FABER POLYNOMIALS COEFFICIENT ESTIMATES FOR BI-UNIVALENT SAKAGUCHI TYPE FUNCTIONS

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Abstract. In this work, considering a general subclass of bi-univalent Sakaguchi type functions, we determine estimates for the general Taylor-Maclaurin coefficients of the functions in these classes. For this purpose we use the Faber polynomial expansions, and in certain cases our estimates improve some of those existing coefficient bounds.

1. Introduction

Let \( A \) denote the class of all functions of the form
\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n
\]
which are analytic in the open unit disk \( U := \{ z \in \mathbb{C} : |z| < 1 \} \). We also denote by \( S \) the class of all functions in the normalized analytic function class \( A \) which are univalent in \( U \).

It is well known that every function \( f \in S \) has an inverse \( f^{-1} \), which satisfy \( f^{-1}(f(z)) = z \) for all \( z \in U \) and \( f \left(f^{-1}(w) \right) = w \) for all \( |w| < r_0(f) \), with \( r_0(f) \geq \frac{1}{4} \).

In fact, the inverse function \( g := f^{-1} \) is given by
\[
g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \ldots
\]
\[= w + \sum_{n=2}^{\infty} A_n w^n. \tag{1.2}
\]

A function \( f \in A \) is said to be bi-univalent in \( U \) if both \( f \) and \( f^{-1} \) are univalent in \( U \), and let \( \Sigma \) denote the class of bi-univalent functions in \( U \) given by (1.1).

The class of analytic bi-univalent functions was first introduced and studied by Lewin \[15\], where it was proved that \( |a_2| < 1.51 \). Netanyahu \[16\] proved that \( |a_2| \leq \frac{1}{3} \). Brannan and Taha \[4\] also investigated certain subclasses of bi-univalent functions.
functions and found non-sharp estimates on the first two Taylor-Maclaurin coefficients \(|a_2|\) and \(|a_3|\). For a brief history and interesting examples of functions in the class \(\Sigma\), see [19]. In fact, the aforecited work of Srivastava et al. [19] essentially revived the investigation of various subclasses of the bi-univalent function class \(\Sigma\) in recent years; it was followed by such works as those by Frasin and Aouf [8], Xu et al. [21, 22], Hayami and Owa [12].

Not much is known about the bounds on the general coefficient \(|a_n|\) for \(n > 3\). This is because the bi-univalency requirement makes the behaviour of the coefficients of the functions \(f\) and \(f^{-1}\) unpredictable. In this paper we use the Faber polynomial expansions for a general subclass of bi-univalent Sakaguchi type functions.

The Faber polynomials introduced by Faber [6] play an important role in various areas of mathematical sciences, especially in geometric function theory. The recent publications [9] and [11] applying the Faber polynomial expansions to meromorphic bi-univalent functions motivated us to apply this technique to classes of analytic bi-univalent functions.

In the literature, there are only a few works determining the general coefficient bounds \(|a_n|\) for the analytic bi-univalent functions given by (1.1) using Faber polynomial expansions [10, 13, 14]. Hamidi and Jahangiri [10] considered the class of analytic bi-close-to-convex functions. Also, Jahangiri and Hamidi [13] studied the class defined by Frasin and Aouf [8], while Jahangiri et al. [14] investigated the class of analytic bi-univalent functions with positive real-part derivatives.

Motivated by the works of Bulut, we defined and studied the main properties of the following classes. We begin by finding the estimate on the coefficients \(|a_n|\), \(|a_2|\) and \(|a_3|\) for bi-univalent Sakaguchi type functions in the classes \(P_\Sigma(\alpha, \lambda, t)\) and \(Q_\Sigma(\alpha, \lambda, t)\) respectively.

2. The Classes \(P_\Sigma(\alpha, \lambda, t)\) and \(Q_\Sigma(\alpha, \lambda, t)\)

**Definition 2.1.** For \(0 \leq \lambda \leq 1\), \(|t| \leq 1\) and \(t \neq 1\), a function \(f \in \Sigma\) given by (1.1) is said to be in the class \(P_\Sigma(\alpha, \lambda, t)\) if the following conditions are satisfied:

\[
\text{Re} \left( \frac{(1-t)zf'(z)}{(1-\lambda)[f(z) - f(tz)] + \lambda z[f'(z) - tf'(tz)]} \right) > \alpha, \ z \in U
\]

and

\[
\text{Re} \left( \frac{(1-t)wg'(w)}{(1-\lambda)[g(w) - g(tw)] + \lambda w[g'(w) - tg'(tw)]} \right) > \alpha, \ w \in U
\]

where \(0 \leq \alpha < 1\) and \(g := f^{-1}\) is defined by (1.2).

**Definition 2.2.** For \(0 \leq \lambda \leq 1\), \(|t| \leq 1\) and \(t \neq 1\), a function \(f \in \Sigma\) given by (1.1) is said to be in the class \(Q_\Sigma(\alpha, \lambda, t)\) if the following conditions are satisfied:

\[
\text{Re} \left( \frac{(1-t)[\lambda z f''(z) + zf'(z)]}{f(z) - f(tz)} \right) > \alpha, \ z \in U
\]

and

\[
\text{Re} \left( \frac{(1-t)[\lambda w g''(w) + wg'(w)]}{g(w) - g(tw)} \right) > \alpha, \ w \in U
\]

where \(0 \leq \alpha < 1\) and \(g := f^{-1}\) is defined by (1.2).
Remarks. 1. Taking extends to any coefficients of its inverse map and order such that are variables.

Using the Faber polynomial expansion of functions, we get the well-known class \( P_\Sigma(\alpha) := P_\Sigma(\alpha, 0, 0) = Q_\Sigma(\alpha, 0, 0) \) of bi-starlike functions of order \( \alpha \). This class consists of functions \( f \in \Sigma \) satisfying \( \operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha, z \in \mathbb{U}, \) and \( \operatorname{Re} \frac{wg'(w)}{g(w)} > \alpha, w \in \mathbb{U}, \) where \( 0 \leq \alpha < 1 \) and \( g := f^{-1} \) is defined by (1.2).

2. The name of Sakaguchi type functions is motivated by the papers [18] and [7].

3. Coefficient Estimates

Using the Faber polynomial expansion of functions \( f \in A \) of the form (1.1), the coefficients of its inverse map \( g = f^{-1} \) may be expressed like in [3], that is

\[
g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n} (a_2, a_3, \ldots, a_n) w^n,
\]

where

\[
K_{n-1}^{-n} (a_2, a_3, \ldots, a_n) = \frac{(-n)!}{(2n+1)![(n-1)!]!} a_2^{n-1} + \frac{(-n)!}{(2(-n+1))(n-3)!} a_2^{n-3} a_3
\]

\[
+ \frac{(-n)!}{(2n+3)![(n-4)!]!} a_2^{n-4} a_4 + \frac{(-n)!}{(2(-n+2))(n-5)!} a_2^{n-5} [a_5 + (-n+2)a_3^2]
\]

\[
+ \frac{(-n)!}{(2n+5)![(n-6)!]!} a_2^{n-6} [a_6 + (-n+5)a_3a_4] + \sum_{j \geq 7} a_2^{n-j} v_j,
\]

such that \( v_j \), with \( 7 \leq j \leq n \), is a homogenous polynomial of degree \( j \) in the variables \( a_2, a_3, \ldots, a_n \). In particular, the first three terms of \( K_{n-1}^{-n} (a_2, a_3, \ldots, a_n) \) are

\[
K_{-2}^{-2} (a_2) = -2a_2, \quad K_{-3}^{-3} (a_2, a_3) = 3 (2a_2^2 - a_3),
\]

\[
K_{-4}^{-4} (a_2, a_3, a_4) = -4 (5a_2^3 - 5a_2 a_3 + a_4).
\]

For the above formulas we used the fact that for any integer \( p \in \mathbb{Z} \) the expansion of \( K_n^p \) has the form (see [2, p. 349])

\[
K_n^p := K_n^p (b_1, b_2, \ldots, b_n) = \frac{pl}{(p-n)!n!} b_1^n + \frac{pl}{(p-n+1)![(n-2)!]} b_1^{n-2} b_2
\]

\[
+ \frac{pl}{(p-n+2)![(n-4)!]} b_1^{n-3} b_3 + \frac{pl}{(p-n+3)![(n-4)!]} b_1^{n-4} \left[ b_4 + \frac{p-n+3}{2} b_2^2 \right]
\]

\[
+ \frac{pl}{(p-n+4)![(n-5)!]} b_1^{n-5} [b_5 + (p-n+4)b_2 b_3] + \sum_{j \geq 6} b_1^{n-j} v_j,
\]

such that \( v_j \), with \( 6 \leq j \leq n \), is a homogenous polynomial of degree \( j \) in the variables \( b_1, b_2, \ldots, b_n \), and the notation

\[
\frac{pl}{(p-n)!n!} := \frac{(p-n+1)(p-n+2)\ldots p}{n!}
\]

extends to any \( p \in \mathbb{Z} \).
In general, for any \( p \in \mathbb{Z} \) the expansion of \( K^n_k \) has the form (see [3] p. 183)

\[
K^n_k (b_1, b_2, \ldots, b_n) = pb_n + \frac{p(p - 1)}{2} D_n^2 + \frac{p!}{(p - 3)!3!} D_n^3 + \cdots + \frac{p!}{(p - n)!n!} D_n^n,
\]

where \( D_n^p \) are given by (see [20, p. 268–269])

\[
D_n^p := D_n^m (b_1, b_2, \ldots, b_{n-m+1}) = \sum_{i_1, \ldots, i_{n-m+1}} \frac{m!}{i_1! \cdots i_{n-m+1}!} b_1^{i_1} \cdots b_n^{i_n},
\]

and the sum is taken over all non-negative integers \( i_1, \ldots, i_{n-m+1} \) satisfying

\[
i_1 + i_2 + \cdots + i_{n-m+1} = m,
\]
\[
i_1 + 2i_2 + \cdots + (n - m + 1)i_{n-m+1} = n.
\]

It is obvious that \( D_n^m (b_1, b_2, \ldots, b_n) = b_n^m \).

Consequently, for any function \( f \in P_2(\alpha, \lambda, t) \) of the form (1.1), we can write

\[
\frac{(1 - t) [zf'(z)]}{(1 - \lambda) [f(z) - f(tz)] + \lambda z [f'(z) - tf'(tz)]} = 1 + \sum_{n=2}^{\infty} F_{n-1} (a_2, a_3, \ldots, a_n) z^{n-1},
\]

where \( F_{n-1} \) is the Faber polynomial of degree \( (n - 1) \) and

\[
F_1 = [2(1 - \lambda) - u_2(1 - \lambda + 2\lambda t)] \frac{a_2}{1 + \lambda t},
\]
\[
F_2 = \frac{1}{1 + \lambda t} \left\{ [3(1 - \lambda) - u_3(1 - \lambda + 3\lambda t)] a_3 - F_1 [2\lambda + u_2(1 - \lambda + 2\lambda t)] a_2 \right\}
\]
\[
= [3(1 - \lambda) - u_3(1 - \lambda + 3\lambda t)] \frac{a_3}{1 + \lambda t}
\]
\[
- [2(1 - \lambda) - u_2(1 - \lambda + 2\lambda t)] [2\lambda + u_2(1 - \lambda + 2\lambda t)] \frac{a_2^2}{(1 + \lambda t)^2},
\]
\[
F_3 = \frac{1}{1 + \lambda t} \left\{ [4(1 - \lambda) - u_4(1 - \lambda + 4\lambda t)] a_4 - F_2 [2\lambda + u_2(1 - \lambda + 2\lambda t)] a_2
\]
\[
- F_1 [3\lambda + u_3(1 - \lambda + 3\lambda t)] a_3 \right\}
\]
\[
= [4(1 - \lambda) - u_4(1 - \lambda + 4\lambda t)] \frac{a_4}{1 + \lambda t}
\]
\[
- [2(1 - \lambda) - u_2(1 - \lambda + 2\lambda t)] [3\lambda + u_3(1 - \lambda + 3\lambda t)] \frac{a_2 a_3}{(1 + \lambda t)^2}
\]
\[
- [3(1 - \lambda) - u_3(1 - \lambda + 3\lambda t)] [2\lambda + u_2(1 - \lambda + 2\lambda t)] \frac{a_2 a_3}{(1 + \lambda t)^2}
\]
\[
+ [2(1 - \lambda) - u_2(1 - \lambda + 2\lambda t)] [2\lambda + u_2(1 - \lambda + 2\lambda t)]^2 \frac{a_3^2}{(1 + \lambda t)^3}, \text{ etc.}
\]

where

\[
u_n := \frac{1 - t^n}{1 - t}, \ n \in \mathbb{N}.
\]

In general
where

\[ F_{n-1}(a_2, a_3, \ldots, a_n) = \frac{1}{(1 + \lambda t)} \left\{ n(1 - \lambda) - u_n(1 - \lambda + n\lambda) \right\} a_n \]

\[-F_{n-2}[2\lambda + u_2(1 - \lambda + 2\lambda)]a_2 - F_{n-3}[3\lambda + u_3(1 - \lambda + 2\lambda)]a_3 \]

\[ \cdots - F_1[(n - 1)\lambda + u_{n-1}(1 - \lambda + (n - 1)t)]a_{n-1} \].

Similarly, if the functions \( f \in Q_{\Sigma}(\alpha, \lambda, t) \) has the form (1.1), we can write

\[ \frac{(1 - t) \left[ \lambda z^2 f''(z) + zf'(z) \right]}{f(z) - f(tz)} = 1 + \sum_{n=2}^{\infty} F_{n-1}(a_2, a_3, \ldots, a_n) z^{n-1}, \tag{3.7} \]

where \( F_{n-1} \) is the Faber polynomial of degree \((n - 1)\) and

\[ F_1 = [2(\lambda + 1) - u_2]a_2, \]

\[ F_2 = [3(2\lambda + 1) - u_3]a_3 - F_1u_2a_2 \]

\[ = [3(2\lambda + 1) - u_3]a_3 - [2(\lambda + 1) - u_2]u_2a_2^2, \]

\[ F_3 = [4(3\lambda + 1) - u_4]a_4 - F_2u_2a_2 - F_1u_3a_3 \]

\[ = [4(3\lambda + 1) - u_4]a_4 - [2(\lambda + 1)u_3 + 3(2\lambda + 1)u_2 - 2u_2u_3]a_2a_3 \]

\[ + [2(\lambda + 1) - u_2]u_2a_2^3, \text{ etc.} \]

where \( u_n \) is given by (3.6). In general

\[ F_{n-1}(a_2, a_3, \ldots, a_n) = [(n(1 - \lambda) + 1) - u_n]a_n - F_{n-2}u_2a_2 - F_{n-3}u_3a_3 \]

\[ \cdots - F_1u_{n-1}a_{n-1}. \]

In our first theorem, for some special cases, we obtained an upper bound for the coefficients \(|a_n|\) of bi-univalent Sakaguchi type functions in the class \( P_{\Sigma}(\alpha, \lambda, t) \).

**Theorem 3.1.** For \( 0 \leq \lambda \leq 1, |t| \leq 1 \) with \( t \neq 1 \), and \( 0 \leq \alpha < 1 \), let the function \( f \in P_{\Sigma}(\alpha, \lambda, t) \) be given by (1.1). If \( a_k = 0 \) for all \( 2 \leq k \leq n - 1 \), then

\[ |a_n| \leq \frac{2(1 - \alpha) |1 + \lambda t|}{n(1 - \lambda) - u_n(1 - \lambda + n\lambda)}, \quad n \geq 4. \]

**Proof.** For the functions \( f \in P_{\Sigma}(\alpha, \lambda, t) \) of the form (1.1), we have the expansion (3.5), and for the inverse map \( g = f^{-1} \), according to (1.2), (3.1), we obtain

\[ (1 - t)wg'(w) \]

\[ \frac{(1 - \lambda) [g(w) - g(tw)] + \lambda w [g'(w) - tg'(tw)]}{\infty} \]

\[ = 1 + \sum_{n=2}^{\infty} F_{n-1}(A_2, A_3, \ldots, A_n) w^{n-1}, \quad z \in U, \tag{3.8} \]

where

\[ A_n = \frac{1}{n} K_{n-1}(a_2, a_3, \ldots, a_n). \]

On the other hand, since \( f \in P_{\Sigma}(\alpha, \lambda, t) \) and \( g = f^{-1} \in P_{\Sigma}(\alpha, \lambda, t) \), from the Definition 2.1 there exist two analytic functions \( p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \) and \( q(w) = \)
1 + \sum_{n=1}^{\infty} d_n w^n$, with $\text{Re} p(z) > 0$, $z \in \mathbb{U}$ and $\text{Re} q(w) > 0$, $w \in \mathbb{U}$, such that
\[
\frac{(1-t)zf'(z)}{(1-\lambda)[f(z) - f(tz)] + \lambda z [f'(z) - tf'(tz)]} = \alpha + (1-\alpha)p(z)
\]
= 1 + (1-\alpha) \sum_{n=1}^{\infty} K_n^1 (c_1, c_2, \ldots, c_n) z^n, \ z \in \mathbb{U}, \quad (3.9)

and
\[
\frac{(1-t)wg'(w)}{(1-\lambda)[g(w) - g(tw)] + \lambda w [g'(w) - tg'(tw)]} = \alpha + (1-\alpha)q(w)
\]
= 1 + (1-\alpha) \sum_{n=1}^{\infty} K_n^1 (d_1, d_2, \ldots, d_n) w^n, \ w \in \mathbb{U}. \quad (3.10)

Comparing the corresponding coefficients of (3.5) and (3.9), we get
\[
F_{n-1} (a_2, a_3, \ldots, a_n) = (1-\alpha)K_{n-1}^1 (c_1, c_2, \ldots, c_{n-1}), \ n \geq 2, \quad (3.11)
\]
and similarly, from (3.8) and (3.10) we find
\[
F_{n-1} (A_2, A_3, \ldots, A_n) = (1-\alpha)K_{n-1}^1 (d_1, d_2, \ldots, d_{n-1}), \ n \geq 2. \quad (3.12)
\]
Assuming that $a_k = 0$ for all $2 \leq k \leq n-1$, we obtain $A_n = -a_n$, and therefore
\[
\frac{n(1-\lambda) - u_n(1 - \lambda + n\lambda t)}{(1 + \lambda t)} a_n = (1-\alpha)c_{n-1},
\]
\[
\frac{n(1-\lambda) - u_n(1 - \lambda + n\lambda t)}{(1 + \lambda t)} a_n = (1-\alpha)d_{n-1}.
\]
From Carathéodory lemma (see, e.g., (3.5)) we have $|c_n| \leq 2$ and $|d_n| \leq 2$ for all $n \in \mathbb{N}$, and taking the absolute values of the above equalities, we obtain
\[
|a_n| = \frac{(1-\alpha)|c_{n-1}|(1 + \lambda t)}{|n(1-\lambda) - u_n(1 - \lambda + n\lambda t)|} = \frac{(1-\alpha)|d_{n-1}|(1 + \lambda t)}{|n(1-\lambda) - u_n(1 - \lambda + n\lambda t)|}
\]
\[
\leq \frac{2(1-\alpha)}{|n(1-\lambda) - u_n(1 - \lambda + n\lambda t)|},
\]
where $u_n$ is given by (3.6), which completes the proof of our theorem. \hfill \square

**Theorem 3.2.** For $0 \leq \lambda \leq 1$, $|t| \leq 1$ with $t \neq 1$, and $0 \leq \alpha < 1$, let the function $f \in Q_{\Sigma}(\alpha, \lambda, t)$ be given by (1.1). If $a_k = 0$ for all $2 \leq k \leq n-1$, then
\[
|a_n| \leq \frac{2(1-\alpha)}{n [(n-1)\lambda + 1] - u_n]}, \ n \geq 4,
\]
where $u_n$ is given by (3.6).

**Proof.** For the functions $f \in Q_{\Sigma}(\alpha, \lambda, t)$ of the form (1.1) we have the expansion (3.7), and for the inverse map $g = f^{-1}$, according to (1.2), (3.1), we obtain
\[
\frac{(1-t) [\lambda w^2 g''(w) + wg'(w)]}{g(w) - g(tw)}
\]
= 1 + \sum_{n=2}^{\infty} F_{n-1} (A_2, A_3, \ldots, A_n) w^{n-1}, \ w \in \mathbb{U}, \quad (3.13)
Theorem 3.3. For $0 \leq \lambda \leq 1$, $|t| \leq 1$, $t \neq 1$, $0 \leq \alpha < 1$, let the function $f \in P_2(\alpha, \lambda, t)$ be given by (3.1). Then, the following inequalities hold:

\[
|a_2| \leq \begin{cases} 
\frac{2(1-\alpha)|1+\lambda t|^2}{|B|}, & \text{for } 0 \leq \alpha < \frac{|A|}{2|B|}, \\
\frac{2(1-\alpha)|1+\lambda t|}{|2(1-\lambda)-u_2(1-\lambda+2\lambda t)|}, & \text{for } \frac{|A|}{2|B|} \leq \alpha < 1,
\end{cases}
\]
From (3.21) and (3.23), according to Carathéodory lemma we get
\[
|a_3| \leq \min\left\{ \frac{4(1-\alpha)^2(1+\lambda t)^2}{|2(1-\lambda) - u_2(1-\lambda + 2\lambda t)|^2} + \frac{2(1-\alpha)(1+\lambda t)}{3(1-\lambda) - u_3(1-\lambda + 3\lambda t)} \right\}, \quad \text{for} \quad 0 \leq \lambda < 1,
\]
\[
\frac{2(1-\alpha)|1+\lambda t|}{|3(1-\lambda) - u_3(1-\lambda + 3\lambda t)|}, \quad \text{for} \quad \lambda = 1,
\]
(3.19)

and
\[
|a_3 - \frac{C}{3(1-\lambda) - u_3(1-\lambda + 3\lambda t)}(1+\lambda t)^2| \leq \frac{2(1-\alpha)|1+\lambda t|}{|3(1-\lambda) - u_3(1-\lambda + 3\lambda t)|},
\]
where
\[
A = 2[3(1-\lambda) - u_3(1-\lambda + 3\lambda t)](1+\lambda t)
- [2(1-\lambda) - u_2(1-\lambda + 2\lambda t)][2(1+\lambda) + u_2(1-\lambda + 3\lambda t)],
\]
\[
B = [3(1-\lambda) - u_3(1-\lambda + 3\lambda t)](1+\lambda t)
- [2(1-\lambda) - u_2(1-\lambda + 2\lambda t)][2\lambda + u_2(1-\lambda + 2\lambda t)],
\]
\[
C = 2[3(1-\lambda) - u_3(1-\lambda + 3\lambda t)](1+\lambda t)
- [2(1-\lambda) - u_2(1-\lambda + 2\lambda t)][2\lambda + u_2(1-\lambda + 2\lambda t)].
\]
(3.20)

**Proof.** Setting \( n = 2 \) and \( n = 3 \) in (3.11) and (3.12) we get, respectively,
\[
[2(1-\lambda) - u_2(1-\lambda + 2\lambda t)] \frac{a_2}{1+\lambda t} = (1-\alpha)c_1,
\]
(3.21)
\[
[3(1-\lambda) - u_3(1-\lambda + 3\lambda t)] \frac{a_3}{1+\lambda t}
- [2(1-\lambda) - u_2(1-\lambda + 2\lambda t)][2\lambda + u_2(1-\lambda + 2\lambda t)] \frac{a_2^2}{(1+\lambda t)^2} = (1-\alpha)c_2,
\]
(3.22)
\[
[2(1-\lambda) - u_2(1-\lambda + 2\lambda t)] \frac{a_2}{1+\lambda t} = (1-\alpha)d_1,
\]
(3.23)
\[
\left\{ \frac{2}{2} \frac{[3(1-\lambda) - u_3(1-\lambda + 3\lambda t)](1+\lambda t)}(1+\lambda t)
- [2(1-\lambda) - u_2(1-\lambda + 2\lambda t)][2\lambda + u_2(1-\lambda + 2\lambda t)] \right\} \frac{a_2^2}{(1+\lambda t)^2}
- [3(1-\lambda) - u_3(1-\lambda + 3\lambda t)] \frac{a_3}{1+\lambda t} = (1-\alpha)d_2.
\]
(3.24)

From (3.21) and (3.23), according to Carathéodory lemma we get
\[
|a_2| = \frac{(1-\alpha)c_1|1+\lambda t|}{2(1-\lambda) - u_2(1-\lambda + 2\lambda t)} = \frac{(1-\alpha)d_1|1+\lambda t|}{2(1-\lambda) - u_2(1-\lambda + 2\lambda t)}
\]
\[
\leq \frac{2(1-\alpha)(1+\lambda t)}{|2(1-\lambda) - u_2(1-\lambda + 2\lambda t)|},
\]
(3.25)
Also, from (3.22) and (3.24) we obtain
\[ 2B \frac{a_2^2}{(1 + \lambda t)^2} = (1 - \alpha)(c_2 + d_2), \]
then, from Carathéodory lemma we get
\[ |a_2| \leq \sqrt{\frac{2(1 - \alpha)^2 |1 + \lambda t|^2}{|B|}}, \]
and combining this with inequality (3.25), we obtain the desired estimate on the coefficient \(|a_2|\) as asserted in (3.18).

In order to find the bound for the coefficient \(|a_3|\), subtracting (3.24) from (3.22), we get
\[ [3(1 - \lambda) - u_3(1 - \lambda + 3\lambda t)] (-2a_2^2 + 2a_3) = (1 - \alpha)(c_2 - d_2)(1 + \lambda t), \]
or
\[ a_3 = a_2^2 + \frac{(1 - \alpha)(c_2 - d_2)(1 + \lambda t)}{2[3(1 - \lambda) - u_3(1 - \lambda + 3\lambda t)]}. \]

Upon substituting the value of \(a_2^2\) from (3.21) into (3.27), it follows that
\[ a_3 = \frac{(1 - \alpha)^2 c_2^2 (1 + \lambda t)^2 + (1 - \alpha)(c_2 - d_2)(1 + \lambda t)}{2[3(1 - \lambda) - u_2(1 - \lambda + 3\lambda t)]^2 + 2[3(1 - \lambda) - u_3(1 - \lambda + 3\lambda t)]}, \]
and thus, from Carathéodory lemma we obtain that
\[ |a_3| \leq \frac{4(1 - \alpha)^2 |1 + \lambda t|^2}{2[3(1 - \lambda) - u_2(1 - \lambda + 3\lambda t)]^2 + 2[3(1 - \lambda) - u_3(1 - \lambda + 3\lambda t)]}. \]

On the other hand, upon substituting the value of \(a_2^2\) from (3.26) into (3.27) it follows that
\[ a_3 = \frac{(1 - \alpha)(1 + \lambda t) \left\{ c_2 C + d_2 \left[ \frac{[2(1 - \lambda) - u_2(1 - \lambda + 2\lambda t)] [2\lambda + u_2(1 - \lambda + 2\lambda t)]}{2B [3(1 - \lambda) - u_3(1 - \lambda + 3\lambda t)]} \right] \right\}}{2B [3(1 - \lambda) - u_3(1 - \lambda + 3\lambda t)]}, \]
and consequently, by Carathéodory lemma we have
\[ |a_3| \leq \frac{2(1 - \alpha) |1 + \lambda t|^2}{|B|}. \]

Combining (3.28) and (3.30), we get the desired estimate on the coefficient \(|a_3|\) as asserted in (3.19).

Finally, from (3.24), by using Carathéodory lemma we deduce that
\[ \left| a_3 - \frac{C}{[3(1 - \lambda) - u_3(1 - \lambda + 3\lambda t)](1 + \lambda t)} a_2^2 \right| \leq \frac{2(1 - \alpha) |1 + \lambda t|}{[3(1 - \lambda) - u_3(1 - \lambda + 3\lambda t)]}, \]
where \(A, B\) and \(C\) are given by (3.20). \(\square\)
Theorem 3.4. For $0 \leq \lambda \leq 1$, $|t| \leq 1$, $t \neq 1$, $0 \leq \alpha < 1$, let the function $f \in Q_S(\alpha, \lambda, t)$ be given by \[1.1\]. Then, the following inequalities hold:

$$|a_2| \leq \begin{cases} \displaystyle \frac{2(1-\alpha)}{|3(2\lambda + 1) - u_3 - [2(\lambda + 1) - u_2]u_2|}, & \text{for} \quad 0 \leq \alpha < \frac{2 |3(2\lambda + 1) - u_3 - [2(\lambda + 1) - u_2]u_2|}{2 \{3(2\lambda + 1) - u_3 - [2(\lambda + 1) - u_2]u_2\}}, \\ \displaystyle \frac{2(1-\alpha)}{|2(\lambda + 1) - u_2|}, & \text{for} \quad \lambda = 1, \end{cases}$$

$$(3.31)$$

$$|a_3| \leq \begin{cases} \displaystyle \min \left\{ \frac{4(1-\alpha)^2}{|2(1-\alpha) - u_2|} + \frac{2(1-\alpha)}{|3(2\lambda + 1) - u_3|}, \right. \\ \left. \frac{2(1-\alpha)}{|3(2\lambda + 1) - u_3 - [2(\lambda + 1) - u_2]u_2|}, \right. \end{cases}$$

$$0 \leq \lambda \leq 1,$$

$$(3.32)$$

and

$$|a_3 - \frac{2 |3(2\lambda + 1) - u_3 - [2(\lambda + 1) - u_2]u_2| a_2^2}{3(2\lambda + 1) - u_3} | \leq \frac{2(1-\alpha)}{|3(2\lambda + 1) - u_3|}.$$

Proof. Setting $n = 2$ and $n = 3$ in (3.16) and (3.17) we get, respectively,

$$[2(\lambda + 1) - u_2] a_2 = (1- \alpha)c_1,$$  

$$(3.33)$$

$$[3(2\lambda + 1) - u_3] a_3 - [2(\lambda + 1) - u_2] u_2 a_2^2 = (1- \alpha)c_2,$$  

$$(3.34)$$

$$- [2(\lambda + 1) - u_2] a_2 = (1- \alpha)d_1,$$  

$$(3.35)$$

$$\{2 |3(2\lambda + 1) - u_3 - [2(\lambda + 1) - u_2]u_2| a_2^2 - [3(2\lambda + 1) - u_3] a_3 = (1- \alpha)d_2.$$

$$(3.36)$$

From (3.33) and (3.35), according to Carathéodory lemma, we find

$$|a_2| = \frac{(1- \alpha)|c_1|}{|2(\lambda + 1) - u_2|} = \frac{(1- \alpha)|d_1|}{|2(\lambda + 1) - u_2|} \leq \frac{2(1-\alpha)}{|2(\lambda + 1) - u_2|}.$$

$$(3.37)$$

Also, from (3.34) and (3.36) we obtain

$$2 \{3(2\lambda + 1) - u_3 - [2(\lambda + 1) - u_2]u_2\} a_2^2 = (1- \alpha)(c_2 + d_2),$$

$$(3.38)$$

then, from Carathéodory lemma we get

$$|a_2| \leq \sqrt{\frac{2(1-\alpha)}{|3(2\lambda + 1) - u_3 - [2(\lambda + 1) - u_2]u_2|}}.$$
and combining this with inequality (3.37), we obtain the desired estimate on the coefficient \( |a_2| \) as asserted in (3.31).

In order to find the bound for the coefficient \( |a_3| \), subtracting (3.36) from (3.34), we get

\[
[3(2\lambda + 1) - u_3] (-2a_2^2 + 2a_3) = (1 - \alpha) (c_2 - d_2),
\]

or

\[
a_3 = a_2^2 + \frac{(1 - \alpha) (c_2 - d_2)}{2[3(2\lambda + 1) - u_3]}, \tag{3.39}
\]

Upon substituting the value of \( a_2^2 \) from (3.32) into (3.37), it follows that

\[
a_3 = \frac{(1 - \alpha)^2 c_2^2}{2(\lambda + 1 - u_2)^2} + \frac{(1 - \alpha) (c_2 - d_2)}{2[3(2\lambda + 1) - u_3]},
\]

and thus, from Carathéodory lemma we obtain that

\[
|a_3| \leq \frac{4(1 - \alpha)^2}{[2(\lambda + 1) - u_2]^2} + \frac{2(1 - \alpha)}{[3(2\lambda + 1) - u_3]}. \tag{3.40}
\]

On the other hand, upon substituting the value of \( a_2^2 \) from (3.38) into (3.39) it follows that

\[
a_3 = \frac{(1 - \alpha) c_2}{2\lambda + 1 - u_2} \left\{ c_2 \left[ 2[3(2\lambda + 1) - u_3] - [2(\lambda + 1) - u_2] u_2 \right] + d_2 \left[ 2(\lambda + 1) - u_2 \right] u_2 \right\}
\]

and consequently, by Carathéodory lemma we have

\[
|a_3| \leq \frac{2(1 - \alpha)}{[3(2\lambda + 1) - u_3] - [2(\lambda + 1) - u_2] u_2]. \tag{3.41}
\]

Combining (3.40) and (3.41), we get the desired estimate on the coefficient \( |a_3| \) as asserted in (3.32).

Finally, from (3.36), by using Carathéodory lemma we deduce that

\[
\left| a_3 - \frac{2[3(2\lambda + 1) - u_3] - [2(\lambda + 1) - u_2] a_2}{3(2\lambda + 1) - u_3} a_2^2 \right| \leq \frac{2(1 - \alpha)}{[3(2\lambda + 1) - u_3]}. \tag{3.37}
\]

\( \square \)

**Remark.** For the special case \( t = 0 \) and \( \lambda = 0 \), the relations (3.18) and (3.19), or (3.31) and (3.32), yield that

\[
|a_2| \leq \sqrt{2(1 - \alpha)}
\]

and

\[
|a_3| \leq 4(1 - \alpha)^2 + 1 - \alpha,
\]

which are the bounds for the coefficients of the functions of the well-known class \( P_\Sigma(\alpha) \), and were previously given by S. Prema and B. Srutha Keerthi \[17\].
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