{z \to p} \frac{f(z)}{|z - p|^k} = 0, \quad k \in \mathbb{N}, \tag{1.3}
\]
(where the case \( k = 0 \) is due to the fact that \( |f(z)| \leq C_1 |z - p| \to 0 \) as \( z \to p \)).

In the case of a generic embedded \( CR \) submanifolds \( M \subset \mathbb{C}^n \), there exists (contrary to the case of complex manifolds) choices of \( M \) allowing for smooth \( CR \) functions which vanish to infinite order at a point \( p \in M \), but not identically, see e.g. Schmalz [25]. In our main result we shall use so-called truncated double cones at a point in a hypersurface.

**Definition 1.3.** Let \( M \) be a \( C^1 \)-smooth \( N \)-dimensional real manifold. We define a set \( C(q) \subset M \) to be a \textit{truncated double cone} in \( M \) at \( q \in M \) if there exists a parametrization of \( M \) by local Euclidean coordinates \((x_1, \ldots, x_N)\) centered at \( q \), such that \( C(q) \) is parametrized, in the variables \((x_1, \ldots, x_N)\), by an open nonempty truncated double cone at \( q \) in \( \mathbb{R}^N \).

For a smooth hypersurface \( M \subset \mathbb{C}^n \), where \( n \geq 2 \), we denote \( T^cM := T_pM \cap J_pT_pM \), where \( J \) is the complex structure map on \( T\mathbb{C}^n \) defined by \( J_p \) on each \( T_p\mathbb{C}^n \). It is still an open problem, to determine necessary and sufficient conditions, under which a \( CR^\infty \) function (by which we mean a \( C^\infty \)-smooth \( CR \) function) on a \( C^\infty \)-smooth hypersurface, such that the function vanishes to infinite order along a curve, is forced to vanish identically. The work of Nirenberg [22] initiated the following question on unique continuation, see Fornaess & Sibony [12]. Let \( \Omega \subset \mathbb{C}^n \), \( n \geq 2 \), be a domain with smooth boundary. Let \( \gamma \) be a smooth curve in \( \partial \Omega \), transverse to \( T_p(\partial \Omega) \), for every \( p \in \gamma \). Does it hold true that if \( \gamma \subset C^\infty(\Omega) \), holomorphic on \( \Omega \) and vanishes to infinite order on \( \gamma \), then \( f \equiv 0 \)?

In our main result we provide some additional conditions under which we have an affirmative answer to the question.

\footnote{By which we mean that \( Xf = 0 \), for all sections \( X \), of \( H^{0,1}M \).}
Theorem 1.4 (Main result). Let \( M \subset \mathbb{C}^n \), be a \( C^\infty \)-smooth real hypersurface. Let \( \gamma \subset M \) be a real \( C^\infty \)-smooth curve such that,
\[
T^\gamma_z M + T^\gamma_z \gamma = T^\gamma_z M, \quad z \in \gamma.
\]
Let \( f \in C^{\infty}(M) \) such that \( f \) vanishes to infinite order along \( \gamma \). Assume that for each \( q \in \gamma \), there exists a truncated double cone \( C \) at \( q \) in \( M \), such that at least one of the following holds true:

(a) There is a constant \( \theta \in \mathbb{R} \), such that \( C \subset \{|\text{Re}(e^{i\theta} f)| \leq |\text{Im}(e^{i\theta} f)|\} \).

(b) \( C \subset \{Re f \geq 0\} \).

(c) \(|f(z)|^{z-q} \to 0, z \to q, z \in C\).

Then \( f \) vanishes on an \( M \)-open neighborhood of \( \gamma \).

2. Preliminary definitions and remarks

Remark 2.1. For a smooth vector field \( X \) on an open \( \Omega \subset \mathbb{R}^n \) and any point \( p \in \Omega \) there exists a unique integral curve, \( \kappa \), satisfying \( \kappa : [0, T) \to \Omega \), (for a maximal \( T \)) \( \kappa(t) = X(\kappa(t)) \), of \( X \) which passes through \( p \) when \( t = 0 \) i.e., \( \kappa(0) = p \).

We shall denote this integral curve by \( t \mapsto \Phi_{X,t}(p) \). It is further known that if \( X = X(\vartheta) =: X_\vartheta \), i.e., \( X \) depends upon a parameter \( \vartheta \) then \( T = T(p, \vartheta) \) is a lower semi-continuous function of \((p, \vartheta)\) and \( t \mapsto \Phi_{X}(p) \) is continuous on the set \( 0 < t < T(p, \vartheta) \), as \((p, \vartheta)\) vary on an open neighborhood of \((p, 0)\).

Definition 2.2. Let \( H \) be a collection of smooth vector fields on \( \Omega \). By a *polygonal path of a finite number of integral curves*, of vector fields in \( H \) joining \( q' \in \Omega \) to \( q \in \Omega \) we mean a piecewise smooth curve \( \kappa : [0, 1] \to \Omega \) such that \( \kappa(0) = q, \kappa(1) = q' \) and \( 0 = s_0 < s_1 < \cdots < s_k = 1 \) such that,
\[
\kappa(s) = \Phi_{X^j, t_{j,s}(\kappa(s_{j-1}))}, \quad s_{j-1} \leq s \leq s_j, \quad 1 \leq j \leq k,
\]
(2.1)
where \( X^j \in H \) and \( t_{j,s}(\kappa) \) is a smooth diffeomorphism of \([s_{j-1}, s_j]\) onto some closed interval of \( \mathbb{R} \) with \( t_{j,s}(s_{j-1}) = 0 \). For \( t = (t_1, \ldots, t_k) \in \mathbb{R}^k \) one may use the notation,
\[
q' = \Phi_{X^1, t_1}(\Phi_{X^2, t_2}(\cdots \Phi_{X^k, t_k}(q) \cdots)),
\]
(2.2)
for expressing that \( q' \) can be reached from \( q \) by a polygonal path of integral curves of the vector fields \( X^j \) (in the given order). This gives a mapping \( \mathbb{R}^k \times \Omega \ni (t, q) \mapsto \Omega \), which for fixed choice of \( X^1, \ldots, X^k \) and for \( t \) near \( 0 \) in \( \mathbb{R}^k \), is given by,
\[
(t, q) \mapsto \Phi_{X^1, t_1}(\Phi_{X^2, t_2}(\cdots \Phi_{X^k, t_k}(q) \cdots)) =: \Phi_{X, t}(q),
\]
(2.3)
(2.3)
where we are using the notation \( X = (X^1, \ldots, X^k) \) for more details on this map, see Baouendi et al. [9], p.69.

Definition 2.3. Let \( M \subset \mathbb{R}^n \), for a positive integer \( n \), be a submanifold and let \( p \in M \). We say that a submanifold \( M' \subset \mathbb{R}^n \) is equivalent to \( M \) at \( p \), denoted \( M \sim_p M' \), if: \( p \in M' \) and there exists an open neighborhood \( V \subset \mathbb{R}^n \) of \( p \), such that \( V \cap M = V \cap M' \). The equivalence class of \( M \), under the equivalence relation \( \sim_p \), is called the *germ of the submanifold\( M \) at \( p \). If \( N \subset M \) is a submanifold and \( p \in N \), then a submanifold \( N' \subset \mathbb{R}^n \), is said to belong to the germ of \( N \) at \( p \) in \( M \),

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This is a consequence of the fundamental theorem of ODE, see e.g. Hartmann [16], p.94, which is usually stated in terms of a unique solution \( \gamma(t) = \eta(t, t_0, \gamma_0, \xi) \), (defined for a maximal interval, which may depend on \( t_0, \gamma_0 \) and the parameters \( \xi \), i.e. \( t \in (\alpha(t_0, \gamma_0, \xi), b(t_0, \gamma_0, \xi)) \)) to the initial value problem \( \gamma'(t) = f(t, \gamma, \xi), \gamma(t_0) = \gamma_0 \). In our case \( f(t, \gamma, \vartheta) = (X_0 \gamma)(t) \), where \( X \) is a vector field.
if $N' \subset M$ and belongs to the germ of the submanifold $N$ at $p$. Any submanifold of $\mathbb{R}^n$ that belongs to the germ of a submanifold $M$ at a point $p \in M$, will be called a representative (or member) of the germ of the submanifold $M$ at $p$.

**Definition 2.4** (See Baouendi et al. [5], p.94). Let $M$ be a smooth CR manifold and let $p \in M$. By a known theorem (see Baouendi et al. [3], p.68) there exists a $C^\infty$-smooth submanifold $W \subset M$, $p \in W$, satisfying (i) if $p \in W'$, where $W'$ is another $C^\infty$-smooth submanifold to which all vector fields of $T'M$ are tangent at every point then there is an open $V \subset M$, $p \in V$, with $W \cap V \subset W' \cap V$, (ii) for every open $U \subset M$, $p \in U$, there exists $N \in \mathbb{Z}_+$, and open $V_1 \subset V_2 \subset U$, with $p \in V_1$, such that any $q \in V_1 \cap W$ can be reached by a polygonal path of $N$ integral curves, of vector fields in $T'M$, contained in $W \cap V_2$.

We denote by $\mathcal{O}(p)$, the members of the germ of $W$ at $p$, in $M$, such that the tangent space at each point of the member contains $T'_q M$. We call $\mathcal{O}(p)$ the local CR-orbit at $p$.

Any representative of $\mathcal{O}(p)$ contains a CR submanifold of $M$ which passes through $p$ and whose CR dimension equals the CR dimension of $M$.

**Definition 2.5** (See e.g. Baouendi et al. [5], p.20). Let $M \subset \mathbb{C}^n$ be an embedded CR submanifold and $p_0 \in M$. $M$ is said to be minimal at $p_0$, if there is no real submanifold $S \subset M$, $p_0 \in S$, such that the following two conditions hold true simultaneously: (1) $T'_p M$ is tangent to $S$ at every $p \in S$. (2) $\dim_{\mathbb{R}} S < \dim_{\mathbb{R}} M$.

If a CR submanifold $M \subset \mathbb{C}^n$, is not minimal at a point $p_0 \in M$, then we shall say that $p_0$ is a non-minimal point of $M$.

### 3. Some known results used in the proof of Theorem 1.4

The problem of unique continuation for CR functions has been studied by many authors, see e.g. Rosay [23], Airapetyan & Khenkin [1], Hunt et al. [18], Baouendi & Treves [7], Alinhac et al. [3], Grachev [14], Schmalz [25], Berhanu & Mendoza [8], Huang et al. [17] and Baouendi & Rothschild [6], Alexander [2] and very recently (in relation to growth conditions) Della Sala & Lamel [11]. Here we mention just a few, which we shall make use of.

**Theorem 3.1** (Alinhac et al. [3], p.635). Let $W \subset \mathbb{C}$ be an open neighborhood of $0$, let $W^+ := W \cap \{\text{Im } z > 0\}$, and let $A \subset \mathbb{C}^n$ be a totally real $C^2$-smooth submanifold. Let $F \in \mathcal{O}(W^+)$ and continuous up to the boundary such that $F$ maps $W \cap \{\text{Im } z = 0\}$ into $A$. If $F$ vanishes to infinite order at the origin then $F$ vanishes identically in the connected component of the origin in $W^+$.

**Theorem 3.2** (Huang et al. [17] and Baouendi & Rothschild [6]). If $f(z)$ is a holomorphic function in the intersection, with the upper half plane, of a domain containing $0$, $f$ continuous up to the boundary, vanishing to infinite order at $0$ (in the sense that $f(z) = O(|z|^N)$ for every $N \in \mathbb{N}$) and $\Re f(x) \geq 0$, $x := \text{Re } z$, then $f$ must vanish identically.

**Theorem 3.3** (Huang et al. [17]. See Remark 3.6 regarding stronger version). If $f = u + iv$ is holomorphic in $H^+ := \{z \in \mathbb{C} : \text{Im } z > 0\}$, and continuous up to $(-1,1) \subset \partial H^+$, such that $|v(t)| \leq |u(t)|$ for $t \in (-1,1)$, and if $f$ vanishes to infinite order at $0$, then $f \equiv 0$. 

Given any of the conditions in the last two theorems, there is a certain technique of reducing to one-variable, to be applied for obtaining a uniqueness result (see Lemma 4.1). The following is a version of a uniqueness theorem for hypersurfaces due to Shafin [26], where the author originally requires that the hypersurface has a positive eigenvalue of the Levi form at 0 and that the growth condition in the theorem is independent of direction, but the proof reveals that the result holds true given one-sided holomorphic extension and that the growth conditions are only required with respect to an open non-empty double cone at 0.

**Theorem 3.4.** Let $M \subset \mathbb{C}^n$ be a $C^\infty$-smooth hypersurface, $0 \in M$. Let $f$ be a $C^\infty$-smooth CR function near 0 such that $f$ has holomorphic extension to one side at 0 and there is a double cone, $C$, at 0 in $M$, $\lim_{z \to 0} |f(z)|^|z| = 0$, $z \in \mathbb{C}$. Then $f \equiv 0$ on an $M$-neighborhood of the origin.

We shall use the following results, in the proof of Theorem 1.4 (see the proof of Claim 4.3).

**Theorem 3.5** (See Treves [30], proof of Theorem II.3.3, p.91). Let $M$ be a $C^\infty$-smooth real manifold equipped with a locally integrable structure $L$, let $\mathcal{V} \subset M$ be an open subset and $X$ a $C^1$-smooth section of $L$ over $\mathcal{V}$ (denoted $X \in \Gamma^1(\mathcal{V}, L)$). Let $\gamma : [0, 1] \to \Omega$ be an integral curve of $\text{Re} X$ and let $f$ be a distribution solution to the system of equations induced by $L$ (i.e. $Xf = 0$ on $\mathcal{V}$, for each $X \in \Gamma^1(\mathcal{V}, L)$).

If $f \equiv 0$ on an open neighborhood of $\gamma(0)$, then $f \equiv 0$ on an open neighborhood of $\gamma(1)$.

We have the following special case, where $M \subset \mathbb{C}^n$ is a $C^\infty$-smooth hypersurface, $L = H^{0,1}M$, i.e. tangential CR vector fields$^3$(where we identify $\text{Re} L$ with $T^c M$).

**Corollary 3.6** (to Theorem 3.5). Let $M \subset \mathbb{C}^n$ be a $C^\infty$-smooth hypersurface and let $p_0' \in M$. Assume there is an integral curve of a CR vector field, such that the curve originates at $p_0'$ and whose end point is $p_0$. If $f$ is a continuous CR function on $M$ which vanishes on an open $M$-neighborhood of $p_0'$, then $f$ vanishes on an open $M$-neighborhood of $p_0$.

4. Proof of Theorem 1.4

We begin with the following lemma.

**Lemma 4.1.** Let $M \subset \mathbb{C}^n$, be a $C^\infty$-smooth real hypersurface, $0 \in M$. Let $\gamma \subset M$ be a real $C^\infty$-smooth curve, $0 \in \gamma$, such that,

$$T^z_\gamma M + T\gamma = T_\gamma M, \quad z \in \gamma.$$ (4.1)

Let $f \in CR^\infty(M)$ such that $f$ vanishes to infinite order on $\gamma$. Assume $f$ has holomorphic extension to at least one side of $M$, near 0. Assume that there is a truncated double cone, $C$, at 0 in $M$ such that at least one of the following holds true:

1. There is a constant $\theta \in \mathbb{R}$, such that $C \subset \{|\text{Re}(e^{i\theta} f)| \leq |\text{Im}(e^{i\theta} f)|\}$.
2. $C \subset \{|\text{Re} f| \geq 0\}$.  

$^3$For the fact that $H^{0,1} M$ is integrable see e.g. Baouendi et al. [5], p.36 (a short proof of the, in itself not sufficient, involutivity can be found in e.g. Boggess [19]).

$^4$This is done via the identification $X \mapsto \frac{X + iY}{2}, \quad X \in T^c M$, with inverse $Y \mapsto Y + \Gamma$, $Y \in T^{0,1} M$, see e.g. Zampieri [36], p.112 and p.116.
Then $f \equiv 0$ on an $M$-open neighborhood of $0$.

Proof. Let $\mathcal{U}$ be an open subset of $\mathbb{C}^n$, such that $V := \partial \mathcal{U} \cap M$ is open in $M$, $0 \in M$, and there exists a function $F \in \mathcal{O}(\mathcal{U}) \cap C^0(\mathcal{U} \cup V)$, $F|_V = f|_V$. Let $D \subset \mathbb{C}^n$ be a complex line passing through $0$ such that $T_0D + T_0M = T_0\mathbb{C}^n$, and, for any sufficiently small open $B \subset \mathbb{C}^n$, $0 \in B$, such that $B \cap D \cap M$ is a connected $C^\infty$-smooth curve. Since we know that $F$ has $C^\infty$-extension to the boundary, then for each $\alpha \in \mathbb{N}^n$, the function $\frac{\partial^n F}{\partial z^\alpha}$ has continuous extension to the boundary, thus vanishing to infinite order on $\gamma$ of $f$, implies,

$$\lim_{p \in \partial \mathcal{U}, p \to 0} \frac{\partial^n F}{\partial z^\alpha}(p) = 0$$

(4.2)

Next, pick a sufficiently small open subset $B \subset \mathbb{C}^n$, $0 \in B$, such that $U := B \cap \mathcal{U}$, satisfies that $M \cap \partial \mathcal{U} \cap D$ is a $C^\infty$-smooth curve. Assuming further that $\mathcal{U}$ is a bounded domain and that $U \cap D$ is a bounded simply connected domain, bounded by a finite union of smooth curves, there is a bijective holomorphic map $\mathcal{S}$ from the (non-empty, bounded and simply connected one-dimensional complex) domain $U^+ := U \cap D$, onto an open half-set, $W^+ \subset \mathbb{C}$ as in Theorem 3.1 and furthermore $\mathcal{S}$ extends to a homeomorphism up to the boundary, see e.g. Taylor [23], p.342 (for a short proof of the fact that a bijective holomorphic map of a domain necessarily has holomorphic inverse, see e.g. Rudin [24], p.217). So we can assume $\mathcal{S}$ is a biholomorphism of $U \cap D$ and a homeomorphism of $U \cap D$. Since $\mathcal{S}$ is an open mapping of $U \cap D$ with open inverse we can assume $\mathcal{S}(0)$ is the origin, belonging to the boundary of $W^+$ in $\mathbb{C}$. Now given a holomorphic coordinate $z$ centered at $0$, for $D$ near $0$, and setting $\mathcal{S}(z) = \zeta$, we can (by the chain rule) for any $j \in \mathbb{N}$, and any $q \in W^+$, express $\frac{\partial^j F}{\partial z^\alpha}(q)$ as finite sum of multiples of $\frac{\partial^j F}{\partial \zeta^\alpha}(\mathcal{S}^{-1}(q))$, and $\frac{\partial^j \mathcal{S}^{-1}}{\partial \zeta^\alpha}(q)$, $k,l \in \{1,\ldots,j\}$. Here we are considering the restriction of $F$ to $D \cap \mathcal{U}$, so there is only one complex coordinate $z$ (which explains why we write $j$ instead of a multi-index $\alpha$). By (4.2), we obtain that $(F \circ \mathcal{S}^{-1})$ is a continuous map of $U \cap D$ (holomorphic on $U \cap D$) which vanishes to infinite order at $\mathcal{S}(0) \in \partial W^+$. Since $M$ is smooth, $D$ a complex one-dimensional manifold We can (by appropriate choice of $B$, if necessary we replace $U \cap D$ but keep the same notation) assume that both $U \cap D$ and $\mathcal{S}(U \cap D)$ have $C^\infty$-smooth boundary. In particular we can assume $(F \circ \mathcal{S}^{-1})$ is smooth up to the boundary. If condition (c) holds true then, by Theorem 3.4 $f$ vanishes on an open $M$-neighborhood of $0$. If condition (c) does not hold true, then by assumption one of (a) or (b) must hold true. Then we are able to choose $D$ such that $D \cap V$ (for sufficiently small $V$) belongs to the intersection with $M$ of a double cone as in (a) or (b). We obtain that $(F \circ \mathcal{S}^{-1})$ maps an interval containing 0 into (a) $\{|\text{Re } z| \leq \text{Im } z\}$ (if necessary after a fixed rotation of its image, by some $\theta \in \mathbb{R}$), or (b) $\{|\text{Re } z| \geq 0\}$. In the case (a) Theorem 4.3 applies and in the case (b) Theorem 3.2 applies, in each case implying that the function $(F \circ \mathcal{S}^{-1})$ vanishes on the connected component of the origin in $\mathcal{S}(D \cap V)$, which implies that it vanishes on an open subset of $W^+$ so using that $\mathcal{S}$ has open inverse we obtain (by the identity theorem) $F \equiv 0$ on $\mathcal{U} \cap D$, and by continuity $f = 0$ on $D \cap V$. Now $D$ was an arbitrary complex one-dimensional manifold which sufficiently near 0 had intersection with $M$ belonging to a certain double cone near 0. This can be repeated for all one-dimensional complex $D$ which are perturbations of $D$, each passing through 0, and whose intersection with $M$ belong, near 0, to the
given double cone. So \( F \) vanishes on the union of intersections \( \tilde{D} \cap \mathcal{U} \), as \( \tilde{D} \) varies over such complex one dimensional manifolds. The union of all such \( \tilde{D} \) covers an open subset of \( \mathcal{U} \), so again by the identity theorem \( f \equiv 0 \) near 0.

We shall use the following observation.

**Observation 4.2.** Any representative, \( W \), of \( \mathfrak{a}(p_0) \), \( p_0 \in M \), is an embedded \( CR \) submanifold (see e.g. Baouendi et al. [5], p.95). This implies \( T_z^c W \subset T_z^c M \) for each \( z \in W \). Assume \( p_0 \) (for the remainder of this observation) is non-minimal. By definition of non-minimality at \( p_0 \), \( CR \text{dim}(M) \leq \dim\mathfrak{a}(p_0) < \dim\mathfrak{a} M \), and since the real codimension of \( M \) is one, \( CR - \text{dim}(M) = \dim\mathfrak{a}(p_0) \), thus \( T_z^c W = T_z^c M \) for each \( z \in W \), which implies that \( W \) is a complex \((n-1)\)-dimensional manifold containing \( p_0 \). By the transversality condition \((4.1)\) for \( \gamma \), it must be transversal to any member of the local \( CR \) orbit at a point of \( \gamma \). If \( W \subset M \) is a small open neighborhood of \( p_0 \), then every point in \( W \) which also belongs to \( \gamma \) is associated to a family of complex \((n-1)\)-dimensional manifolds (each a member of a different local \( CR \) orbit). In the case of real codimension one, the restriction of the \( CR \) function \( f \) to any complex submanifold \( \mathfrak{M}_{p_0} \subset M \), passing through \( p_0 \in \gamma \) is a holomorphic function, hence \( f \) vanishes within \( \mathfrak{M}_{p_0} \), as soon as \( \mathfrak{M}_{p_0} \) is a member of the local orbit at \( p_0 \in \gamma \). This concludes the observation.

Given a reference point \( p_0 \in \gamma \), we parametrize \( \gamma \), locally near a sufficiently small neighborhood \( \mathcal{W} \subset M \) of \( p_0 \), by introducing smooth local coordinates,

\[
\gamma \cap \mathcal{W} = \{(\phi_1, \ldots, \phi_{2n-2}, \varphi) : \phi_1 = \cdots = \phi_{2n-2} = 0\},
\]

where \( p_0 = (0, \hat{\varphi}) \) is a point of \( \gamma \cap \mathcal{W} \).

**The strategy of the proof is as follows:**

- We construct a open \( M \)-neighborhood, denoted \( \mathcal{C}_{\hat{\varphi}} \) (see \((4.8)\)), of \((0, \hat{\varphi})\), such that every point of \( \mathcal{C}_{\hat{\varphi}} \) belongs to the global Sussmann orbit, \( S(0, \varphi) \), of some point \((0, \varphi)\) (with \( \varphi \) near \( \hat{\varphi} \)).
- We then proceed to prove that \( f \equiv 0 \) on \( \mathcal{C}_{\hat{\varphi}} \), see Claim \(4.3\) and the proof of the latter claim is divided into two main cases based upon minimality.
- In the first case (denoted (i)) appearing in the proof of Claim \(4.3\) Lemma \(4.1\) is invoked.
- The second case (denoted (ii)) is divided into two subcases (based upon existence and non-existence respectively, of minimal points in a given global orbit passing \( \gamma \) near the reference point). Observation \(4.2\) is used to handle the easy subcase when all points of a given global orbit are non-minimal. The second subcase requires more work in terms of invoking known propagation results in fusion with the properties of \( \mathcal{C}_{\hat{\varphi}} \).

Remark \(2.1\) shows that if we pick a nonzero vector field \( Z \in \Gamma(\mathcal{W}, T^c M) \), and we introduce the parameter \( \vartheta \), (to be further specified later) on which \( Z \) depends, i.e. \( Z = Z_\vartheta \), then there is a unique integral curve, \( \eta(t, (0, \varphi), \vartheta) =: \Phi_{Z_\vartheta, t}((0, \varphi), \vartheta) \), of \( Z \) originating at \((0, \varphi)\) defined for \( t \in [0, T((0, \varphi), \vartheta)) \), where \( T((0, \varphi), \vartheta)) \) is a lower semi-continuous function near \((0, \varphi), 0)\), (i.e \( T \) is the maximal time parameter as in Remark \(2.1\), specifically, given any \( \epsilon > 0 \), \((0, \hat{\varphi}) \in \gamma \cap \mathcal{W} \), and any \( \vartheta_0 \), there
exists a $\delta(\hat{\varphi}, \partial_{\varphi}, \epsilon)$ such that,

$$(((\varphi, \varphi) - (0, \hat{\varphi})) < \delta(\hat{\varphi}, \partial_{\varphi}, \epsilon)) \land (\left|\partial - \hat{\varphi}\right| < \delta(\hat{\varphi}, \partial_{\varphi}, \epsilon)) \Rightarrow T((\varphi, \varphi), \partial) \geq T((0, \hat{\varphi}), \partial_{\varphi}) - \epsilon,$$  \hspace{1cm} (4.4)

hence $T$ is bounded from below as $((\varphi, \varphi), \partial)$ varies on an open box-neighborhood of $(\hat{\varphi}, \partial_{\varphi})$. This in turn implies that $\Phi_{Z_{\alpha}, \ell}((0, \varphi))$, which defines the end point of the integral curve of the vector field $Z_{\phi}$ passing through $(0, \varphi)$, varies smoothly with respect to the base point, near $(0, \hat{\varphi})$. For each $z \in \gamma$, we let $\mathcal{S}$ denote the set of points of $M$ which can be reached from $q$ by a polygonal path (see the definition in the preliminaries) of integral curves to sections of $T_{c}M$ ($\mathcal{S}$ is called the global Sussmann orbit at $q$). Let $\mathcal{W}$ be sufficiently small such that there is a basis, of vector fields $v_{1}, \ldots, v_{2n-2}$ (we assume each $v_{k}$ is normalized), for the set of sections of $T_{c}M$ over $\mathcal{W}$. Let,

$$Z_{\phi} := Z_{\alpha} + \sum_{k=1}^{2n-2} \partial_{\varphi}v_{k}. \hspace{1cm} (4.5)$$

We shall use $Z_{\alpha} = 0$, in which case we know that each $Z_{\phi}$ is a section of $T_{c}M$, and we shall use $\partial_{\varphi} = 0$.

Given $\partial_{\varphi} = 0$, we set $T(\hat{\varphi}) := T((0, \hat{\varphi}), 0)$. We complement $v_{1}, \ldots, v_{2n-2}$ to full basis by adjoining a vector field $v_{2n-1}$ which along $\gamma$ coincides with $\partial_{\varphi}$.

Next we consider the map,

$$\Psi: (\partial, \varphi) \mapsto \Phi_{((Z_{\alpha} + (\varphi - \hat{\varphi})v_{2n-1}), 1)((0, \hat{\varphi}))}. \hspace{1cm} (4.6)$$

Since $v_{1}, \ldots, v_{2n-1}$ form a basis for $T\mathcal{W}$, the map $\Psi$ has nonzero determinant at $(0, \hat{\varphi})$ (see e.g. Baouendi et al. [3], p.65) so let $\{\varphi v_{2n-1} + \sum_{j=1}^{2n-2} \partial_{\varphi}v_{j}: |\varphi - \hat{\varphi}| < \nu, |\partial_{\varphi} - \hat{\varphi}| < \nu\} =: B_{\nu} \subset TM$ be such that the image of any subdomain of $B_{\nu}$ containing $(0, \hat{\varphi})$, under $\Psi$, is an open subset of $M$, and such that the maximal time parameter $T$ above is bounded from below on $B_{\nu}$ by $T(\hat{\varphi})/8$. In particular we must chose $\epsilon < T(\hat{\varphi})/8$ above and $\nu < \delta(\hat{\varphi}, 0, \epsilon)$. Let $a = \min\{1/8, T(\hat{\varphi})/8, \nu/8\}$. Define the following sets,

$$C(\varphi) = \bigcup_{|\partial_{\varphi} < a} \Phi_{Z_{\alpha}, \varphi}((0, \varphi)), \hspace{1cm} (4.7)$$

$$C_{\varphi} = \bigcup_{\varphi \in \{s: |s - \hat{\varphi}| < a\}} C(\varphi). \hspace{1cm} (4.8)$$

Now for fixed $\varphi$ (sufficiently near $\hat{\varphi}$ as above) and $|\partial_{\varphi}| < a$, we have,

$$\Phi_{Z_{\alpha}, \varphi}((0, \varphi)) = \Phi_{Z_{\alpha}, \varphi}(\Phi_{v_{2n-1}, \varphi - \hat{\varphi}}((0, \hat{\varphi})) = \Psi((a, \varphi, \varphi)). \hspace{1cm} (4.9)$$

Also for fixed $\varphi$ such that $|\varphi - \hat{\varphi}| < a$, the union, $\bigcup_{|\partial_{\varphi} < a} \Phi_{Z_{\alpha}, \varphi}(0, \varphi)$ belongs to $S_{\varphi}(0, \hat{\varphi})$ (the global Sussmann orbit). Since we already know that its image under $\Psi$ is an open subset of $M$ containing $(0, \hat{\varphi})$ we obtain that the union, $\bigcup_{|\varphi - \hat{\varphi}| < a} S_{\varphi}$ contains an $M$-open neighborhood of $(0, \hat{\varphi})$.

**Claim 4.3.** $f \equiv 0$ on $C_{\varphi}$.

**Proof.** Indeed, there are two cases which can occur given a $\varphi \in \{|\varphi - \hat{\varphi}| < a\}$:

(i) $(0, \varphi)$ is a minimal point of $M$. It is a known result (due to Trepreau [33] and generalized by Tumanov [34], for our precise formulation, see Trepreau [32], p.409)
that minimality at a point implies holomorphic extension of $f$ to one side of $M$ near that point i.e. we assume that $f$ has holomorphic extension to one side of $M$, near $(0, \varphi)$. By Lemma 4.1 we obtain that $f \equiv 0$ on an $M$-neighborhood of $(0, \varphi)$. This however, by definition implies that $(0, \varphi)$ does not belong to $\text{supp} f$, which in turn by the known result of Treves [30], p.91, this implies that $\mathcal{S}_{(0,\varphi)} \cap \text{supp} f = \emptyset$, so $f$ vanishes on $\mathcal{C}(\varphi)$ because the latter set is a subset of $\mathcal{S}_{(0,\varphi)}$.

(ii) $(0, \varphi)$ is a non-minimal point of $M$. If all points of $\mathcal{S}_{(0,\varphi)}$ are non-minimal then there passes through each, a complex $(n - 1)$- dimensional manifold and the vanishing of $f$ near $(0, \varphi)$ (in the sense of Remark 4.2) propagates along each such manifold, from $(0, \varphi)$, so $f$ must vanish on $\mathcal{S}_{(0,\varphi)}$. Assume instead that there is a minimal point, $q$, belonging to $\mathcal{S}_{(0,\varphi)}$. By definition $q$ can be reached from $q_0 = (0, \varphi)$ by a polygonal path of $\mathcal{C}R$ curves in $\mathcal{S}_{(0,\varphi)}$. For a $C^\infty$-smooth hypersurface $M \subset \mathbb{C}^n$, it is a known result that holomorphic extension to one side of $M$, at a given point $q \in M$, of continuous $\mathcal{C}R$ functions, holds true if and only if there does not pass a germ of a complex $(n - 1)$-dimensional submanifold of $M$ through $q$, and the last condition is equivalent to minimality at $q$ (see e.g. Baouendi et al. [5], Theorem 1.5.15, p.20).

Also, in the case of $C^\infty$-smooth hypersurfaces holomorphic extension to one side of $M$ coincides with holomorphic wedge-extension, which in turn propagates along a given $\mathcal{C}R$ curve (the direct consequence of the latter result, stated in the terms we shall use it, can be found in Trepreau [32], Theorem 2, p.409; the more detailed cause of propagation can be found in Trepreau [32], p.418, and information about directionality in Tumanov [33]-[35]). Hence, we can assume $f$ has holomorphic extension to one side of $M$, near each point of $\mathcal{S}_{(0,\varphi)}$. Let $\mathcal{U}$ be an open subset of $\mathbb{C}^n$, such that $V := \partial \mathcal{U} \cap M$ contains $q_0$, is open in $M$, and such that there exists a function $F \in \mathcal{O}(\mathcal{U}) \cap C^0(\mathcal{U} \cup V)$, $F|_V = f|_V$. By Lemma 4.1 $f \equiv 0$ on an $M$-neighborhood of $q_0$. Theorem 3.5 implies that $f$ vanishes at all points of $\mathcal{S}_{(0,\varphi)}$. This completes the proof of Claim 4.3.

By Claim 4.3, $f \equiv 0$ on an open $M$-neighborhood of $p_0$ and since the latter point was an arbitrary point of $\gamma$ this also completes the proof of Theorem 1.4.

5. SOME EXAMPLES ON GEOMETRIC CONDITIONS ON $M$ WITH REDUCED GROWTH CONDITIONS

Example 5.1 (The Levi flat case). Let $M \subset \mathbb{C}^n$ be a $C^\infty$-smooth hypersurface and let $\gamma \subset M$ be a $C^\infty$-smooth curve which is not locally the intersection with a complex line, but satisfies the condition of 4.1. Assume $M$ is Levi flat on an $M$-open neighborhood, $U$, of $\gamma$.

Then any smooth $\mathcal{C}R$ function which vanishes to infinite order along $\gamma$ must vanish on an open $M$-neighborhood of $\gamma$: Let $p_0 \in \gamma$, and assume w.l.o.g., $p_0$ coincides with the origin, in $\gamma$ and in $M$. It is a known consequence of the complex version of Frobenius theorem (see Freeman [13]) that there passes through each point of $U \cap \gamma$, a complex manifold of complex dimension $n - 1$ (i.e. the $\mathcal{C}R$ dimension of $M$). In particular every point of $U$ is a non-minimal point of $M$. Hence we can apply the proof of (i), to the $C^\infty$-smooth $\mathcal{C}R$ function $f$, vanishing to infinite order along $\gamma$, in the $C^\infty$-smooth hypersurface $U$ (the reason being that in Claim 4.3 the requirements (a)-(c) are not invoked). This will yield that $f$ vanishes on the Sussmann orbit of each point of $\gamma \cap U$, in $U$. As the proof of our main result shows, the union of such Sussmann orbits cover an $M$-open neighborhood of $p_0$. 

\[ \square \]
Example 5.2. When $M \subset \mathbb{C}^n$ is a $C^\infty$-smooth hypersurface then it was proved by Rosay [23] (the proof uses a result of Andreotti & Hill [4]), that if $U \cap \gamma = U \cap D$ for a complex ambient line $D$, transversal to $U \cap M$, for some small open $U$, then any $f \in CR^\infty(M)$ which vanishes to infinite order along $U \cap \gamma$, must vanish on an open $M$-neighborhood of $U \cap \gamma$. This example does not require that the origin is a minimal point, and does not have additional growth conditions compared to Theorem 1.4.

Example 5.3 (The real-analytic case). When $M \subset \mathbb{C}^n$ is a $C^\infty$-smooth hypersurface and $\gamma \subset M$ a real-analytic curve, then any Lipschitz continuous $CR$ function that vanishes to infinite order along $\gamma$, vanishes on an $M$-open neighborhood of $\gamma$. This result is due to Baouendi & Treves [7] (see Treves [30], Theorem II.8.1, together with Corollary II.8.1, p.118, for a textbook version). Let $(z_1, \ldots, z_n)$ denote holomorphic coordinates, $z_n = x_n + i y_n$, let $0 \in \gamma$ and $U$ be open in $M$ such that, $0 \in U$, $U \cap M = U \cap \{y_n = h(z_1, \ldots, z_{n-1}, x_n)\}$ for a real-analytic graphing function $h$. The unique continuation result of Baouendi & Treves [7] in this real-analytic case, is a consequence of the so called compact cocycle property for $\{y_n = h(0, x_n)\}$.

Definition 5.4 (see Treves [30], p.115). Let $\Lambda$ be a maximally real submanifold of $\mathbb{C}^n$, $p \in \Lambda$. $\Lambda$ is said to have the compact cocycle property at $p$ if there is a basis of neighborhoods of $p$ such that, if $N$ is any one of these neighborhoods, then there is $F \in \mathcal{O}(N)$ with $F(p) \neq 0$ and $\{w \in \Lambda \cap N : F(w) \neq 0\} \subset \Lambda \cap N$.

Here is an example, covered by Theorem 1.4 of this paper, where $\Sigma \neq \emptyset$.

Example 5.5. The following function on $\mathbb{R}$ is known to be $C^\infty$-smooth but nowhere real-analytic (see e.g. Kim & Kwon [21]),

$$\rho(x) := \sum_{k=1}^{\infty} \frac{1}{k!} \theta \left( 2^k (x - \lfloor x \rfloor) \right),$$

(5.1)

where $\lfloor \cdot \rfloor$ denotes the least upper integer, $\theta(x) := \exp \left( -\frac{1}{x^2} \right) \exp \left( -\frac{1}{(x-1)^2} \right)$, $0 < x < 1$, and $\theta(x) = 0$, $x \notin (0,1)$. Let $(z_1 = x_1 + i y_1, z_2 = x_2 + i y_2) \in \mathbb{C}^2$ be holomorphic coordinates and define for each $j \in \mathbb{Z}_+$, $\chi_j(z_1, x_2) \in C^\infty_c(B_j)$ (where $B_j := B_{\frac{1}{2}}(0, \frac{1}{j})$, and $B_r(p)$ denotes the ball in $\mathbb{C} \times \mathbb{R}$, of radius $r$, and center $p$) such that $\chi_j = 1$ on $C_j := B_{\frac{1}{j}}(0, \frac{1}{j})$, (see e.g. HÃºrmander [19], Theorem 1.4.1, p.25, for the existence of such $\chi_j$). Let $B := \bigcup_{j \in \mathbb{Z}_+} B_j$, $C := \bigcup_{j \in \mathbb{Z}_+} C_j$, and define $M := \{(z_1, z_2) \in \mathbb{C}^2 : y_2 = \rho(z_1, x_2)\}$, where,

$$h(z_1, x_2) := \begin{cases} \rho(x_2) + \chi(x_2, z_1) \left| z_1 \right|^2 - \rho(x_2) \\
\rho(x_2) \end{cases},$$
onumber

(5.2)

on $B$, otherwise.

By construction $M \subset \mathbb{C}^2$ is: (i) a smooth hypersurface $(M = \{(z_1, z_2) \in \mathbb{C}^2 : \psi = 0\})$ where $\psi(z_1, z_2) := y_2 - h(z_1, x_2)$, with $\frac{\partial \psi}{\partial z_2} \equiv 1$, so $\partial \psi |_{\Sigma} \neq 0$ near $0$, (ii) not real-analytic on any open subset, (iii) strictly pseudoconvex on the subset $\mathcal{C} \subset M$, and (iv) Levi flat at all points of $M \setminus \overline{B}$. Then, for any $C^\infty$-smooth curve $\gamma \subset \{0\} \cup (M \setminus \overline{B})$ with $0 \in \gamma$ we have a decomposition $\gamma = \Sigma \cup (\gamma \setminus \Sigma)$, where $\Sigma$.

\[^5\] $H^{1,0}M$ spanned by (see Boggess [19], p.144), $\mathcal{L} = -2i \left( \frac{1}{1+z_1 \bar{z_2}} \right) \frac{\partial \rho}{\partial z_2} \frac{\partial z_2}{\partial \bar{z_2}} + \frac{\partial \rho}{\partial z_1} = \frac{\partial}{\partial z_1}$, since $h$ is (recall that we are speaking of the set $M \setminus \overline{B}$) independent of $Re z_1, Im z_1$. Thus $[L, \mathcal{L}] = 0$. 
denotes the set of points \( z \in \gamma \) such that every \( M \)-neighborhood of \( z \) contains a point where the Levi form \( \mathcal{L} \) of \( M \) is nonzero.

**Remark 5.6.** After the completion of this paper we were informed that Alexander [2], Theorem 1, p.2, proved a stronger version of Theorem 3.3 indeed the condition that \( f \) map the part of the real axis as in Theorem 3.3 into a non-spiraling set will imply that either \( f \equiv 0 \) or \( f \) cannot vanish to infinite order at 0. In fact by Alexander’s result we may replace the condition \( |v(t)| \leq |u(t)| \) for \( t \in (-1,1) \) with the condition \( |v(t)| \leq C|u(t)| \) for \( t \in (-1,1) \), and a non-negative constant \( C \). We have chosen to state our results using the weaker version found in Theorem 3.3.

**References**


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\[6\] This paper only deals with the case of a smooth hypersurface \( M \) in which case the Levi form at \( p_0 \in M \) is particularly easy to describe. Let \( U \) be an open subset containing \( p_0 \), such that \( M \cap U = \{ \rho = 0 \} \), for some \( \rho : U \to \mathbb{R} \) with \( |\nabla \rho(p_0)| = 1 \). The Levi form at \( p_0 \in M \), is (a constant multiple of), \( L_{M,p_0}(W) = - \left( \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial \zeta_j \partial \zeta_k}(p_0) \zeta_j \zeta_k \right) \nabla \rho(p_0) \), \( W = \sum_{k=1}^{n} \zeta_k \frac{\partial}{\partial \zeta_k} \in H^1_p M \).

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